Math 4707 Spring 2018 (Darij Grinberg): midterm 2 with solutions [preliminary version]

Contents

Recall the following:

- If $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$.
- We use the Iverson bracket notation.

Also, here is a collection of identities that we shall use:

- We have
$$\binom{m}{n} = 0 \tag{1}$$
for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m < n$.
- We have
$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n} \tag{2}$$
for any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. (This is the recurrence relation of the binomial coefficients.)
- We have
$$\binom{m}{n} = \binom{m}{m-n} \tag{3}$$
for any $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m \geq n$.
- We have
$$\binom{m+n}{m} = \binom{m+n}{n} \tag{4}$$
for any $m \in \mathbb{N}$ and $n \in \mathbb{N}$. (This follows by applying (3) to $m+n$ and m instead of m and n.)

- We have
$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!} \tag{5}$$
for any $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m \geq n$. (This is [Grinbe16, Proposition 3.4].)

- We have
$$\binom{m}{n} \neq 0 \tag{6}$$
for any $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m \geq n$. (This follows immediately from (5), since $m! \neq 0$.)

- We have
$$\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1} \tag{7}$$
for any $m \in \mathbb{Q}$ and $n \in \{1,2,3,\ldots\}$. (This is the *absorption identity*, and is precisely [Grinbe16, Proposition 3.22]. Also, it is very easy to check directly.)

- We have
$$\binom{m}{n} = \frac{m-n+1}{n}\binom{m}{n-1} \tag{8}$$
for any $m \in \mathbb{Q}$ and $n \in \{1,2,3,\ldots\}$. (This is easy to check[1].)

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[1] *Proof of (8):* Let $m \in \mathbb{Q}$ and $n \in \{1,2,3,\ldots\}$. Then, $n \neq 0$ (since $n \in \{1,2,3,\ldots\}$), so that the fraction $\frac{m-n+1}{n}$ is well-defined.

We have $n \in \{1,2,3,\ldots\}$; in other words, n is a positive integer. Hence, $m(m-1)\cdots(m-n+1) = (m(m-1)\cdots(m-n+2))\cdot(m-n+1)$ and $n! = n\cdot(n-1)!$.

We have $n \in \{1,2,3,\ldots\} \subseteq \mathbb{N}$. Thus, the definition of $\binom{m}{n}$ yields

$$\begin{aligned}\binom{m}{n} &= \frac{m(m-1)\cdots(m-n+1)}{n!} = \frac{(m(m-1)\cdots(m-n+2))\cdot(m-n+1)}{n\cdot(n-1)!} \\ &\qquad \left(\begin{array}{c}\text{since } m(m-1)\cdots(m-n+1) = (m(m-1)\cdots(m-n+2))\cdot(m-n+1) \\ \text{and } n! = n\cdot(n-1)!\end{array}\right) \\ &= \frac{m-n+1}{n}\cdot\frac{m(m-1)\cdots(m-n+2)}{(n-1)!}. \end{aligned} \tag{9}$$

Moreover, $n-1 \in \mathbb{N}$ (since $n \in \{1,2,3,\ldots\}$), so that the definition of $\binom{m}{n-1}$ yields

$$\binom{m}{n-1} = \frac{m(m-1)\cdots(m-(n-1)+1)}{(n-1)!} = \frac{m(m-1)\cdots(m-n+2)}{(n-1)!}$$
$$(\text{since } m-(n-1)+1 = m-n+2).$$

Multiplying this equality by $\frac{m-n+1}{n}$, we obtain

$$\frac{m-n+1}{n}\binom{m}{n-1} = \frac{m-n+1}{n}\cdot\frac{m(m-1)\cdots(m-n+2)}{(n-1)!}.$$

- Every $n \in \mathbb{N}$ satisfies

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \qquad (10)$$

(This is Corollary 1.16b in the classwork from 22 January 2018, or [Grinbe16, Proposition 3.39 **(b)**].)

- Every $n \in \mathbb{N}$ satisfies

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n = 0]. \qquad (11)$$

(This is Corollary 3.3 in the classwork from 14 February 2018, or [Grinbe16, Proposition 3.39 **(c)**].)

- If $m \in \mathbb{N}$ and $n \in \mathbb{N}$, and if S is an m-element set, then

$$\binom{m}{n} \text{ is the number of all } n\text{-element subsets of } S. \qquad (12)$$

(This is the combinatorial interpretation of the binomial coefficients.)

0.1. Counting first-even tuples

Exercise 1. Let n and d be two positive integers.

An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ will be called *first-even* if its first entry x_1 occurs in it an even number of times (i.e., the number of $i \in [n]$ satisfying $x_i = x_1$ is even). (For example, the 3-tuples $(1, 5, 1)$ and $(2, 2, 3)$ are first-even, while the 3-tuple $(4, 1, 1)$ is not.)

Prove that the number of first-even n-tuples in $[d]^n$ is $\frac{1}{2} d \left(d^{n-1} - (d-2)^{n-1} \right)$.

Our solution for this exercise will rely on the following definition:

Definition 0.1. Let $n \in \mathbb{N}$ and $d \in \mathbb{N}$. Let $k \in [d]$. An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ will be called *k-even* if the number k occurs in it an even number of times (i.e., the number of $i \in [n]$ satisfying $x_i = k$ is even). (For example, the 3-tuple $(1, 4, 4)$ is 4-even and 3-even but not 1-even.)

This definition generalizes the concept of "1-even" defined in Homework set 3. Exercise 5 on Homework set 3 claimed the following:

Comparing this with (9), we obtain $\binom{m}{n} = \frac{m-n+1}{n} \binom{m}{n-1}$. This proves (8).

> **Proposition 0.2.** Let $n \in \mathbb{N}$, and let d be a positive integer. Then, the number of 1-even n-tuples in $[d]^n$ is $\frac{1}{2}\left(d^n + (d-2)^n\right)$.

The same argument proves the following:

> **Proposition 0.3.** Let $n \in \mathbb{N}$, and let d be a positive integer. Let $k \in [d]$. Then, the number of k-even n-tuples in $[d]^n$ is $\frac{1}{2}\left(d^n + (d-2)^n\right)$.

Indeed, Proposition 0.2 is the particular case of Proposition 0.3 for $k = 1$; but conversely, Proposition 0.3 can be derived from Proposition 0.2 by "renaming 1 as k".

To make this rigorous, you can argue as follows:

Proof of Proposition 0.3 (sketched). There is clearly some permutation $\sigma \in S_d$ such that $\sigma(1) = k$. (For example, we can let σ be the transposition swapping 1 with k when $k \neq 1$, and otherwise we can just set $\sigma = \operatorname{id}$.) Fix such a σ. Then, there is a bijection

$$\begin{aligned} \{1\text{-even } n\text{-tuples in } [d]^n\} &\to \{k\text{-even } n\text{-tuples in } [d]^n\}, \\ (x_1, x_2, \ldots, x_n) &\mapsto (\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n)). \end{aligned}$$

(This is well-defined, because the occurrences of the number 1 in an n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ clearly correspond to the occurrences of the number $\sigma(1) = k$ in the n-tuple $(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n))$.) This bijection shows that

$$\begin{aligned} \left|\{k\text{-even } n\text{-tuples in } [d]^n\}\right| &= \left|\{1\text{-even } n\text{-tuples in } [d]^n\}\right| \\ &= \left(\text{the number of 1-even } n\text{-tuples in } [d]^n\right) \\ &= \frac{1}{2}\left(d^n + (d-2)^n\right) \qquad \text{(by Proposition 0.2)}. \end{aligned}$$

In other words, the number of k-even n-tuples in $[d]^n$ is $\frac{1}{2}\left(d^n + (d-2)^n\right)$. This proves Proposition 0.3. □

Solution to Exercise 1 (sketched). We first make the following claim:

> *Observation 1:* Let $k \in [d]$. Then, the number of n-tuples in $[d]^{n-1}$ that are **not** k-even is $\frac{1}{2}\left(d^{n-1} - (d-2)^{n-1}\right)$.

[*Proof of Observation 1:* Proposition 0.3 (applied to $n-1$ instead of n) shows that the number of k-even $(n-1)$-tuples in $[d]^{n-1}$ is $\frac{1}{2}\left(d^{n-1} + (d-2)^{n-1}\right)$. Hence, the number of $(n-1)$-tuples in $[d]^{n-1}$ that are **not** k-even is $d^{n-1} - \frac{1}{2}\left(d^{n-1} + (d-2)^{n-1}\right)$ (since the total number of $(n-1)$-tuples in $[d]^{n-1}$ is d^{n-1}). In view of $d^{n-1} - \frac{1}{2}\left(d^{n-1} + (d-2)^{n-1}\right) = \frac{1}{2}\left(d^{n-1} - (d-2)^{n-1}\right)$, this rewrites as follows: The number of $(n-1)$-tuples in $[d]^{n-1}$ that are **not** k-even is $\frac{1}{2}\left(d^{n-1} - (d-2)^{n-1}\right)$. This proves Observation 1.]

We can construct each first-even n-tuple $(x_1, x_2, \ldots, x_n)$ in $[d]^n$ as follows:

- First, we choose the value of x_1. We denote this value by k. There are d choices at this step (since this value must belong to $[d]$).
- Next, we choose the $(n-1)$-tuple $(x_2, x_3, \ldots, x_n)$. Note that the entry $k = x_1$ must occur an **odd** number of times in this $(n-1)$-tuple $(x_2, x_3, \ldots, x_n)$ (because we want the n-tuple $(x_1, x_2, \ldots, x_n)$ to be first-even, so that x_1 must occur an **even** number of times in this n-tuple; but the $(n-1)$-tuple $(x_2, x_3, \ldots, x_n)$ is missing its very first occurrence, and thus must contain it an **odd** number of times). In other words, the $(n-1)$-tuple $(x_2, x_3, \ldots, x_{n-1})$ must **not** be k-even. Thus, there are $\frac{1}{2}\left(d^{n-1} - (d-2)^{n-1}\right)$ choices at this step (since Observation 1 yields that the number of $(n-1)$-tuples in $[d]^{n-1}$ that are **not** k-even is $\frac{1}{2}\left(d^{n-1} - (d-2)^{n-1}\right)$).

Hence, the total number of first-even n-tuples $(x_1, x_2, \ldots, x_n)$ in $[d]^n$ is

$$d \cdot \frac{1}{2}\left(d^{n-1} - (d-2)^{n-1}\right) = \frac{1}{2} d \left(d^{n-1} - (d-2)^{n-1}\right).$$

This solves Exercise 1. □

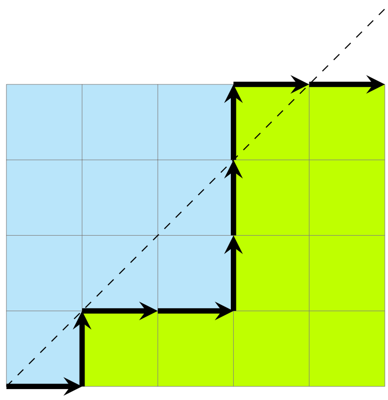
0.2. Counting legal paths (generalization of Catalan numbers)

Recall the notion of a *lattice path*, defined in Midterm 1. (Lattice paths have up-steps and right-steps.)

We say that a point $(x, y) \in \mathbb{Z}^2$ is *off-limits* if $y > x$. (Thus, the off-limits points are the ones that lie strictly above the $x = y$ diagonal in Cartesian coordinates.)

A lattice path $(v_0, v_1, \ldots, v_n)$ is said to be *legal* if none of the points $v_0, v_1, \ldots, v_n$ is off-limits.

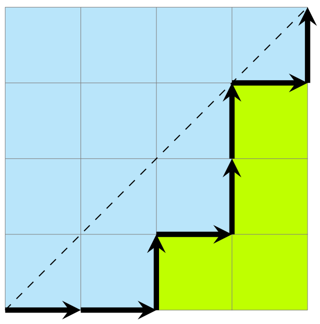
For example, the lattice path drawn from $(0, 0)$ to $(4, 5)$ drawn in the picture



[2] is not legal, since it contains the off-limits point $(3, 4)$. Meanwhile, the lattice path

[2]Formally speaking, this lattice path is the list $((0,0), (1,0), (1,1), (2,1), (3,1), (3,2), (3,3), (3,4), (4,4), (5,4))$.

from $(0,0)$ to $(4,4)$ drawn in the picture



is legal.

For any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we let $L_{n,m}$ be the number of all legal lattice paths from $(0,0)$ to (n,m).

For each point $v = (x,y) \in \mathbb{Z}^2$, we let $\mathbf{x}(v)$ denote the x-coordinate x of v, and we let $\mathbf{y}(v)$ denote the y-coordinate y of v. For example, $\mathbf{x}((5,9)) = 5$ and $\mathbf{y}((5,9)) = 9$.

The following facts are easy:

Lemma 0.4. Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

(a) If $(v_0, v_1, \ldots, v_p)$ is a lattice path from $(0,0)$ to (n,m), then

$$0 = \mathbf{x}(v_0) \leq \mathbf{x}(v_1) \leq \cdots \leq \mathbf{x}(v_p) = n \qquad \text{and}$$
$$0 = \mathbf{y}(v_0) \leq \mathbf{y}(v_1) \leq \cdots \leq \mathbf{y}(v_p) = m.$$

(b) If $(v_0, v_1, \ldots, v_p)$ is a lattice path from $(0,0)$ to (n,m), then

$$\mathbf{x}(v_i) + \mathbf{y}(v_i) = i \qquad \text{for each } i \in \{0, 1, \ldots, p\}.$$

(c) If $(v_0, v_1, \ldots, v_p)$ is a lattice path from $(0,0)$ to (n,m), then $p = n + m$.

(d) The lattice path $((0,0))$ (consisting of the single point $(0,0)$) is the only lattice path from $(0,0)$ to $(0,0)$.

(e) We have $L_{0,0} = 1$.

(f) We have $L_{n,m} = 0$ if at least one of the numbers n and m is negative.

(g) We have $L_{n,m} = 0$ if $m > n$.

The following obnoxiously long argument just formalizes the obviousness:

Proof of Lemma 0.4. **(a)** Let $(v_0, v_1, \ldots, v_p)$ be a lattice path from $(0,0)$ to (n,m). Thus, the definition of a lattice path shows that $v_0 = (0,0)$ and $v_p = (n,m)$. From $v_0 = (0,0)$, we obtain $\mathbf{x}(v_0) = 0$ and $\mathbf{y}(v_0) = 0$. From $v_p = (n,m)$, we obtain $\mathbf{x}(v_p) = n$ and $\mathbf{y}(v_p) = m$.

Let $i \in [p]$. Then, the definition of a lattice path shows that the difference vector $v_i - v_{i-1}$ is either $(0,1)$ or $(1,0)$ (because $(v_0, v_1, \ldots, v_p)$ is a lattice path). In other words, either $v_i - v_{i-1} = (0,1)$ or $v_i - v_{i-1} = (1,0)$. Thus, $\mathbf{x}(v_{i-1}) \leq \mathbf{x}(v_i)$ [3].

[3] *Proof.* We know that either $v_i - v_{i-1} = (0,1)$ or $v_i - v_{i-1} = (1,0)$. Hence, we are in one of the

Now, forget that we fixed i. We thus have proven that $\mathbf{x}(v_{i-1}) \leq \mathbf{x}(v_i)$ for each $i \in [p]$. In other words, $\mathbf{x}(v_0) \leq \mathbf{x}(v_1) \leq \cdots \leq \mathbf{x}(v_p)$. Combining this with $\mathbf{x}(v_0) = 0$ and $\mathbf{x}(v_p) = n$, we obtain

$$0 = \mathbf{x}(v_0) \leq \mathbf{x}(v_1) \leq \cdots \leq \mathbf{x}(v_p) = n.$$

Similarly,

$$0 = \mathbf{y}(v_0) \leq \mathbf{y}(v_1) \leq \cdots \leq \mathbf{y}(v_p) = m.$$

Thus, Lemma 0.4 **(a)** is proven.

(b) Let $(v_0, v_1, \ldots, v_p)$ be a lattice path from $(0,0)$ to (n,m). Thus, the definition of a lattice path shows that $v_0 = (0,0)$ and $v_p = (n,m)$. From $v_0 = (0,0)$, we obtain $\mathbf{x}(v_0) = 0$ and $\mathbf{y}(v_0) = 0$.

For each $i \in \{0, 1, \ldots, p\}$, we define an integer z_i by $z_i = \mathbf{x}(v_i) + \mathbf{y}(v_i) - i$.

Let $i \in [p]$. Then, the definition of a lattice path shows that the difference vector $v_i - v_{i-1}$ is either $(0,1)$ or $(1,0)$ (because $(v_0, v_1, \ldots, v_p)$ is a lattice path). In other words, either $v_i - v_{i-1} = (0,1)$ or $v_i - v_{i-1} = (1,0)$. Thus, $z_i = z_{i-1}$ [4]. In other words, $z_{i-1} = z_i$.

Now, forget that we fixed i. We thus have proven that $z_{i-1} = z_i$ for each $i \in [p]$. In other words, $z_0 = z_1 = \cdots = z_p$.

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following two cases:

Case 1: We have $v_i - v_{i-1} = (0,1)$.

Case 2: We have $v_i - v_{i-1} = (1,0)$.

Let us first consider Case 1. In this case, we have $v_i - v_{i-1} = (0,1)$. Thus, $\mathbf{x}(v_i - v_{i-1}) = 0$. But subtraction of vectors in $\mathbb{Z}^2$ is done coordinatewise. Thus, $\mathbf{x}(v_i - v_{i-1}) = \mathbf{x}(v_i) - \mathbf{x}(v_{i-1})$. Hence, $\mathbf{x}(v_i) - \mathbf{x}(v_{i-1}) = \mathbf{x}(v_i - v_{i-1}) = 0$, so that $\mathbf{x}(v_i) = \mathbf{x}(v_{i-1})$. Therefore, $\mathbf{x}(v_{i-1}) = \mathbf{x}(v_i) \leq \mathbf{x}(v_i)$. Thus, $\mathbf{x}(v_{i-1}) \leq \mathbf{x}(v_i)$ is proven in Case 1.

Let us next consider Case 2. In this case, we have $v_i - v_{i-1} = (1,0)$. Thus, $\mathbf{x}(v_i - v_{i-1}) = 1$. But subtraction of vectors in $\mathbb{Z}^2$ is done coordinatewise. Thus, $\mathbf{x}(v_i - v_{i-1}) = \mathbf{x}(v_i) - \mathbf{x}(v_{i-1})$. Hence, $\mathbf{x}(v_i) - \mathbf{x}(v_{i-1}) = \mathbf{x}(v_i - v_{i-1}) = 1$, so that $\mathbf{x}(v_i) = \mathbf{x}(v_{i-1}) + 1 \geq \mathbf{x}(v_{i-1})$. Therefore, $\mathbf{x}(v_{i-1}) \leq \mathbf{x}(v_i)$. Thus, $\mathbf{x}(v_{i-1}) \leq \mathbf{x}(v_i)$ is proven in Case 2.

We have now proven $\mathbf{x}(v_{i-1}) \leq \mathbf{x}(v_i)$ in each of the two Cases 1 and 2. Hence, $\mathbf{x}(v_{i-1}) \leq \mathbf{x}(v_i)$ always holds. Qed.

[4] *Proof.* The definition of z_i yields $z_i = \mathbf{x}(v_i) + \mathbf{y}(v_i) - i$. But the definition of z_{i-1} yields $z_{i-1} = \mathbf{x}(v_{i-1}) + \mathbf{y}(v_{i-1}) - (i-1)$.

We know that either $v_i - v_{i-1} = (0,1)$ or $v_i - v_{i-1} = (1,0)$. Hence, we are in one of the following two cases:

Case 1: We have $v_i - v_{i-1} = (0,1)$.

Case 2: We have $v_i - v_{i-1} = (1,0)$.

Let us first consider Case 1. In this case, we have $v_i - v_{i-1} = (0,1)$. Thus, $\mathbf{x}(v_i - v_{i-1}) = 0$ and $\mathbf{y}(v_i - v_{i-1}) = 1$. But subtraction of vectors in $\mathbb{Z}^2$ is done coordinatewise. Thus, $\mathbf{x}(v_i - v_{i-1}) = \mathbf{x}(v_i) - \mathbf{x}(v_{i-1})$. Hence, $\mathbf{x}(v_i) - \mathbf{x}(v_{i-1}) = \mathbf{x}(v_i - v_{i-1}) = 0$. Hence, $\mathbf{x}(v_i) = \mathbf{x}(v_{i-1})$.

Also, subtraction of vectors in $\mathbb{Z}^2$ is done coordinatewise. Thus, $\mathbf{y}(v_i - v_{i-1}) = \mathbf{y}(v_i) - \mathbf{y}(v_{i-1})$. Hence, $\mathbf{y}(v_i) - \mathbf{y}(v_{i-1}) = \mathbf{y}(v_i - v_{i-1}) = 1$. Hence, $\mathbf{y}(v_i) = 1 + \mathbf{y}(v_{i-1})$.

Thus,

$$\begin{aligned} z_i &= \underbrace{\mathbf{x}(v_i)}_{=\mathbf{x}(v_{i-1})} + \underbrace{\mathbf{y}(v_i)}_{=1+\mathbf{y}(v_{i-1})} - i = \mathbf{x}(v_{i-1}) + (1 + \mathbf{y}(v_{i-1})) - i \\ &= \mathbf{x}(v_{i-1}) + \mathbf{y}(v_{i-1}) - i + 1 = \mathbf{x}(v_{i-1}) + \mathbf{y}(v_{i-1}) - (i-1) = z_{i-1} \end{aligned}$$

(since $z_{i-1} = \mathbf{x}(v_{i-1}) + \mathbf{y}(v_{i-1}) - (i-1)$). Thus, $z_i = z_{i-1}$ is proven in Case 1.

Similarly, we can prove $z_i = z_{i-1}$ in Case 2.

We have now proven $z_i = z_{i-1}$ in each of the two Cases 1 and 2. Hence, $z_i = z_{i-1}$ always holds. Qed.

Now, let $i \in \{0, 1, \ldots, p\}$ be arbitrary. Then, from $z_0 = z_1 = \cdots = z_p$, we obtain

$$z_i = z_0 = \underbrace{\mathbf{x}(v_0)}_{=0} + \underbrace{\mathbf{y}(v_0)}_{=0} - 0 \qquad \text{(by the definition of } z_0\text{)}$$
$$= 0.$$

Hence, $0 = z_i = \mathbf{x}(v_i) + \mathbf{y}(v_i) - i$ (by the definition of z_i). In other words, $\mathbf{x}(v_i) + \mathbf{y}(v_i) = i$. This proves Lemma 0.4 **(b)**.

(c) Let $(v_0, v_1, \ldots, v_p)$ be a lattice path from $(0,0)$ to (n,m). Thus, the definition of a lattice path shows that $v_0 = (0,0)$ and $v_p = (n,m)$. From $v_p = (n,m)$, we obtain $\mathbf{x}(v_p) = n$ and $\mathbf{y}(v_p) = m$.

But Lemma 0.4 **(b)** (applied to $i = p$) yields $\mathbf{x}(v_p) + \mathbf{y}(v_p) = p$. Hence, $p = \underbrace{\mathbf{x}(v_p)}_{=n} + \underbrace{\mathbf{y}(v_p)}_{=m} = n + m$. This proves Lemma 0.4 **(c)**.

(d) Clearly, the lattice path $((0,0))$ (containing just the single point $(0,0)$) is a lattice path from $(0,0)$ to $(0,0)$.

Let $(v_0, v_1, \ldots, v_p)$ be a lattice path from $(0,0)$ to $(0,0)$. Thus, the definition of a lattice path shows that $v_0 = (0,0)$ and $v_p = (0,0)$. But Lemma 0.4 **(c)** (applied to 0 and 0 instead of n and m) yields $p = 0 + 0 = 0$. Hence, $(v_0, v_1, \ldots, v_p) = (v_0, v_1, \ldots, v_0) = (v_0) = ((0,0))$ (since $v_0 = (0,0)$).

Now, forget that we fixed $(v_0, v_1, \ldots, v_p)$. We thus have shown that if $(v_0, v_1, \ldots, v_p)$ is a lattice path from $(0,0)$ to $(0,0)$, then $(v_0, v_1, \ldots, v_p) = ((0,0))$. In other words, every lattice path from $(0,0)$ to $(0,0)$ must be equal to $((0,0))$. Hence, the path $((0,0))$ is the only lattice path from $(0,0)$ to $(0,0)$ (because we already know that $((0,0))$ is a lattice path from $(0,0)$ to $(0,0)$). This proves Lemma 0.4 **(d)**.

(e) We must prove that $L_{0,0} = 1$. In other words, we must prove that there is exactly one legal lattice path from $(0,0)$ to $(0,0)$ (because $L_{0,0}$ was defined as the number of all legal lattice paths from $(0,0)$ to $(0,0)$). So let us prove this.

Clearly, the lattice path $((0,0))$ (containing just the single point $(0,0)$) is a lattice path from $(0,0)$ to $(0,0)$, and is legal (since the point $(0,0)$ is not off-limits). Hence, there exists at least one legal lattice path from $(0,0)$ to $(0,0)$ (namely, this lattice path $((0,0))$).

But Lemma 0.4 **(d)** yields that the lattice path $((0,0))$ is the only lattice path from $(0,0)$ to $(0,0)$. Hence, the lattice path $((0,0))$ is the only **legal** lattice path from $(0,0)$ to $(0,0)$ as well. Thus, there is **exactly one** legal lattice path from $(0,0)$ to $(0,0)$ (namely, $((0,0))$). In other words, $L_{0,0} = 1$ (since $L_{0,0}$ is the number of all legal lattice paths from $(0,0)$ to $(0,0)$). This proves Lemma 0.4 **(e)**.

(f) Assume that at least one of the numbers n and m is negative. We must prove that $L_{n,m} = 0$. In other words, we must prove that there are no legal lattice paths from $(0,0)$ to (n,m) (because $L_{n,m}$ was defined as the number of all legal lattice paths from $(0,0)$ to (n,m)). So let us prove this.

Let $(v_0, v_1, \ldots, v_p)$ be a legal lattice path from $(0,0)$ to (n,m). Then, Lemma 0.4 **(a)** yields

$$0 = \mathbf{x}(v_0) \leq \mathbf{x}(v_1) \leq \cdots \leq \mathbf{x}(v_p) = n \qquad \text{and}$$
$$0 = \mathbf{y}(v_0) \leq \mathbf{y}(v_1) \leq \cdots \leq \mathbf{y}(v_p) = m.$$

Hence, $0 \leq n$ and $0 \leq m$. Thus, n is nonnegative (since $0 \leq n$) and m is nonnegative (since $0 \leq m$). Hence, both n and m are nonnegative. This contradicts the fact that at least one of n and m is negative.

Now, forget that we fixed $(v_0, v_1, \ldots, v_p)$. We thus have obtained a contradiction for each legal lattice path $(v_0, v_1, \ldots, v_p)$ from $(0,0)$ to (n,m). Hence, there are no legal lattice paths from $(0,0)$ to (n,m). In other words, $L_{n,m} = 0$ (since $L_{n,m}$ is the number of all legal lattice paths from $(0,0)$ to (n,m)). This proves Lemma 0.4 **(f)**.

(g) Assume that $m > n$. We must prove that $L_{n,m} = 0$. In other words, we must prove that there are no legal lattice paths from $(0,0)$ to (n,m) (because $L_{n,m}$ was defined as the number of all legal lattice paths from $(0,0)$ to (n,m)). So let us prove this.

Let $(v_0, v_1, \ldots, v_p)$ be a legal lattice path from $(0,0)$ to (n,m). Thus, the definition of a lattice path shows that $v_0 = (0,0)$ and $v_p = (n,m)$. Moreover, the definition of "legal" shows that none of the points $v_0, v_1, \ldots, v_p$ is off-limits (since the lattice path $(v_0, v_1, \ldots, v_p)$ is legal). Hence, in particular, the point v_p is not off-limits.

But $m > n$. Thus, the point (n, m) is off-limits. In other words, the point v_p is off-limits (since $v_p = (n, m)$). This contradicts the fact that the point v_p is not off-limits.

Now, forget that we fixed $(v_0, v_1, \ldots, v_p)$. We thus have obtained a contradiction for each legal lattice path $(v_0, v_1, \ldots, v_p)$ from $(0,0)$ to (n,m). Hence, there are no legal lattice paths from $(0,0)$ to (n,m). In other words, $L_{n,m} = 0$ (since $L_{n,m}$ is the number of all legal lattice paths from $(0,0)$ to (n,m)). This proves Lemma 0.4 **(g)**. $\square$

Remark 0.5. If you know the notion of a directed graph (a.k.a. digraph), you will immediately recognize the legal lattice paths as the paths in a certain (infinite) directed graph, whose vertices are the pairs $(x,y) \in \mathbb{Z}^2$ satisfying $y \leq x$, and whose arcs are $(x,y) \to (x, y+1)$ and $(x,y) \to (x+1, y)$.

Exercise 2. **(a)** Prove that $L_{n,m} = L_{n-1,m} + L_{n,m-1}$ for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n \geq m$ and $(n,m) \neq (0,0)$.

(b) Prove that

$$L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m - 1$.

[The requirement $n \geq m-1$ as opposed to $n \geq m$ is not a typo; the equality still holds for $n = m-1$, albeit for fairly simple reasons.]

(c) Prove that $L_{n,m} = \dfrac{n+1-m}{n+1}\dbinom{n+m}{m}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m-1$.

(d) Prove that $L_{n,n} = \dfrac{1}{n+1}\dbinom{2n}{n}$ for any $n \in \mathbb{N}$.

Remark 0.6. Exercise 2 **(c)** can be rewritten as follows:

$$L_{n,m} = \frac{n+1-m}{n+1+m}\binom{n+1+m}{n+1}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m-1$.

This is a particular case of the so-called *ballot theorem* (see, e.g., [Renaul07]), obtained by setting $a = n+1$, $b = m$ and $k = 1$. Indeed, a legal lattice path from $(0,0)$ to (n,m) corresponds to a way to count $n+1$ votes for candidate A and m votes for candidate B during an (anonymous) election in such a way that candidate A leads (i.e., has more votes than candidate B) throughout the counting process (at least after the first vote has been counted). (If you have such a vote counting process, you can construct the corresponding lattice path as follows: Ignore the first vote (which is necessarily a vote for A, since otherwise A would lose the lead right away). Every time a vote for A is counted, take a right-step; every time a vote for B is counted, take an up-step.)

Exercise 2 **(d)** is, of course, equivalent to the well-known fact that the Catalan numbers $\dfrac{1}{n+1}\dbinom{2n}{n}$ count Dyck words and Dyck paths. Vic Reiner proved this in two different ways (once using generating functions and once combinatorially) in one of the classes he substituted.

We shall solve Exercise 2 **(b)** by a (more or less) straightforward induction on $n + m$. Exercise 2 **(d)** will follow from Exercise 2 **(b)** (via Exercise 2 **(c)**). However, it is rather difficult to prove Exercise 2 **(d)** directly by induction. Thus, if one wants to prove Exercise 2 **(d)** by induction, one is more or less forced to generalize it to Exercise 2 **(b)**. This illustrates an important phenomenon in mathematics: A more general statement is often easier to prove than a less general one (particularly when the proof uses induction). Thus, generalizing is a problem-solving skill.

Solution to Exercise 2 (sketched). In the following, the word "point" will always mean a pair $(x, y) \in \mathbb{Z}^2$ (and will be regarded as a point in the Euclidean plane $\mathbb{R}^2$). The word "path" will always mean a lattice path. Moreover, if n and m are two integers, then "path to (n, m)" shall always mean "path from $(0, 0)$ to (n, m)". (So, paths start at $(0, 0)$ by default.)

For any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we have

$$\begin{aligned} L_{n,m} &= (\text{the number of all legal lattice paths from } (0,0) \text{ to } (n,m)) \\ &\qquad (\text{by the definition of } L_{n,m}) \\ &= (\text{the number of all legal paths to } (n,m)) \end{aligned} \tag{13}$$

(because we abbreviate "lattice paths from $(0, 0)$ to (n, m)" as "paths to (n, m)").

A *step* in a path $(v_0, v_1, \ldots, v_n)$ means a pair of the form (v_{i-1}, v_i) for some $i \in [n]$. More precisely, this pair (v_{i-1}, v_i) will be called the *i-th step* of the path.

We say that a path $(v_0, v_1, \ldots, v_n)$ *passes* through a point w if $w \in \{v_0, v_1, \ldots, v_n\}$.

(a) Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ be such that $n \geq m$ and $(n, m) \neq (0, 0)$. If the point (n, m) was off-limits, then we would have $m > n$, which would contradict $n \geq m$. Thus, the point (n, m) is not off-limits.

The equality (13) (applied to $m - 1$ instead of m) yields

$$L_{n,m-1} = (\text{the number of all legal paths to } (n, m-1)). \tag{14}$$

Any path to (n, m) contains at least one step (since otherwise, we would have $(n, m) = (0, 0)$, which would contradict $(n, m) \neq (0, 0)$), and thus has a **last** step. This last step must be either an up-step, or a right-step. Hence,

$$\begin{aligned} &(\text{the number of all legal paths to } (n,m)) \\ &= (\text{the number of all legal paths to } (n,m) \text{ whose last step is an up-step}) \\ &\qquad + (\text{the number of all legal paths to } (n,m) \text{ whose last step is a right-step}). \end{aligned} \tag{15}$$

Let us now compute the two numbers on the right hand side.

Any legal path $\mathbf{p}$ to (n, m) whose last step is an up-step must pass through $(n, m - 1)$ (because this is the point from which an up-step leads to (n, m)). Thus, this path $\mathbf{p}$ consists of two parts: the first part is a path to $(n, m-1)$; the second part is a single up-step from $(n, m-1)$ to (n, m). Let us denote the first part by $L(\mathbf{p})$; this first part $L(\mathbf{p})$ is still legal (because any off-limits point on it would also be contained in $\mathbf{p}$). Hence, we have defined a map

$$L : \{\text{legal paths to } (n, m) \text{ whose last step is an up-step}\} \\ \to \{\text{legal paths to } (n, m-1)\}$$

(which simply removes the last step from a path). This map L is a bijection (indeed, the inverse map simply adds an up-step at the end of a path[5]). Thus,

$$\begin{aligned} &|\{\text{legal paths to } (n, m-1)\}| \\ &= |\{\text{legal paths to } (n, m) \text{ whose last step is an up-step}\}| \\ &= (\text{the number of all legal paths to } (n, m) \text{ whose last step is an up-step}). \end{aligned}$$

Comparing this with

$$\begin{aligned} |\{\text{legal paths to } (n, m-1)\}| &= (\text{the number of all legal paths to } (n, m-1)) \\ &= L_{n,m-1} \qquad (\text{by (14)}), \end{aligned}$$

we obtain

$$(\text{the number of all legal paths to } (n, m) \text{ whose last step is an up-step}) = L_{n,m-1}.$$

Similarly,

$$(\text{the number of all legal paths to } (n, m) \text{ whose last step is a right-step}) = L_{n-1,m}.$$

Hence, (15) becomes

$$\begin{aligned} &(\text{the number of all paths to } (n, m)) \\ &= \underbrace{(\text{the number of all paths to } (n, m) \text{ whose last step is an up-step})}_{=L_{n,m-1}} \\ &\qquad + \underbrace{(\text{the number of all paths to } (n, m) \text{ whose last step is a right-step})}_{=L_{n-1,m}} \\ &= L_{n,m-1} + L_{n-1,m} = L_{n-1,m} + L_{n,m-1}. \end{aligned}$$

[5] Why is this inverse map well-defined?

We must show that if $\mathbf{q}$ is a legal path to $(n, m-1)$, then adding an up-step at the end of $\mathbf{q}$ results in a legal path to (n, m) whose last step is an up-step. It is clear that adding an up-step at the end of $\mathbf{q}$ results in a path to (n, m) whose last step is an up-step; let us denote this latter path by $\mathbf{q}'$. All we need to check is that this new path $\mathbf{q}'$ is legal.

The path $\mathbf{q}$ is legal; in other words, none of the points on $\mathbf{q}$ is off-limits. Also, the point (n, m) is not off-limits.

Recall that the path $\mathbf{q}'$ is obtained by adding an up-step at the end of $\mathbf{q}$. Thus, the points on this path $\mathbf{q}'$ are the points on $\mathbf{q}$ and the new point (n, m) (which is where the newly added up-step leads). Since neither the points on $\mathbf{q}$ nor the new point (n, m) are off-limits, we thus conclude that none of the points on $\mathbf{q}'$ is off-limits. In other words, the path $\mathbf{q}'$ is legal. This is exactly what we wanted to show.

Hence, (13) yields

$$L_{n,m} = (\text{the number of all legal paths to } (n,m)) = L_{n-1,m} + L_{n,m-1}.$$

This solves Exercise 2 **(a)**.

(b) We shall solve Exercise 2 **(b)** by strong induction on $n+m$:

Induction step: Let $k \in \mathbb{N}$. Assume (as the induction hypothesis) that Exercise 2 **(b)** holds whenever $n+m<k$. We must prove that Exercise 2 **(b)** holds when $n+m=k$.

We have assumed that Exercise 2 **(b)** holds whenever $n+m<k$. In other words, if $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy $n \geq m-1$ and $n+m<k$, then

$$L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}. \tag{16}$$

Now, let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be such that $n \geq m-1$ and $n+m=k$. We are going to prove that

$$L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}. \tag{17}$$

Indeed, (17) is true when $n = m-1$ [6]. Hence, for the rest of this proof, we WLOG assume that we don't have $n = m-1$. In other words, we have $n \neq m-1$.

We have $n > m-1$ (since we have $n \geq m-1$ but $n \neq m-1$). Since n and $m-1$ are integers, this shows that $n \geq (m-1)+1 = m$.

Furthermore, (17) is true when $(n,m) = (0,0)$ [7]. Hence, for the rest of this proof, we WLOG assume that we don't have $(n,m) = (0,0)$. In other words, we have $(n,m) \neq (0,0)$. Hence, Exercise 2 **(a)** yields $L_{n,m} = L_{n-1,m} + L_{n,m-1}$.

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[6] *Proof.* Assume that $n = m-1$. Thus, $m > m-1 = n$. Hence, Lemma 0.4 **(g)** shows that $L_{n,m} = 0$. But (4) yields $\binom{m+n}{m} = \binom{m+n}{n} = \binom{n+m}{m-1}$ (since $m+n = n+m$ and $n = m-1$). Therefore, $\binom{n+m}{m-1} = \binom{m+n}{m} = \binom{n+m}{m}$. In other words, $\binom{n+m}{m} - \binom{n+m}{m-1} = 0$. Comparing this with $L_{n,m} = 0$, we obtain $L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$. In other words, (17) holds. Thus, we have proven that (17) is true when $n = m-1$.

[7] *Proof.* Assume that $(n,m) = (0,0)$. Thus, $n = 0$ and $m = 0$. Hence, $L_{n,m} = L_{0,0} = 1$ (by Lemma 0.4 **(e)**). Comparing this with

$$\binom{n+m}{m} - \binom{n+m}{m-1} = \underbrace{\binom{0+0}{0}}_{=1} - \underbrace{\binom{0+0}{0-1}}_{=0} \qquad (\text{since } n = 0 \text{ and } m = 0)$$
$$= 1,$$

we obtain $L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$. In other words, (17) holds. Thus, we have proven that (17) is true when $(n,m) = (0,0)$.

Moreover, (17) is true when $m = 0$ [8]. Hence, for the rest of this proof, we WLOG assume that we don't have $m = 0$. In other words, we have $m \neq 0$. Since $m \in \mathbb{N}$, we thus obtain $m > 0$, so that $m - 1 \in \mathbb{N}$ (since $m \in \mathbb{N}$). From $n \geq m > 0$, we also obtain $n - 1 \in \mathbb{N}$ (since $n \in \mathbb{N}$).

The numbers $n-1 \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy $\underbrace{n}_{\geq m} - 1 \geq m - 1$ and $\underbrace{(n-1)}_{<n} + m < n + m = k$. Hence, (16) (applied to $n-1$ instead of n) yields

$$L_{n-1,m} = \binom{(n-1)+m}{m} - \binom{(n-1)+m}{m-1} = \binom{n+m-1}{m} - \binom{n+m-1}{m-1}$$

(since $(n-1)+m = n+m-1$).

On the other hand, the numbers $n \in \mathbb{N}$ and $m - 1 \in \mathbb{N}$ satisfy $n \geq (m-1) - 1$

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[8] *Proof.* Assume that $m = 0$. If we had $n = 0$, then we would thus have $\left(\underbrace{n}_{=0}, \underbrace{m}_{=0}\right) = (0,0)$, which would contradict $(n,m) \neq (0,0)$. Thus, we cannot have $n = 0$. Hence, $n \neq 0$, so that $n > 0$ (since $n \in \mathbb{N}$) and therefore $n - 1 \in \mathbb{N}$.

The number $m - 1$ is negative (since $\underbrace{m}_{=0} - 1 = -1 < 0$). Hence, at least one of the integers n and $m - 1$ is negative. Thus, Lemma 0.4 **(f)** (applied to $m - 1$ instead of m) yields $L_{n,m-1} = 0$.

Also, $n - 1 \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy $\underbrace{(n-1)}_{<n} + m < n + m = k$ and $\underbrace{n}_{\geq 0 = m} - 1 \geq m - 1$. Hence, (16) (applied to $n-1$ instead of n) yields

$$\begin{aligned} L_{n-1,m} &= \binom{(n-1)+m}{m} - \binom{(n-1)+m}{m-1} \\ &= \underbrace{\binom{(n-1)+0}{0}}_{=1} - \underbrace{\binom{(n-1)+0}{0-1}}_{=\binom{n-1}{-1}=0} \qquad \text{(since } m = 0\text{)} \\ &= 1. \end{aligned}$$

Now,

$$L_{n,m} = \underbrace{L_{n-1,m}}_{=1} + \underbrace{L_{n,m-1}}_{=0} = 1.$$

Comparing this with

$$\begin{aligned} \binom{n+m}{m} - \binom{n+m}{m-1} &= \underbrace{\binom{n+0}{0}}_{=1} - \underbrace{\binom{n+0}{0-1}}_{=\binom{n}{-1}=0} \qquad \text{(since } m = 0\text{)} \\ &= 1, \end{aligned}$$

we obtain $L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$. In other words, (17) holds. Thus, we have proven that (17) is true when $m = 0$.

(since $n \geq m-1 \geq (m-1)-1$) and $n+\underbrace{(m-1)}_{<m} < n+m = k$. Hence, (16) (applied to $m-1$ instead of m) yields

$$L_{n,m-1} = \binom{n+(m-1)}{m-1} - \binom{n+(m-1)}{(m-1)-1} = \binom{n+m-1}{m-1} - \binom{n+m-1}{(m-1)-1}$$

(since $n+(m-1) = n+m-1$).

But

$$\begin{aligned} L_{n,m} &= \underbrace{L_{n-1,m}}_{=\binom{n+m-1}{m}-\binom{n+m-1}{m-1}} + \underbrace{L_{n,m-1}}_{=\binom{n+m-1}{m-1}-\binom{n+m-1}{(m-1)-1}} \\ &= \binom{n+m-1}{m} - \binom{n+m-1}{m-1} + \binom{n+m-1}{m-1} - \binom{n+m-1}{(m-1)-1}. \end{aligned}$$

Comparing this with

$$\begin{aligned} &\underbrace{\binom{n+m}{m}}_{\substack{=\binom{n+m-1}{m-1}+\binom{n+m-1}{m} \\ \text{(by (2) (applied to } n+m \text{ and } m \\ \text{instead of } m \text{ and } n))}} - \underbrace{\binom{n+m}{m-1}}_{\substack{=\binom{n+m-1}{(m-1)-1}+\binom{n+m-1}{m-1} \\ \text{(by (2) (applied to } n+m \text{ and } m-1 \\ \text{instead of } m \text{ and } n))}} \\ &= \binom{n+m-1}{m-1} + \binom{n+m-1}{m} - \left(\binom{n+m-1}{(m-1)-1} + \binom{n+m-1}{m-1}\right) \\ &= \binom{n+m-1}{m} - \binom{n+m-1}{m-1} + \binom{n+m-1}{m-1} - \binom{n+m-1}{(m-1)-1}, \end{aligned}$$

we obtain $L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$. Thus, (17) is proven.

Now, forget that we fixed n and m. We thus have shown that if $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy $n \geq m-1$ and $n+m = k$, then (17) holds. In other words, Exercise 2 **(b)** holds when $n+m = k$. This completes the induction step. Hence, Exercise 2 **(b)** is solved by induction.

(c) Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be such that $n \geq m-1$. Then, the fraction $\frac{m}{n+1}$ is well-defined (since $n+1 \neq 0$ (since $n \in \mathbb{N}$)).

Also, (8) (applied to $n+m$ and m instead of m and n) yields

$$\binom{n+m}{m} = \frac{(n+m)-m+1}{m}\binom{n+m}{m-1} = \frac{n+1}{m}\binom{n+m}{m-1}$$

(since $(n+m)-m+1 = n+1$). Multiplying this equality by $\frac{m}{n+1}$, we obtain

$$\frac{m}{n+1}\binom{n+m}{m} = \frac{m}{n+1} \cdot \frac{n+1}{m}\binom{n+m}{m-1} = \binom{n+m}{m-1}. \tag{18}$$

But Exercise 2 **(b)** yields

$$L_{n,m} = \binom{n+m}{m} - \underbrace{\binom{n+m}{m-1}}_{\substack{=\frac{m}{n+1}\binom{n+m}{m} \\ \text{(by (18))}}} = \binom{n+m}{m} - \frac{m}{n+1}\binom{n+m}{m}$$

$$= \underbrace{\left(1 - \frac{m}{n+1}\right)}_{=\frac{n+1-m}{n+1}} \binom{n+m}{m} = \frac{n+1-m}{n+1}\binom{n+m}{m}.$$

This solves Exercise 2 **(c)**.

(d) Let $n \in \mathbb{N}$. Then, $n \geq n-1$. Hence, Exercise 2 **(c)** (applied to $m = n$) yields

$$L_{n,n} = \frac{n+1-n}{n+1}\binom{n+n}{n} = \frac{1}{n+1}\binom{2n}{n}$$

(since $n+1-n = 1$ and $n+n = 2n$). This solves Exercise 2 **(d)**. □

0.3. Scary fractions

Exercise 3. Let k, a and b be three positive integers such that $k \leq a \leq b$. Prove that

$$\frac{k-1}{k}\sum_{n=a}^{b}\frac{1}{\binom{n}{k}} = \frac{1}{\binom{a-1}{k-1}} - \frac{1}{\binom{b}{k-1}}.$$

Exercise 3 may look scary, but it is a straightforward exercise on induction (on b). To make our life a little bit easier, we shall slightly relax the condition $a \leq b$ to $b \geq a-1$ (so that we can use the case $b = a-1$ instead of $b = a$ as an induction base):

Proposition 0.7. Let k be a positive integer. Let a be a positive integer such that $k \leq a$. Let $b \in \{a-1, a, a+1, \ldots\}$. Then,

$$\frac{k-1}{k}\sum_{n=a}^{b}\frac{1}{\binom{n}{k}} = \frac{1}{\binom{a-1}{k-1}} - \frac{1}{\binom{b}{k-1}}.$$

(In particular, all fractions appearing in this equality are well-defined.)

Proof of Proposition 0.7. All fractions appearing in Proposition 0.7 are well-defined.

[*Proof:* We have $k \neq 0$ (since k is a positive integer); thus, the fraction $\frac{k-1}{k}$ is well-defined.

Also, $a-1 \in \mathbb{N}$ (since a is a positive integer) and $k-1 \in \mathbb{N}$ (since k is a positive integer) and $a-1 \geq k-1$ (since $\underbrace{k}_{\leq a} -1 \leq a-1$). Thus, (6) (applied to $a-1$ and $k-1$ instead of m and n) yields $\binom{a-1}{k-1} \neq 0$. Hence, the fraction $\frac{1}{\binom{a-1}{k-1}}$ is well-defined.

Also, $b \in \{a-1, a, a+1, \ldots\}$, so that $b \geq a-1 \geq k-1 \geq 0$ (since $k-1 \in \mathbb{N}$). Hence, $b \in \mathbb{N}$ (since $b \geq 0$ and $b \in \{a-1, a, a+1, \ldots\} \subseteq \mathbb{Z}$) and $k-1 \in \mathbb{N}$ and $b \geq k-1$. Thus, (6) (applied to b and $k-1$ instead of m and n) yields $\binom{b}{k-1} \neq 0$. Hence, the fraction $\frac{1}{\binom{b}{k-1}}$ is well-defined.

Now, let $n \in \{a, a+1, \ldots, b\}$. Thus, $n \geq a > a-1 \geq 0$, so that $n \in \mathbb{N}$ (since $n \in \{a, a+1, \ldots, b\} \subseteq \mathbb{Z}$). Also, $k \in \mathbb{N}$. Furthermore, $n \geq a \geq k$ (since $k \leq a$). Hence, (6) (applied to n and k instead of m and n) yields $\binom{n}{k} \neq 0$. Hence, the fraction $\frac{1}{\binom{n}{k}}$ is well-defined.

Now, forget that we fixed n. We thus have shown that the fraction $\frac{1}{\binom{n}{k}}$ is well-defined for each $n \in \{a, a+1, \ldots, b\}$. We now have proven that the fractions $\frac{k-1}{k}$, $\frac{1}{\binom{a-1}{k-1}}$ and $\frac{1}{\binom{b}{k-1}}$ and also the fractions $\frac{1}{\binom{n}{k}}$ for all $n \in \{a, a+1, \ldots, b\}$ are well-defined. In other words, all fractions appearing in Proposition 0.7 are well-defined.]

Let us now prove Proposition 0.7 by induction on b:

Induction base: Comparing

$$\frac{k-1}{k} \underbrace{\sum_{n=a}^{a-1} \frac{1}{\binom{n}{k}}}_{=(\text{empty sum})=0} = \frac{k-1}{k} \cdot 0 = 0$$

with $\frac{1}{\binom{a-1}{k-1}} - \frac{1}{\binom{a-1}{k-1}} = 0$, we conclude that $\frac{k-1}{k} \sum_{n=a}^{a-1} \frac{1}{\binom{n}{k}} = \frac{1}{\binom{a-1}{k-1}} - \frac{1}{\binom{a-1}{k-1}}$. In other words, Proposition 0.7 holds for $b = a-1$. This completes the induction base.

Induction step: Let $\beta \in \{a, a+1, a+2, \ldots\}$. Assume that Proposition 0.7 holds for $b = \beta - 1$. We must prove that Proposition 0.7 holds for $b = \beta$.

We have assumed that Proposition 0.7 holds for $b = \beta - 1$. In other words, we have

$$\frac{k-1}{k}\sum_{n=a}^{\beta-1}\frac{1}{\dbinom{n}{k}}=\frac{1}{\dbinom{a-1}{k-1}}-\frac{1}{\dbinom{\beta-1}{k-1}}. \tag{19}$$

(In particular, all fractions appearing in this equality are well-defined.)

We have $k\in\{1,2,3,\ldots\}$ (since k is a positive integer). Hence, (7) (applied to $m=\beta$ and $n=k$) yields $\dbinom{\beta}{k}=\dfrac{\beta}{k}\dbinom{\beta-1}{k-1}$. Multiplying this equality by k, we obtain

$$k\binom{\beta}{k}=k\cdot\frac{\beta}{k}\binom{\beta-1}{k-1}=\beta\binom{\beta-1}{k-1}. \tag{20}$$

Now,

$$\begin{aligned}
&\frac{k-1}{k}\underbrace{\sum_{n=a}^{\beta}\frac{1}{\dbinom{n}{k}}}_{=\sum_{n=a}^{\beta-1}\frac{1}{\binom{n}{k}}+\frac{1}{\binom{\beta}{k}}}\\
&=\frac{k-1}{k}\left(\sum_{n=a}^{\beta-1}\frac{1}{\dbinom{n}{k}}+\frac{1}{\dbinom{\beta}{k}}\right)\\
&=\underbrace{\frac{k-1}{k}\sum_{n=a}^{\beta-1}\frac{1}{\dbinom{n}{k}}}_{\substack{=\frac{1}{\binom{a-1}{k-1}}-\frac{1}{\binom{\beta-1}{k-1}}\\ \text{(by (19))}}}+\underbrace{\frac{k-1}{k}\cdot\frac{1}{\dbinom{\beta}{k}}}_{\substack{=\frac{k-1}{k\binom{\beta}{k}}=\frac{k-1}{\beta\binom{\beta-1}{k-1}}\\ \text{(by (20))}}}\\
&=\frac{1}{\dbinom{a-1}{k-1}}-\frac{1}{\dbinom{\beta-1}{k-1}}+\frac{k-1}{\beta\dbinom{\beta-1}{k-1}}=\frac{1}{\dbinom{a-1}{k-1}}-\underbrace{\left(\frac{1}{\dbinom{\beta-1}{k-1}}-\frac{k-1}{\beta\dbinom{\beta-1}{k-1}}\right)}_{=\frac{\beta-(k-1)}{\beta\binom{\beta-1}{k-1}}=\frac{\beta-k+1}{\beta\binom{\beta-1}{k-1}}}\\
&=\frac{1}{\dbinom{a-1}{k-1}}-\frac{\beta-k+1}{\beta\dbinom{\beta-1}{k-1}}.
\end{aligned} \tag{21}$$

Also, (8) (applied to $m = \beta$ and $n = k$) yields $\dbinom{\beta}{k} = \dfrac{\beta - k + 1}{k} \dbinom{\beta}{k-1}$. Multiplying this equality by k, we obtain

$$k \binom{\beta}{k} = k \cdot \frac{\beta - k + 1}{k} \binom{\beta}{k-1} = (\beta - k + 1) \binom{\beta}{k-1}.$$

Comparing this with (20), we obtain

$$\beta \binom{\beta - 1}{k-1} = (\beta - k + 1) \binom{\beta}{k-1}.$$

Therefore,

$$\frac{\beta - k + 1}{\beta \dbinom{\beta - 1}{k-1}} = \frac{\beta - k + 1}{(\beta - k + 1) \dbinom{\beta}{k-1}} = \frac{1}{\dbinom{\beta}{k-1}}.$$

Hence, (21) becomes

$$\frac{k-1}{k} \sum_{n=a}^{\beta} \frac{1}{\dbinom{n}{k}} = \frac{1}{\dbinom{a-1}{k-1}} - \underbrace{\frac{\beta - k + 1}{\beta \dbinom{\beta - 1}{k-1}}}_{= \frac{1}{\binom{\beta}{k-1}}} = \frac{1}{\dbinom{a-1}{k-1}} - \frac{1}{\dbinom{\beta}{k-1}}$$

(and, in particular, all fractions appearing in this equality are well-defined). In other words, Proposition 0.7 holds for $b = \beta$. This completes the induction step. Thus, Proposition 0.7 is proven by induction. $\square$

Solution to Exercise 3. From $b \geq a \geq a - 1$, we obtain $b \in \{a-1, a, a+1, \ldots\}$ (since b is an integer). Thus, Proposition 0.7 yields $\dfrac{k-1}{k} \sum\limits_{n=a}^{b} \dfrac{1}{\dbinom{n}{k}} = \dfrac{1}{\dbinom{a-1}{k-1}} - \dfrac{1}{\dbinom{b}{k-1}}$.

This solves Exercise 3. $\square$

0.4. Derangements that are involutions

Definition 0.8. Let σ be a permutation of a set X.

(a) We say that σ is a *derangement* if and only if each $x \in X$ satisfies $\sigma(x) \neq x$.

(b) We say that σ is an *involution* if and only if $\sigma \circ \sigma = \mathrm{id}$ (that is, each $x \in X$ satisfies $\sigma(\sigma(x)) = x$).

For example, the permutation α of the set $[5]$ that sends $1, 2, 3, 4, 5$ to $3, 5, 1, 4, 2$ is an involution (it satisfies $\alpha \left(\underbrace{\alpha(1)}_{=3} \right) = \alpha(3) = 1$ and $\alpha \left(\underbrace{\alpha(2)}_{=5} \right) = \alpha(5) = 2$ and similarly $\alpha(\alpha(x)) = x$ for all other $x \in [5]$), but not a derangement (since $\alpha(4) = 4$).

On the other hand, the permutation β of the set $[6]$ that sends $1,2,3,4,5,6$ to $3,4,2,1,6,5$ is a derangement (it satisfies $\beta(x) \neq x$ for all $x \in [6]$), but not an involution (since $\beta(\beta(1)) \neq 1$).

Exercise 4. Let $n \in \mathbb{N}$. Prove the following:

(a) If n is odd, then there exist no derangements of $[n]$ that are involutions.

(b) If n is even, then the number of derangements of $[n]$ that are involutions is $\dfrac{n!}{2^{n/2}(n/2)!}$.

[**Hint:** What does the number $\dfrac{n!}{2^{n/2}(n/2)!}$ remind you of?]

0.5. Hypergreen permutations

Exercise 5. Let $n \in \mathbb{N}$ be such that $n \geq 2$. We shall call a permutation $\pi \in S_n$ *hypergreen* if it satisfies both $\pi(1) < \pi(2)$ and $\pi^{-1}(1) < \pi^{-1}(2)$.

(a) Prove that any $\pi \in S_n$ satisfying $\pi(1) = 1$ must be hypergreen.

(b) Prove that the number of hypergreen permutations $\pi \in S_n$ that **do not** satisfy $\pi(1) = 1$ is $\dbinom{n-2}{2}^2 (n-4)!$. (Here, $\dbinom{n-2}{2}^2 (n-4)!$ is understood to be 0 when $n < 4$.)

[**Hint:** For **(b)**, argue first that if $\pi \in S_n$ is hypergreen but does not satisfy $\pi(1) = 1$, then the four numbers $1, 2, \pi(1), \pi(2)$ are distinct.]

0.6. Counting the parts of all compositions

Recall that if $n \in \mathbb{N}$, then a *composition* of n means a finite list $(a_1, a_2, \ldots, a_k)$ of positive integers such that $a_1 + a_2 + \cdots + a_k = n$.

For example, the compositions of 3 are (3), $(2,1)$, $(1,2)$ and $(1,1,1)$.

The *length* of a composition $(a_1, a_2, \ldots, a_k)$ of n is defined to be k.

Exercise 6. Let n be a positive integer. Prove that the sum of the lengths of all compositions of n is $(n+1)\,2^{n-2}$.

See [19fco, solution to Exercise 2.10.9] for a solution to this exercise.

References

[19fco] Darij Grinberg, *Enumerative Combinatorics (Drexel Fall 2019 Math 222 notes)*, 18 September 2020.
http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.

http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10 .

[Renaul07] Marc Renault, *Four Proofs of the Ballot Theorem*, Mathematics Magazine, vol. 80, no. 5, 2007, pp. 345–352.
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