University of Minnesota, School of Mathematics

Math 4707: Combinatorics, Spring 2018 Midterm 2

Brady Olson (edited by Darij Grinberg)

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1 Exercise 1

1.1 Problem

Let n and d be two positive integers.

An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ will be called *first-even* if its first entry x_1 occurs in it an even number of times (i.e., the number of $i \in [n]$ satisfying $x_i = x_1$ is even). (For example, the 3-tuples $(1, 5, 1)$ and $(2, 2, 3)$ are first-even, while the 3-tuple $(4, 1, 1)$ is not.)

Prove that the number of first-even n-tuples in $[d]^n$ is $\dfrac{1}{2} d \left(d^{n-1} - (d-2)^{n-1} \right)$.

1.2 Solution

Remark 1.1. This proof is incredibly similar to that of HW3 Exe5. It follows the same form and uses the same lemma (labeled Lemma 0.17 in HW3), stated below.

Lemma 1.2. *Let $n \in \mathbb{N}$ and $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$. Then,*

$$(x+y)^n + (-x+y)^n = 2 \sum_{\substack{k \in \{0,1,\ldots,n\}; \\ k \text{ is even}}} \binom{n}{k} x^k y^{n-k}.$$

This lemma was proven in the HW3 solutions using the Binomial Formula.

Similarly, we can show the following:

Lemma 1.3. *Let $n \in \mathbb{N}$ and $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$. Then,*

$$(x+y)^n - (-x+y)^n = 2 \sum_{\substack{k \in \{0,1,\ldots,n\}; \\ k \text{ is odd}}} \binom{n}{k} x^k y^{n-k}.$$

Solution to Exercise 1. The problem of finding a first-even n-tuple can be decomposed into selecting the first element x_1 and then building a $(n-1)$-tuple that contains x_1 an odd number of times.

Set $e = d-1$. Then, we can construct any first-even n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ using the following algorithm:

- First, fix some $x_1 \in [d]$, of which there are d choices.
- Next, we choose the number k of times the entry x_1 will appear in the $(n-1)$-tuple $(x_2, x_3, \ldots, x_{n-1})$. This number k must be odd (since we want our n-tuple to be first-even), and must belong to $\{0, 1, \ldots, n-1\}$.
- Then, we choose the k positions in which the $(n-1)$-tuple $(x_2, x_3, \ldots, x_n)$ will have the entry x_1 (in other words, choose the k indices $i \in \{2, 3, \ldots, n\}$ that will satisfy $x_i = x_1$). This choice can be made in $\binom{n-1}{k}$ many ways (since we are choosing k out of $n-1$ possible indices).
- Next, choose the entries in the remaining $(n-1)-k$ positions of our $(n-1)$-tuple. The entries can be arbitrary, except that they must be distinct from x_1 (since we have already chosen the entries that will equal x_1). Thus, there are $d-1 = e$ choices for each entry, and therefore $e^{(n-1)-k}$ choices altogether in this step.

Thus, since choosing x_1 is independent of choosing $(x_2, x_3, \ldots, x_n)$, the total number of first-even n-tuples is $d \sum\limits_{\substack{k \in \{0,1,\ldots,n-1\};\\ k \text{ is odd}}} \binom{n-1}{k} e^{(n-1)-k}$.

Lemma 1.3 (applied to 1, e and $n-1$ instead of x, y and n) yields

$$\begin{aligned}(1+e)^{n-1} - (-1+e)^{n-1} &= 2 \sum_{\substack{k \in \{0,1,\ldots,n-1\};\\ k \text{ is odd}}} \binom{n-1}{k} \underbrace{1^k}_{=1} e^{(n-1)-k} \\ &= 2 \sum_{\substack{k \in \{0,1,\ldots,n-1\};\\ k \text{ is odd}}} \binom{n-1}{k} e^{(n-1)-k}.\end{aligned}$$

Turning this equality around and multiplying both sides by $d/2$, we obtain

$$\begin{aligned}d \sum_{\substack{k \in \{0,1,\ldots,n-1\};\\ k \text{ is odd}}} \binom{n-1}{k} e^{(n-1)-k} &= \frac{1}{2} d \left(\left(1 + \underbrace{e}_{=d-1} \right)^{n-1} - \left(-1 + \underbrace{e}_{=d-1} \right)^{n-1} \right) \\ &= \frac{1}{2} d \left(\left(\underbrace{1+d-1}_{=d} \right)^{n-1} - \left(\underbrace{-1+d-1}_{=d-2} \right)^{n-1} \right) \\ &= \frac{1}{2} d \left(d^{n-1} - (d-2)^{n-1} \right).\end{aligned}$$

But we have already proven that the total number of first-even n-tuples is the left hand side of this equality. Hence, the total number of first-even n-tuples is $\frac{1}{2} \left(d^{n-1} - (d-2)^{n-1} \right)$. □

Remark 1.4. In the case when $n = 1$, the second step of the above algorithm offers no valid choices. But this is not surprising: In fact, there are no first-even 1-tuples, since the first element x_1 will always appear exactly once, making it not even.

2 Exercise 5

2.1 Problem

Let $n \in \mathbb{N}$ be such that $n \geq 2$. We shall call a permutation $\pi \in S_n$ *hypergreen* if it satisfies both $\pi(1) < \pi(2)$ and $\pi^{-1}(1) < \pi^{-1}(2)$.

(a) Prove that any $\pi \in S_n$ satisfying $\pi(1) = 1$ must be hypergreen.

(b) Prove that the number of hypergreen permutations $\pi \in S_n$ that **do not** satisfy $\pi(1) = 1$ is $\dbinom{n-2}{2}^2 (n-4)!$.

2.2 Solution

Proof of (a). Let $\pi \in S_n$ be such that $\pi(1) = 1$. We must show that π is hypergreen.

Since $\pi(1) = 1$ and π is a permutation, $\pi(2) \neq 1$. Since $\pi \in S_n$, we thus have $\pi(2) > 1$, so $\pi(2) > \pi(1)$. Since $\pi(1) = 1$, we have $\pi^{-1}(1) = 1$ and by the same line of logic we have $\pi^{-1}(2) > \pi^{-1}(1)$. So, π is hypergreen. □

Proof of (b). We begin with the following:

> *Observation 1:* Let $\pi \in S_n$ be a hypergreen permutation that does not satisfy $\pi(1) = 1$. Then, $\pi(2) > \pi(1) > 2 > 1$.

[*Proof of Observation 1:* Since $\pi \in S_n$ and $\pi(1) \neq 1$, the number $\pi(1)$ must be at least 2. Since $\pi(2) > \pi(1)$, this entails that $\pi(2)$ must be greater than 2. Similarly, since $\pi^{-1}(1) \neq 1$, the number $\pi^{-1}(1)$ must be at least 2, and therefore $\pi^{-1}(2)$ must be greater than 2. Hence, $\pi^{-1}(2) \neq 1$, so that $\pi(1) \neq 2$ and thus $\pi(1) > 2$ (since $\pi(1) \geq 2$). Hence, $\pi(2) > \pi(1) > 2 > 1$.]

When a permutation $\pi \in S_n$ is hypergreen, its inverse π^{-1} also is hypergreen. Moreover, if $\pi \in S_n$ does not satisfy $\pi(1) = 1$, then its inverse π^{-1} doesn't either. Hence, we can apply Observation 1 to π^{-1} instead of π, and conclude the following:

> *Observation 2:* Let $\pi \in S_n$ be a hypergreen permutation that does not satisfy $\pi(1) = 1$. Then, $\pi^{-1}(2) > \pi^{-1}(1) > 2 > 1$.

Now, we can construct any hypergreen $\pi \in S_n$ that does not satisfy $\pi(1) = 1$ by the following method:

- First, choose the values $\pi(1)$ and $\pi(2)$. These two values must satisfy $\pi(2) > \pi(1) > 2 > 1$ (by Observation 1); thus, they must be chosen from the set $\{3, 4, \ldots, n\}$, and the second is larger than the first. Hence, we have $\dbinom{n-2}{2}$ choices for them.
- Next, choose the preimages $\pi^{-1}(1)$ and $\pi^{-1}(2)$. These two preimages must satisfy $\pi^{-1}(2) > \pi^{-1}(1) > 2 > 1$ (by Observation 2); thus, they must be chosen from the set $\{3, 4, \ldots, n\}$, and the second is larger than the first. Hence, we have $\dbinom{n-2}{2}$ choices for them.

- At this point, four values of π are already chosen: $\pi(1)$, $\pi(2)$, $\pi(\pi^{-1}(1)) = 1$ and $\pi(\pi^{-1}(2)) = 2$. And these four values don't contradict each other, because they are distinct (since $\pi(2) > \pi(1) > 2 > 1$) and their positions also are distinct (since $\pi^{-1}(2) > \pi^{-1}(1) > 2 > 1$). It remains to choose the values of π at the remaining $n-4$ positions. Since π wants to be a permutation, this boils down to choosing a bijection between two given $(n-4)$-element sets; thus, there are $(n-4)!$ options at this step.

Thus, the total number of hypergreen permutations built this way is given by the product $\binom{n-2}{2}\binom{n-2}{2}(n-4)! = \binom{n-2}{2}^2 (n-4)!$. □

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3 Exercise 6

3.1 Problem

Let n be a positive integer. Prove that the sum of the lengths of all compositions of n is $(n+1)\,2^{n-2}$.

3.2 Notation

Throughout the solution below, the symbol "$\underset{0}{=}$" means "equals, by removing addends which are zero", and the symbol "$\underset{\text{Thm } x}{=}$" means "equals, using Theorem x" (and similar for other logical holdings).

3.3 Solution

First, we introduce a closed-form representation for the number of compositions of a given positive integer n whose length is a given positive integer k. This will then be used to sum over all possible lengths k, to get the total sum of composition lengths.

Lemma 3.1. *Let n and k be positive integers. Then, there are exactly $\binom{n-1}{k-1}$ compositions of n with length k.*

Proof of Lemma 3.1. [1] Construct a list of n 1's, with a box between each adjacent pair:

$$(\underbrace{1 \overbrace{\square\ 1\ \square\ \cdots\ \square}^{(n-1)\text{ boxes}}\ 1}_{n\ \text{1's}})$$

Then, if we replace each of the boxes either by a plus sign or by a comma, we obtain a composition of n. Each combination of pluses and commas creates a unique composition. As an example, take $n = 3$, so the list is $(1 \square 1 \square 1)$. The choices for compositions are

$$\begin{array}{ccc} (1 \boxed{,} 1 \boxed{,} 1) & \to & (1,1,1), \\ (1 \boxplus 1 \boxed{,} 1) & \to & (2,1), \\ (1 \boxed{,} 1 \boxplus 1) & \to & (1,2), \\ (1 \boxplus 1 \boxplus 1) & \to & (3). \end{array}$$

[1] An elaboration of the proof found at https://en.wikipedia.org/wiki/Composition_(combinatorics)#Number_of_compositions

Let us explain this in more detail.

Let $\mathfrak{B}$ be the set of all ways to fill the $n-1$ boxes with $k-1$ commas and $n-k$ pluses. We call these ways "box fillings".

Let $\mathfrak{C}$ be the set of all compositions of n having length k.

We can construct a map $F : \mathfrak{B} \to \mathfrak{C}$ as follows: Each box filling in $\mathfrak{B}$ uniquely constructs a composition by additively collapsing terms separated with pluses. This construction leaves you with k terms, since they are separated by $k-1$ commas. This is a composition of n (because of the associativity of addition and because $\sum_{i=1}^{n} 1 = n$). This composition is the image of our box filling under F.

This map F is invertible. Its inverse F^{-1} sends every composition of n to the box filling that is obtained by replacing each entry a_i of the composition by $\underbrace{1+1+\cdots+1}_{a_i \text{ addends}}$ (and putting each comma and each plus sign into a box).

Thus, the map F is a bijection. Hence, $|\mathfrak{B}| = |\mathfrak{C}|$. But $|\mathfrak{B}| = \binom{n-1}{k-1}$ (since choosing a box filling in $\mathfrak{B}$ is tantamount to deciding which $k-1$ of the $n-1$ boxes will contain a comma). Hence, $|\mathfrak{C}| = |\mathfrak{B}| = \binom{n-1}{k-1}$. Due to how $\mathfrak{C}$ was defined, this means that the number of compositions of n having length k is $\binom{n-1}{k-1}$. This proves Lemma 3.1. □

Next, we need a few identities for binomial coefficients to simplify some of the arithmetic and get rid of summations.

Lemma 3.2. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then,* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

This basic lemma was shown in class.

Lemma 3.3. *Let $n \in \mathbb{N}$. Then,* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

This lemma was also shown in class and derived from the binomial formula.

Solution to Exercise 6. We shall solve Exercise 6 by induction.

For each positive integer n, let $p(n)$ be the logical statement that the sum of the lengths of all compositions of n is $(n+1)2^{n-2}$.

For a base case, take $n = 1$. Clearly, the only composition of 1 is (1), where the sum of lengths is, again, 1. On the other hand, $(1+1)2^{1-2} = 2 \cdot 2^{-1} = 1$, so $p(1)$ holds.

For an inductive hypothesis, fix a positive integer m, and assume $p(m)$:

$$(\text{sum of lengths of compositions of } m) = (m+1)2^{m-2}.$$

Then, to show $p(m+1)$, we first devise a more formal representation of the sum of lengths. The sum of lengths of compositions of $m+1$ equals the sum (over all integers k) of the number of compositions of $m+1$ having length k times their length, which length is k. That is,

$$\begin{aligned}
&(\text{sum of lengths of compositions of } m+1) \\
&= \sum_{k \in \mathbb{Z}} k \cdot (\# \text{ of compositions of } m+1 \text{ having length } k).
\end{aligned}$$

Since there can be no compositions of $m+1$ whose length is ≤ 0 (after all, $m+1$ is positive) and no compositions of $m+1$ whose length is $> m+1$, we can turn the sum on the right hand side into a finite sum by indexing k from 1 to $m+1$. Additionally, Lemma 3.1 (applied to $n = m+1$) shows that (# of compositions of $m+1$ having length k) $= \binom{m}{k-1}$ for every positive integer k. Combining these two facts gives

$$\left(\text{sum of lengths of compositions of } m+1\right) = \sum_{k=1}^{m+1} k \binom{m}{k-1}. \tag{1}$$

So, the sum of lengths of compositions of $m+1$ is given by

$$\sum_{k=1}^{m+1} k \binom{m}{k-1} \underset{\text{Lemma 3.2}}{=} \sum_{k=1}^{m+1} k \left(\binom{m-1}{k-2} + \binom{m-1}{k-1} \right).$$

The sums then can be separated, and we can do some index shifting:

$$\sum_{k=1}^{m+1} k \binom{m-1}{k-2} + \sum_{k=1}^{m+1} k \binom{m-1}{k-1} = \sum_{j=0}^{m} (j+1) \binom{m-1}{j-1} + \sum_{k=1}^{m+1} k \binom{m-1}{k-1},$$

and bump up the lower limit of the first sum (since $\binom{x}{-1} = 0$ for any x):

$$\underset{0}{=} \sum_{j=1}^{m} (j+1) \binom{m-1}{j-1} + \sum_{k=1}^{m+1} k \binom{m-1}{k-1}.$$

We move the upper limit of the second sum down (since $\binom{m-1}{k-1} = 0$ for $k = m+1$), and this becomes

$$\underset{0}{=} \sum_{j=1}^{m} (j+1) \binom{m-1}{j-1} + \sum_{k=1}^{m} k \binom{m-1}{k-1}.$$

The left sum can be separated, again, to yield

$$\sum_{j=1}^{m} \binom{m-1}{j-1} + \sum_{j=1}^{m} j \binom{m-1}{j-1} + \sum_{k=1}^{m} k \binom{m-1}{k-1} = \sum_{j=1}^{m} \binom{m-1}{j-1} + 2 \sum_{k=1}^{m} k \binom{m-1}{k-1}.$$

Finally, rewrite the left sum using $\sum\limits_{j=1}^{m} \binom{m-1}{j-1} = \sum\limits_{k=0}^{m-1} \binom{m-1}{k} = 2^{m-1}$ (this is a consequence of Lemma 3.3), and rewrite the right sum using $\sum\limits_{k=1}^{m} k \binom{m-1}{k-1} = (m+1)\, 2^{m-2}$ (this follows from the inductive hypothesis, because just as we proved (1), we can show that (sum of lengths of compositions of m) $= \sum\limits_{k=1}^{m} k \binom{m-1}{k-1}$). We thus obtain

$$= 2^{m-1} + 2\left((m+1) 2^{m-2}\right) = 2^{m-1} + (m+1) 2^{m-1} = ((m+1)+1) 2^{m-1} = ((m+1)+1) 2^{(m+1)-2}.$$

So, $p(m+1)$ is true, given $p(m)$. This completes the induction step.

Hence, $p(n)$ holds for all positive integers n, by the Principle of Mathematical Induction (with a shifted starting index). So, the sum of the lengths of all compositions of n is $(n+1)2^{n-2}$. $\square$