Math 4707 Spring 2018 (Darij Grinberg): midterm 2 [corrected version]
due date: Wednesday 4 April 2018 at the beginning of class, or before that by email or moodle
Please solve **at most 4** of the 6 exercises!
Collaboration is not allowed!

Contents

Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

Recall the following:

- If $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$.
- We use the Iverson bracket notation.

0.1. Counting first-even tuples

Exercise 1. Let n and d be two positive integers.

An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ will be called *first-even* if its first entry x_1 occurs in it an even number of times (i.e., the number of $i \in [n]$ satisfying $x_i = x_1$ is even). (For example, the 3-tuples $(1, 5, 1)$ and $(2, 2, 3)$ are first-even, while the 3-tuple $(4, 1, 1)$ is not.)

Prove that the number of first-even n-tuples in $[d]^n$ is $\frac{1}{2} d \left(d^{n-1} - (d-2)^{n-1} \right)$.

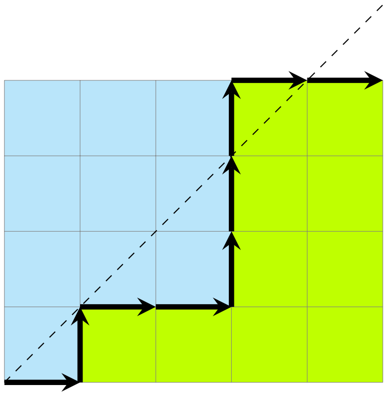
0.2. Counting legal paths (generalization of Catalan numbers)

Recall the notion of a *lattice path*, defined in Midterm 1. (Lattice paths have up-steps and right-steps.)

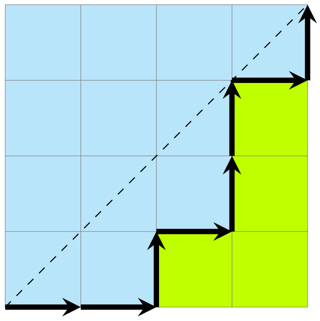
We say that a point $(x, y) \in \mathbb{Z}^2$ is *off-limits* if $y > x$. (Thus, the off-limits points are the ones that lie strictly above the $x = y$ diagonal in Cartesian coordinates.)

A lattice path $(v_0, v_1, \ldots, v_n)$ is said to be *legal* if none of the points $v_0, v_1, \ldots, v_n$ is off-limits.

For example, the lattice path drawn from $(0,0)$ to $(4,5)$ drawn in the picture



[1] is not legal, since it contains the off-limits point $(3,4)$. Meanwhile, the lattice path from $(0,0)$ to $(4,4)$ drawn in the picture



is legal.

For any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we let $L_{n,m}$ be the number of all legal lattice paths from $(0,0)$ to (n,m). Clearly, $L_{n,m} = 0$ if any of n and m is negative. Also, $L_{n,m} = 0$ if $m > n$ (because if $m > n$, then the point (n,m) is off-limits).

Exercise 2. (a) Prove that $L_{n,m} = L_{n-1,m} + L_{n,m-1}$ for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n \geq m$ and $(n,m) \neq (0,0)$.

(b) Prove that

$$L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m-1$.

[The requirement $n \geq m-1$ as opposed to $n \geq m$ is not a typo; the equality still holds for $n = m-1$, albeit for fairly simple reasons.]

(c) Prove that $L_{n,m} = \dfrac{n+1-m}{n+1}\dbinom{n+m}{m}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m-1$.

(d) Prove that $L_{n,n} = \dfrac{1}{n+1}\dbinom{2n}{n}$ for any $n \in \mathbb{N}$.

[1] Formally speaking, this lattice path is the list $((0,0),(1,0),(1,1),(2,1),(3,1),(3,2),(3,3),(3,4),(4,4),(5,4))$.

[You **cannot** use generating functions in this exercise.]

[**Hint:** You may know part **(d)** from Vic's lectures, but the whole exercise can be solved by induction without recourse to any of what Vic did – I even think this is the easiest way to solve it!]

0.3. Scary fractions

Exercise 3. Let k, a and b be three positive integers such that $k \leq a \leq b$. Prove that

$$\frac{k-1}{k} \sum_{n=a}^{b} \frac{1}{\dbinom{n}{k}} = \frac{1}{\dbinom{a-1}{k-1}} - \frac{1}{\dbinom{b}{k-1}}.$$

0.4. Derangements that are involutions

Definition 0.1. Let σ be a permutation of a set X.

(a) We say that σ is a *derangement* if and only if each $x \in X$ satisfies $\sigma(x) \neq x$.

(b) We say that σ is an *involution* if and only if $\sigma \circ \sigma = \operatorname{id}$ (that is, each $x \in X$ satisfies $\sigma(\sigma(x)) = x$).

For example, the permutation α of the set $[5]$ that sends $1, 2, 3, 4, 5$ to $3, 5, 1, 4, 2$ is an involution (it satisfies $\alpha\left(\underbrace{\alpha(1)}_{=3}\right) = \alpha(3) = 1$ and $\alpha\left(\underbrace{\alpha(2)}_{=5}\right) = \alpha(5) = 2$ and similarly $\alpha(\alpha(x)) = x$ for all other $x \in [5]$), but not a derangement (since $\alpha(4) = 4$).

On the other hand, the permutation β of the set $[6]$ that sends $1, 2, 3, 4, 5, 6$ to $3, 4, 2, 1, 6, 5$ is a derangement (it satisfies $\beta(x) \neq x$ for all $x \in [6]$), but not an involution (since $\beta(\beta(1)) \neq 1$).

Exercise 4. Let $n \in \mathbb{N}$. Prove the following:

(a) If n is odd, then there exist no derangements of $[n]$ that are involutions.

(b) If n is even, then the number of derangements of $[n]$ that are involutions is $\dfrac{n!}{2^{n/2}(n/2)!}$.

[**Hint:** What does the number $\dfrac{n!}{2^{n/2}(n/2)!}$ remind you of?]

0.5. Hypergreen permutations

Exercise 5. Let $n \in \mathbb{N}$ be such that $n \geq 2$. We shall call a permutation $\pi \in S_n$ *hypergreen* if it satisfies both $\pi(1) < \pi(2)$ and $\pi^{-1}(1) < \pi^{-1}(2)$.

(a) Prove that any $\pi \in S_n$ satisfying $\pi(1) = 1$ must be hypergreen.

(b) Prove that the number of hypergreen permutations $\pi \in S_n$ that **do not** satisfy $\pi(1) = 1$ is $\binom{n-2}{2}^2 (n-4)!$. (Here, $\binom{n-2}{2}^2 (n-4)!$ is understood to be 0 when $n < 4$.)

[**Hint:** For **(b)**, argue first that if $\pi \in S_n$ is hypergreen but does not satisfy $\pi(1) = 1$, then the four numbers $1, 2, \pi(1), \pi(2)$ are distinct.]

0.6. Counting the parts of all compositions

Recall that if $n \in \mathbb{N}$, then a *composition* of n means a finite list $(a_1, a_2, \ldots, a_k)$ of positive integers such that $a_1 + a_2 + \cdots + a_k = n$.

For example, the compositions of 3 are (3), $(2, 1)$, $(1, 2)$ and $(1, 1, 1)$.

The *length* of a composition $(a_1, a_2, \ldots, a_k)$ of n is defined to be k.

Exercise 6. Let n be a positive integer. Prove that the sum of the lengths of all compositions of n is $(n+1) 2^{n-2}$.