University of Minnesota, School of Mathematics

Math 4707: Combinatorics, Spring 2018 Midterm 2

Nathaniel Gorski (edited by Darij Grinberg)

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Exercise 1

Exercise 0.1. Let n and d be positive integers. Then the number of first-even n-tuples in $[d]^n$ is equal to $\frac{1}{2}d(d^{n-1}-(d-2)^{n-1})$.

Proof. Fix $d \in \mathbb{N}$ with $d \geq 0$. For each $n \geq 1$, let t_n be the number of first-even n-tuples in $[d]^n$.

We will first find a recursive formula for t_n. Let $n \in \mathbb{N}$ with $n \geq 2$. Thus, t_n is the number of first-even n-tuples in $[d]^n$. Let s be one such n-tuple. We examine two cases depending on the value of the last entry of s.

- Case 1: The last entry of s is equal to the first entry of s: If this is the case, then if we consider the first $n-1$ entries of s as an $(n-1)$-tuple, that tuple will have the same first entry as s. And further, that tuple must have an odd number of occurrences of that first entry. Hence, that $(n-1)$-tuple is not first-even. In fact, one can notice that s is formed by taking any $(n-1)$-tuple which is not first-even, and concatenating the first entry of that tuple to the end. Therefore, there must be the same number of first-even n-tuples in $[d]^n$ which have the same first and last entries as there are $(n-1)$-tuples in $[d]^{n-1}$ which are not first-even. Since a tuple must either be first-even or not first-even, and there are d^{n-1} total $(n-1)$-tuples in $[d]^{n-1}$, then the number of tuples which are not first-even in $[d]^{n-1}$ is equal to $d^{n-1} - t_{n-1}$. Hence, there are $d^{n-1} - t_{n-1}$ first-even n-tuples which have the same first and last entry.

- CASE 2: The last entry of s is not equal to the first entry of s: If this is the case, then we consider the first $n-1$ entries of s as an $(n-1)$-tuple. Since the last entry of s is not equal to the first entry of s, then the first $(n-1)$ entries of s form a first-even $(n-1)$-tuple as well, of which there are t_{n-1} of. Clearly, s can be formed by taking any such first-even $(n-1)$-tuple and concatenating an element of $[d]$ to its end which is not equal to the first entry, of which there are $(d-1)$. Hence, there are a total of $(d-1)t_{n-1}$ different first-even n-tuples in $[d]^n$ which have different first and last entries.

And since, for any first-even n-tuple in $[d]^n$, the last entry must either be equal to the first entry, or not equal to the first entry, the total number of first-even n-tuples must be equal to the sum of those which have the same first and last entry, and those which have different first and last entries. So we can therefore conclude that

$$t_n = (d^{n-1} - t_{n-1}) + (d-1)t_{n-1} = d^{n-1} + (d-2)t_{n-1}. \tag{1}$$

Using this fact, we will now verify the claim of the exercise by induction. Let $\mathcal{A}(n)$ represent the statement "The number of first-even n-tuples in $[d]^n$ is equal to $\frac{1}{2}d(d^{n-1} - (d-2)^{n-1})$". We will first show that $\mathcal{A}(1)$ holds. If $n = 1$, then for any 1-tuple in $[d]^1$, there is only one entry, which is equal to the first entry. Hence, there are zero first-even 1-tuples in $[d]^1$. And observe that if $n = 1$, then $\frac{1}{2}d(d^{n-1} - (d-2)^{n-1}) = \frac{1}{2}d(1-1) = 0$. Hence, if $n = 1$, the number of first-even n-tuples in $[d]^n$ is $\frac{1}{2}d(d^{n-1} - (d-2)^{n-1})$, so $\mathcal{A}(1)$ holds.

Now suppose that, for some positive $n \in \mathbb{N}$, $\mathcal{A}(n)$ holds. We will show that $\mathcal{A}(n+1)$ holds. Our induction hypothesis says that $\mathcal{A}(n)$ holds. In other words, the number of first-even n-tuples in $[d]^n$ is equal to $\frac{1}{2}d(d^{n-1} - (d-2)^{n-1})$. Equivalently, $t_n = \frac{1}{2}d(d^{n-1} - (d-2)^{n-1})$.

Now, applying (1) to $n+1$ instead of n, we get

$$\begin{aligned}
t_{n+1} &= d^n + (d-2)t_n \\
&= d^n + (d-2)\frac{1}{2}d(d^{n-1} - (d-2)^{n-1}) \qquad \left(\text{since } t_n = \frac{1}{2}d(d^{n-1} - (d-2)^{n-1})\right) \\
&= d^n + (d-2)\frac{1}{2}dd^{n-1} - (d-2)\frac{1}{2}d(d-2)^{n-1} \\
&= d^n + \frac{1}{2}(d-2)d^n - \frac{1}{2}d(d-2)^n \\
&= \frac{1}{2}dd^n - \frac{1}{2}d(d-2)^n \qquad \left(\text{since } d^n + \frac{1}{2}(d-2)d^n = \frac{1}{2}dd^n\right) \\
&= \frac{1}{2}d(d^n - (d-2)^n) \\
&= \frac{1}{2}d(d^{(n+1)-1} - (d-2)^{(n+1)-1}).
\end{aligned}$$

In other words, the number of first-even $(n+1)$-tuples in $[d]^{n+1}$ is equal to $\frac{1}{2}d(d^{(n+1)-1} - (d-2)^{(n+1)-1})$; hence, $\mathcal{A}(n+1)$ holds.

We now have shown that $\mathcal{A}(1)$ holds, and that $\mathcal{A}(n)$ implies $\mathcal{A}(n+1)$. Therefore, by induction, $\mathcal{A}(n)$ holds for all positive $n \in \mathbb{N}$. In other words, for all positive $n \in \mathbb{N}$, the number of first-even n-tuples in $[d]^n$ is equal to $\frac{1}{2}d(d^{n-1} - (d-2)^{n-1})$.

$\square$

Exercise 3

Exercise 0.2. Let $k \leq a \leq b$ be three positive integers. Then,

$$\frac{k-1}{k} \sum_{n=a}^{b} \frac{1}{\binom{n}{k}} = \frac{1}{\binom{a-1}{k-1}} - \frac{1}{\binom{b}{k-1}}.$$

Proof. For any integer n with $a \leq n \leq b$, we have

$$\begin{aligned}
\frac{1}{\binom{n-1}{k-1}} - \frac{1}{\binom{n}{k-1}} &= \frac{(k-1)!}{(n-1)(n-2)\cdots(n-k+1)} - \frac{(k-1)!}{(n)(n-1)\cdots(n-k+2)} \\
&= \frac{(k-1)!n}{(n)(n-1)\cdots(n-k+1)} - \frac{(k-1)!(n-k+1)}{(n)(n-1)\cdots(n-k+1)} \\
&= \frac{(k-1)!(n-(n-k+1))}{(n)(n-1)\cdots(n-k+1)} \\
&= \frac{(k-1)\cdot(k-1)!}{(n)(n-1)\cdots(n-k+1)} \\
&= \left(\frac{k-1}{k}\right) \frac{k!}{(n)(n-1)\cdots(n-k+1)} \\
&= \left(\frac{k-1}{k}\right) \frac{1}{\binom{n}{k}}.
\end{aligned}$$

Therefore,

$$\frac{k-1}{k} \sum_{n=a}^{b} \frac{1}{\binom{n}{k}} = \sum_{n=a}^{b} \left(\frac{k-1}{k}\right) \frac{1}{\binom{n}{k}} = \sum_{n=a}^{b} \left(\frac{1}{\binom{n-1}{k-1}} - \frac{1}{\binom{n}{k-1}} \right) = \frac{1}{\binom{a-1}{k-1}} - \frac{1}{\binom{b}{k-1}}$$

by the telescope principle. (All the denominators are nonzero, since the relevant values of n satisfy $n \geq a \geq k$ and since $a-1$ and b surpass $k-1$.) □

Exercise 4

The solution will rely on the following lemma:

Lemma 0.3. *Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ be a derangement which is also an involution. Then all of the cycles in the cycle digraph of σ are 2-cycles.*

Proof. Let $i \in [n]$. Then, in the cycle digraph of σ, i is either part of a cycle of length 1, or part of a cycle of length 2, or part of a cycle of length greater than 2. Suppose that i was part of a cycle in σ which had length greater then 2. Then this implies that $\sigma(\sigma(i)) \neq i$. But since σ is an involution, we know that $\sigma(\sigma(i)) = i$, which is a contradiction. Likewise, if i were part of a 1-cycle, then we would have that $\sigma(i) = i$. But since σ is a derangement, we know that $\sigma(i) \neq i$, which is a contradiction.

Therefore, the only valid length of a cycle in the cycle digraph of σ that i could be part of is 2. Since i could represent any element of $[n]$, every element of $[n]$ is in a 2-cycle in the cycle digraph of σ. Hence, every cycle in the cycle digraph of σ is a 2-cycle. □

Part A

Exercise 0.4. Let $n \in \mathbb{N}$ be odd. Then there exist no derangements of $[n]$ which are involutions.

Proof. Let $n \in \mathbb{N}$ be odd. Suppose, to the contrary, that there existed a derangement σ of $[n]$ which was an involution. Since σ is a derangement, it is a permutation. Thus, consider the cycle digraph of σ. Each element of $[n]$ is featured in exactly one cycle of this digraph. Thus, the sum of the lengths of the cycles in the cycle digraph is equal to n. Since n is odd, therefore, the sum of the lengths of the cycles must be odd. From the lemma, however, each cycle must have length 2, so the sum of all their lengths must be a multiple of 2, which is even. This is a contradiction. □

Part B

Exercise 0.5. Let $n \in \mathbb{N}$ be even. Then the number of derangements of $[n]$ which are involutions is $\dfrac{n!}{2^{n/2}(n/2)!}$.

Proof. In the lemma, we showed that the cycle digraph of any derangement which is an involution must contain only 2-cycles. We will now show that any permutation for which every element in the domain is part of a 2-cycle is both a derangement and an involution. Indeed, let $\sigma \in S_n$ be a permutation such that every element in $[n]$ lies in a 2-cycle in the cycle digraph of σ. Each $i \in [n]$ is part of a 2-cycle in the cycle digraph of σ, and thus satisfies $\sigma(i) \neq i$; hence, σ is a derangement. And also, each $i \in [n]$ is part of a 2-cycle in the cycle digraph of σ, and thus satisfies $\sigma(\sigma(i)) = i$; hence, σ is an involution.

Thus, every derangement which is also an involution has a cycle digraph composed only of 2-cycles, and conversely, every permutation which has a cycle digraph composed only of 2-cycles is both a derangement and an involution. Therefore, the number of derangements which are also involutions is equal to the number of permutations which have a cycle digraph composed only of 2-cycles.

Now let A be the set of derangements of $[n]$ which are also involutions, and let B be the set of perfect matchings of $[n]$. (See Homework set #3 for the definition of a perfect matching.) Then we claim that there is a bijection from A to B.

Let $\alpha : A \to B$ be defined such that, if $\sigma \in A$, then α maps σ to the perfect matching

$$\{\text{all cycles in the cycle digraph of } \sigma\}$$

of $[n]$ (where we regard each cycle as the set of all elements belonging to this cycle). This is well-defined, since each cycle in the cycle digraph of σ has exactly 2 elements (because σ is a derangement which is also an involution, and thus has a cycle digraph composed only of 2-cycles), and because each element of $[n]$ belongs to exactly one cycle in the cycle digraph of σ.

[For example: If $n = 6$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 5 & 4 & 2 \end{pmatrix}$, then $\alpha(\sigma) = \{\{1,3\},\{2,6\},\{4,5\}\}$.]

And let $\beta : B \to A$ be defined such that, if $p = \{\{p_1, q_1\}, \{p_2, q_2\}, \ldots, \{p_k, q_k\}\} \in B$ (with $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k$ all being distinct), then β maps p to the derangement of $[n]$ which sends $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k$ to $q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k$, respectively. (This latter derangement can also be computed as the composition $t_{p_1,q_1} \circ t_{p_2,q_2} \circ \cdots \circ t_{p_k,q_k}$ of transpositions.) Because each element of $[n]$ is contained in a pair in p, then each element of $[n]$ will be contained in a 2-cycle of $\beta(p)$, so $\beta(p) \in A$ and β is well defined.

It is easy to see that $\alpha \circ \beta = \text{id}$ and $\beta \circ \alpha = \text{id}$.

Since the maps α and β are well-defined and mutually inverse, they are bijections between the set of derangements of $[n]$ which are also involutions, and the set of perfect matchings of $[n]$. So there are an equal number of each. Since n is even, then there exists some $a \in \mathbb{N}$ such that $n = 2a$. Consider this a. Exercise 3 **(c)** of homework set 3 shows that the number of perfect matchings of $[2a]$ is equal to $\frac{(2a)!}{2^a (a)!}$. In other words, the number of perfect matchings of $[n]$ is equal to $\frac{n!}{2^{(n/2)} (n/2)!}$. And thus, the number of derangements of $[n]$ which are also involutions is equal to $\frac{n!}{2^{(n/2)} (n/2)!}$. □

Exercise 6

Exercise 0.6. Let n be a positive integer. Then the sum of the lengths of all compositions of n is $(n+1)2^{n-2}$.

Proof. Corollary 1.16b from the class of January 22nd says that, for any $m \in \mathbb{N}$, we have

$$\sum_{i=0}^{m} \binom{m}{i} = 2^m. \tag{2}$$

A composition of n can take any size k such that $1 \leq k \leq n$ (since n is positive). If $k \in [n]$ is given, then the number of compositions of n into k parts is $\binom{n-1}{k-1}$. Since a composition of n into k parts has length k, the total length of all compositions of n into k parts is $k\binom{n-1}{k-1}$. And hence, the sum of the lengths of all compositions of n is $\sum_{k=1}^{n} k\binom{n-1}{k-1}$. But

$$\begin{aligned}
\sum_{k=1}^{n} k\binom{n-1}{k-1} &= \sum_{k=1}^{n}(n-(n-k))\frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \\
&= \sum_{k=1}^{n}\Bigg(n\frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \\
&\qquad - (n-k)\frac{(n-1)(n-2)(n-3)\cdots(n-k+1)}{(k-1)!}\Bigg) \\
&= \sum_{k=1}^{n}\left(n\frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} - (n-1)\frac{(n-2)(n-3)\cdots(n-k)}{(k-1)!}\right) \\
&= \sum_{k=1}^{n}\left(n\binom{n-1}{k-1} - (n-1)\binom{n-2}{k-1}\right) \\
&= \sum_{k=1}^{n}\left(n\binom{n-2}{k-1} + n\binom{n-2}{k-2} - (n-1)\binom{n-2}{k-1}\right) \\
&\qquad \left(\text{since } \binom{n-1}{k-1} = \binom{n-2}{k-1} + \binom{n-2}{k-2}\right) \\
&= \sum_{k=1}^{n}\left(n\binom{n-2}{k-2} + \binom{n-2}{k-1}\right) \\
&= n\sum_{k=1}^{n}\binom{n-2}{k-2} + \sum_{k=1}^{n}\binom{n-2}{k-1}.
\end{aligned}$$

If we substitute $i := k-2$ and $j := k-1$ in the two sums, the right hand side of this equality becomes

$$n\sum_{i=-1}^{n-2}\binom{n-2}{i} + \sum_{j=0}^{n-1}\binom{n-2}{j}.$$

And if we set $m := n-2$, then this further becomes

$$\begin{aligned}
n\sum_{i=-1}^{m}\binom{m}{i} + \sum_{j=0}^{m+1}\binom{m}{j} &= n\binom{m}{-1} + n\sum_{i=0}^{m}\binom{m}{i} + \sum_{j=0}^{m}\binom{m}{j} + \binom{m}{m+1} \\
&= 0 + n2^m + 2^m + 0 \qquad \text{(here, we used (2) twice)} \\
&= (n+1)2^m \\
&= (n+1)2^{n-2}.
\end{aligned}$$

Hence, we have that $\sum_{k=1}^{n} k\binom{n-1}{k-1} = (n+1)2^{n-2}$. And since the sum of the lengths of all compositions of n is $\sum_{k=1}^{n} k\binom{n-1}{k-1}$, it is therefore also $(n+1)2^{n-2}$. $\square$