Math 4707 Spring 2018 (Darij Grinberg): midterm 1 with solutions

Contents

Let us recall that

$$\binom{m}{n} = 0 \qquad (1)$$

for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m < n$. (This is exactly [Grinbe16, Proposition 3.6].)

We also recall the *recurrence relation of the binomial coefficients*. It says that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for all $n \in \mathbb{Q}$ and $k \in \mathbb{Z}$.

0.1. More on the Sierpinski triangle in Pascal's triangle

Exercise 1. Let $n \in \mathbb{N}$.

(a) Prove that the integer $\binom{2^n - 1}{b}$ is odd for each $b \in \{0, 1, \ldots, 2^n - 1\}$.

(b) Prove that the integer $\binom{2^n}{b}$ is even for each $b \in \{1, 2, \ldots, 2^n - 1\}$.

[Here, the set $\{0, 1, \ldots, 2^n - 1\}$ means the set of all integers k with $0 \leq k \leq 2^n - 1$, and the set $\{1, 2, \ldots, 2^n - 1\}$ means the set of all integers k with $1 \leq k \leq 2^n - 1$.]

I know of several solutions for Exercise 1. The shortest one (though seemingly somewhat unmotivated) relies on the following binomial-coefficient identity:

Proposition 0.1. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Then,

$$\sum_{r=0}^{m} (-1)^r \binom{n}{r} = (-1)^m \binom{n-1}{m}.$$

Proposition 0.1 is the particular case of [Grinbe16, Lemma 3.47] obtained by setting $k = 0$. But it also falls prey to a straightforward induction, which should be routine by now:

Proof of Proposition 0.1. We shall prove Proposition 0.1 by induction on m:

Induction base: We have $\sum_{r=0}^{0} (-1)^r \binom{n}{r} = \underbrace{(-1)^0}_{=1} \underbrace{\binom{n}{0}}_{=1} = 1$. Comparing this with $\underbrace{(-1)^0}_{=1} \underbrace{\binom{n-1}{0}}_{=1} = 1$, we obtain $\sum_{r=0}^{0} (-1)^r \binom{n}{r} = (-1)^0 \binom{n-1}{0}$. In other words, Proposition 0.1 holds for $m = 0$.

Induction step: Let $\mu \in \mathbb{N}$. Assume that Proposition 0.1 holds for $m = \mu$. We must prove that Proposition 0.1 holds for $m = \mu + 1$.

We have assumed that Proposition 0.1 holds for $m = \mu$. In other words, we have

$$\sum_{r=0}^{\mu} (-1)^r \binom{n}{r} = (-1)^{\mu} \binom{n-1}{\mu}.$$

Now,

$$\begin{aligned}
\sum_{r=0}^{\mu+1} (-1)^r \binom{n}{r} &= \underbrace{\sum_{r=0}^{\mu} (-1)^r \binom{n}{r}}_{=(-1)^{\mu}\binom{n-1}{\mu}} + (-1)^{\mu+1} \underbrace{\binom{n}{\mu+1}}_{\substack{=\binom{n-1}{(\mu+1)-1}+\binom{n-1}{\mu+1} \\ \text{(by the recurrence relation of the binomial coefficients)}}} \\
&= \underbrace{(-1)^{\mu}}_{=-(-1)^{\mu+1}} \binom{n-1}{\mu} + (-1)^{\mu+1} \left(\underbrace{\binom{n-1}{(\mu+1)-1}}_{=\binom{n-1}{\mu}} + \binom{n-1}{\mu+1} \right) \\
&= \left(-(-1)^{\mu+1} \right) \binom{n-1}{\mu} + (-1)^{\mu+1} \left(\binom{n-1}{\mu} + \binom{n-1}{\mu+1} \right) \\
&= -(-1)^{\mu+1} \binom{n-1}{\mu} + (-1)^{\mu+1} \binom{n-1}{\mu} + (-1)^{\mu+1} \binom{n-1}{\mu+1} \\
&= (-1)^{\mu+1} \binom{n-1}{\mu+1}.
\end{aligned}$$

In other words, Proposition 0.1 holds for $m = \mu + 1$. This completes the induction step. Thus, Proposition 0.1 is proven by induction. □

Actually, Proposition 0.1 holds for all $n \in \mathbb{Q}$ (not just for $n \in \mathbb{N}$), and the same proof that we gave above applies in this generality.

Our next is ingredient is Exercise 4 from Math 4707 Homework Set 1, which we restate as a proposition:

Proposition 0.2. Let $n \in \mathbb{N}$. Let a and b be two elements of $\{0, 1, \ldots, 2^n - 1\}$. Then,

$$\binom{2^n + a}{b} \equiv \binom{a}{b} \mod 2 \qquad \text{and} \qquad (2)$$

$$\binom{2^n + a}{2^n + b} \equiv \binom{a}{b} \mod 2. \qquad (3)$$

[Here, "$\{0, 1, \ldots, 2^n - 1\}$" means the set of all integers k satisfying $0 \leq k \leq 2^n - 1$.]

We recall also the following easy fact (which appeared as Lemma 0.12 **(a)** in the solutions to Math 4707 Homework Set 1):

Lemma 0.3. Let $b \in \mathbb{Z}$. Then, $\binom{0}{b} = [b = 0]$. (Here, the Iverson bracket notation is being used.)

We can now easily deal with Exercise 1:

Solution to Exercise 1. **(b)** Let $b \in \{1, 2, \ldots, 2^n - 1\}$. We have $0 \in \{0, 1, \ldots, 2^n - 1\}$ and $b \in \{1, 2, \ldots, 2^n - 1\} \subseteq \{0, 1, \ldots, 2^n - 1\}$. Hence, (2) (applied to $a = 0$) yields $\binom{2^n + 0}{b} \equiv \binom{0}{b} \mod 2$.

Lemma 0.3 yields $\binom{0}{b} = [b = 0]$. But $b \neq 0$ (since $b \in \{1, 2, \ldots, 2^n - 1\}$). Hence, $[b = 0] = 0$. Finally, $2^n = 2^n + 0$. Therefore,

$$\binom{2^n}{b} = \binom{2^n + 0}{b} \equiv \binom{0}{b} = [b = 0] = 0 \mod 2.$$

In other words, the integer $\binom{2^n}{b}$ is even. This solves Exercise 1 **(b)**.

(a) Let $b \in \{0, 1, \ldots, 2^n - 1\}$. Thus, $b \leq 2^n - 1$. Now,

$$\binom{2^n}{r} \equiv 0 \mod 2 \qquad \text{for each } r \in \{1, 2, \ldots, b\}. \qquad (4)$$

[*Proof of (4):* Let $r \in \{1, 2, \ldots, b\}$. Thus, $r \in \{1, 2, \ldots, b\} \subseteq \{1, 2, \ldots, 2^n - 1\}$ (since $b \leq 2^n - 1$). Hence, Exercise 1 **(b)** (applied to r instead of b) yields that the integer $\binom{2^n}{r}$ is even. In other words, $\binom{2^n}{r} \equiv 0 \mod 2$. This proves (4).]

The integer $(-1)^b$ is odd (because it is either 1 or -1). In other words, $(-1)^b \equiv 1 \mod 2$.

Proposition 0.1 (applied to 2^n and b instead of n and m) yields

$$\sum_{r=0}^{b} (-1)^r \binom{2^n}{r} = \underbrace{(-1)^b}_{\equiv 1 \bmod 2} \binom{2^n - 1}{b} \equiv \binom{2^n - 1}{b} \bmod 2.$$

Hence,

$$\binom{2^n - 1}{b} \equiv \sum_{r=0}^{b} (-1)^r \binom{2^n}{r} = \underbrace{(-1)^0}_{=1} \underbrace{\binom{2^n}{0}}_{=1} + \sum_{r=1}^{b} (-1)^r \underbrace{\binom{2^n}{r}}_{\substack{\equiv 0 \bmod 2 \\ \text{(by (4))}}}$$
$$\equiv 1 + \underbrace{\sum_{r=1}^{b} (-1)^r 0}_{=0} = 1 \bmod 2.$$

In other words, the integer $\binom{2^n - 1}{b}$ is odd. This solves Exercise 1 **(a)**. □

Remark 0.4. The above solution of Exercise 1 **(a)** appears to come out of left field; how did I come up with Proposition 0.1, and how did I think of applying it to 2^n and b instead of n and m ?

Here is the motivation: Having solved part **(b)**, I wanted to deduce part **(a)** from it. This necessitates finding a formula for the entries in the $(2^n - 1)$-st row of Pascal's triangle in terms of entries in the 2^n-th row. To get such a formula, I try to reorganize the recurrence relation of the binomial coefficients as

$$\binom{2^n - 1}{b} = \binom{2^n}{b} - \binom{2^n - 1}{b - 1}. \tag{5}$$

But now, I have a $\binom{2^n - 1}{b - 1}$ on the right hand side, which is another entry on the $(2^n - 1)$-st row. I rewrite this $\binom{2^n - 1}{b - 1}$ again using the recurrence relation of the binomial coefficients, obtaining $\binom{2^n - 1}{b - 1} = \binom{2^n}{b - 1} - \binom{2^n - 1}{b - 2}$. Substituting into (5), I obtain

$$\binom{2^n - 1}{b} = \binom{2^n}{b} - \left(\binom{2^n}{b - 1} - \binom{2^n - 1}{b - 2} \right)$$
$$= \binom{2^n}{b} - \binom{2^n}{b - 1} + \binom{2^n - 1}{b - 2}.$$

Now, I have a $\binom{2^n - 1}{b - 2}$ on the right hand side, which I rewrite using the recurrence relation in the same way... and so on. I obtain the following chain of equalities:

$$\binom{2^n - 1}{b} = \binom{2^n}{b} - \binom{2^n - 1}{b - 1}$$
$$= \binom{2^n}{b} - \binom{2^n}{b - 1} + \binom{2^n - 1}{b - 2}$$
$$= \binom{2^n}{b} - \binom{2^n}{b - 1} + \binom{2^n}{b - 2} - \binom{2^n - 1}{b - 3}$$
$$= \cdots .$$

I end this chain once the binomial coefficient with the $2^n - 1$ on top is $\dbinom{2^n - 1}{-1}$. The result is

$$\binom{2^n - 1}{b} = \binom{2^n}{b} - \binom{2^n}{b-1} + \binom{2^n}{b-2} \pm \cdots + (-1)^b \binom{2^n}{0} + (-1)^{b+1} \underbrace{\binom{2^n - 1}{-1}}_{=0}$$
$$= \binom{2^n}{b} - \binom{2^n}{b-1} + \binom{2^n}{b-2} \pm \cdots + (-1)^b \binom{2^n}{0}.$$

This is a binomial identity that doesn't rely on 2^n being a power of 2, so I generalize it to

$$\binom{n-1}{b} = \binom{n}{b} - \binom{n}{b-1} + \binom{n}{b-2} \pm \cdots + (-1)^b \binom{n}{0}$$

for all $n \in \mathbb{N}$. This is just a restatement of Proposition 0.1.

0.2. Counting by symmetry

Recall that if $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$. If $n \in \mathbb{N}$, then S_n shall mean the set of all permutations of the set $[n]$. The number of these permutations is $|S_n| = n!$. (We shall prove this in class soon.) Note that S_n is called the *n-th symmetric group*.

Proposition 0.5. Let $n \geq 4$ be an integer. Then, the number of all permutations $\sigma \in S_n$ satisfying $\sigma(3) > \sigma(4)$ is $n!/2$.

Proof of Proposition 0.5. I say that a permutation $\sigma \in S_n$ is

- *green* if it satisfies $\sigma(3) > \sigma(4)$;
- *red* if it satisfies $\sigma(3) < \sigma(4)$.

Every permutation $\sigma \in S_n$ is either green or red (indeed, every permutation $\sigma \in S_n$ is injective, and thus satisfies $\sigma(3) \neq \sigma(4)$, so that it must satisfy either $\sigma(3) > \sigma(4)$ or $\sigma(3) < \sigma(4)$), but no permutation $\sigma \in S_n$ can be both green and red at the same time (since $\sigma(3) > \sigma(4)$ would contradict $\sigma(3) < \sigma(4)$). Hence, the set S_n is the union of its two disjoint subsets $\{\text{green permutations } \sigma \in S_n\}$ and $\{\text{red permutations } \sigma \in S_n\}$. Thus,

$$|S_n| = |\{\text{green permutations } \sigma \in S_n\}| + |\{\text{red permutations } \sigma \in S_n\}|. \tag{6}$$

On the other hand, I claim that "the colors are equidistributed", i.e., the number of green permutations $\sigma \in S_n$ equals the number of red permutations $\sigma \in S_n$.

To prove this, I will construct a bijection from $\{\text{green permutations } \sigma \in S_n\}$ to $\{\text{red permutations } \sigma \in S_n\}$.

Indeed, let s_3 be the permutation of $[n]$ that swaps the numbers 3 and 4 while leaving all other numbers unchanged. That is, s_3 is given by

$$s_3(i) = \begin{cases} 4, & \text{if } i = 3; \\ 3, & \text{if } i = 4; \\ i, & \text{if } i \notin \{3,4\} \end{cases} \qquad \text{for all } i \in [n].$$

(In one-line notation, s_3 is represented as $(1,2,4,3,5,6,\ldots,n)$, where only the two numbers 3 and 4 are out of order.)

Notice that $s_3 \circ s_3 = \text{id}$. (Visually speaking, this is clear: If we swap 3 and 4, and then swap 3 and 4 again, then all numbers return to their old places.)

If α and β are two permutations of $[n]$, then their composition $\alpha \circ \beta$ is a permutation of $[n]$ as well[1]. Hence, for every permutation $\sigma \in S_n$, the map $\sigma \circ s_3$ is also a permutation of $[n]$.

We now claim that

$$\text{if } \sigma \in S_n \text{ is green, then } \sigma \circ s_3 \in S_n \text{ is red.} \tag{7}$$

[*Proof of (7):* Assume that $\sigma \in S_n$ is green. Thus, $\sigma(3) > \sigma(4)$ (by the definition of "green").

We know $\sigma \circ s_3$ is a permutation of $[n]$. In other words, $\sigma \circ s_3 \in S_n$. We must prove that $\sigma \circ s_3$ is red. In other words, we must prove that $(\sigma \circ s_3)(3) < (\sigma \circ s_3)(4)$ (because this is what it means for $\sigma \circ s_3$ to be red).

But the definition of s_3 shows that $s_3(3) = 4$ and $s_3(4) = 3$. Thus, $(\sigma \circ s_3)(3) = \sigma\left(\underbrace{s_3(3)}_{=4}\right) = \sigma(4)$ and $(\sigma \circ s_3)(4) = \sigma\left(\underbrace{s_3(4)}_{=3}\right) = \sigma(3)$. Hence, $(\sigma \circ s_3)(4) = \sigma(3) > \sigma(4) = (\sigma \circ s_3)(3)$. In other words, $(\sigma \circ s_3)(3) < (\sigma \circ s_3)(4)$. But this is exactly what we wanted to prove. Thus, (7) is proven.]

An analogous argument shows that

$$\text{if } \sigma \in S_n \text{ is red, then } \sigma \circ s_3 \in S_n \text{ is green.} \tag{8}$$

Now, let α be the map

$$\{\text{green permutations } \sigma \in S_n\} \to \{\text{red permutations } \sigma \in S_n\}, \\ \sigma \mapsto \sigma \circ s_3$$

(this is well-defined because of (7)). Let β be the map

$$\{\text{red permutations } \sigma \in S_n\} \to \{\text{green permutations } \sigma \in S_n\}, \\ \sigma \mapsto \sigma \circ s_3$$

[1] because permutations of $[n]$ are just bijective maps $[n] \to [n]$, but the composition of two bijective maps is again bijective

(this is well-defined because of (8)). We have $\alpha \circ \beta = \text{id}$ (since every red permutation $\sigma \in S_n$ satisfies

$$\begin{aligned}(\alpha \circ \beta)(\sigma) &= \alpha\left(\underbrace{\beta(\sigma)}_{\substack{=\sigma \circ s_3 \\ \text{(by the definition of } \beta)}}\right) = \alpha(\sigma \circ s_3) \\ &= (\sigma \circ s_3) \circ s_3 \qquad \text{(by the definition of } \alpha) \\ &= \sigma \circ \underbrace{(s_3 \circ s_3)}_{=\text{id}} = \sigma = \text{id}(\sigma)\end{aligned}$$

) and $\beta \circ \alpha = \text{id}$ (by an analogous computation). Thus, the two maps α and β are mutually inverse. Hence, α is a bijection. Thus, we have found a bijection from $\{\text{green permutations } \sigma \in S_n\}$ to $\{\text{red permutations } \sigma \in S_n\}$ (namely, α). Therefore,

$$|\{\text{green permutations } \sigma \in S_n\}| = |\{\text{red permutations } \sigma \in S_n\}|. \tag{9}$$

Now, (6) becomes

$$\begin{aligned}|S_n| &= |\{\text{green permutations } \sigma \in S_n\}| + \underbrace{|\{\text{red permutations } \sigma \in S_n\}|}_{\substack{=|\{\text{green permutations } \sigma \in S_n\}| \\ \text{(by (9))}}} \\ &= |\{\text{green permutations } \sigma \in S_n\}| + |\{\text{green permutations } \sigma \in S_n\}| \\ &= 2 \cdot |\{\text{green permutations } \sigma \in S_n\}|.\end{aligned}$$

Hence,

$$|\{\text{green permutations } \sigma \in S_n\}| = \frac{1}{2} \underbrace{|S_n|}_{=n!} = \frac{1}{2} n! = n!/2.$$

In other words, the number of all green permutations $\sigma \in S_n$ is $n!/2$. In other words, the number of all permutations $\sigma \in S_n$ satisfying $\sigma(3) > \sigma(4)$ is $n!/2$ (because these permutations are precisely the green permutations $\sigma \in S_n$). This proves Proposition 0.5. □

Our above proof was an example of a "counting by symmetry": We did not count the green permutations directly; instead, we showed that they are in bijection with the remaining (i.e., red) permutations $\sigma \in S_n$ (that is, we matched up each green permutation with a red one), from which we concluded that they make up exactly half of the set S_n; and this told us that there are $\frac{1}{2}|S_n| = n!/2$ of them.

Exercise 2. Let $n \geq 4$ be an integer. Prove the following:

(a) The number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2)$ and $\sigma(3) > \sigma(4)$ is $n!/4$.

(b) The number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2) > \sigma(3)$ is $n!/6$.

[**Hint:** You'll need more than 2 colors...]

Solution to Exercise 2 (sketched). I will be more laconic than in the proof of Proposition 0.5 above, because much of the arguments below simply repeats arguments made in the latter proof.

The main tool for this solution will be the $n-1$ *simple transpositions* $s_1, s_2, \ldots, s_{n-1}$. These are defined as follows: Given $k \in [n-1]$, we let s_k be the permutation of $[n]$ that swaps the numbers k and $k+1$ while leaving all other numbers unchanged.[2] Of course, for $k = 3$, this s_k is precisely the permutation s_3 that was used in the proof of Proposition 0.5. These permutations $s_1, s_2, \ldots, s_{n-1}$ are known as the *simple transpositions* in S_n, and they are in a sense the "building blocks" of all permutations (see Remark 0.6 **(c)** below).

For each $k \in [n-1]$, we have $s_k \circ s_k = \mathrm{id}$. (This generalizes the $s_3 \circ s_3 = \mathrm{id}$ from the proof of Proposition 0.5. Of course, the proof is just as trivial.)

Let us now come to the actual solution.

(a) I say that a permutation $\sigma \in S_n$ is

- *green-green* if it satisfies $\sigma(1) > \sigma(2)$ and $\sigma(3) > \sigma(4)$;
- *green-red* if it satisfies $\sigma(1) > \sigma(2)$ and $\sigma(3) < \sigma(4)$;
- *red-green* if it satisfies $\sigma(1) < \sigma(2)$ and $\sigma(3) > \sigma(4)$;
- *red-red* if it satisfies $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$.

Every permutation $\sigma \in S_n$ is either green-green or green-red or red-green or red-red, but no permutation $\sigma \in S_n$ can have two (or more) of these properties simultaneously. Thus,

$$\begin{aligned} &|S_n| \\ &= |\{\text{green-green permutations } \sigma \in S_n\}| \\ &\qquad + |\{\text{green-red permutations } \sigma \in S_n\}| \\ &\qquad + |\{\text{red-green permutations } \sigma \in S_n\}| \\ &\qquad + |\{\text{red-red permutations } \sigma \in S_n\}|. \end{aligned} \tag{10}$$

Next, I claim that the four "colors" (by which I mean the properties "green-green", "green-red", "red-green" and "red-red") are equidistributed, i.e., the number of permutations of one color equals the number of permutations of any other.

[2] That is, s_k is given by

$$s_k(i) = \begin{cases} k+1, & \text{if } i = k; \\ k, & \text{if } i = k+1; \\ i, & \text{if } i \notin \{k, k+1\} \end{cases} \qquad \text{for all } i \in [n].$$

(In one-line notation, s_k is represented as $(1, 2, \ldots, k-1, k+1, k, k+2, k+3, \ldots, n)$, where only the two numbers k and $k+1$ are out of order.)

It is easy to see that if $\sigma \in S_n$ is green-green, then $\sigma \circ s_3 \in S_n$ is green-red[3]. This lets us define a map

$$\{\text{green-green permutations } \sigma \in S_n\} \to \{\text{green-red permutations } \sigma \in S_n\}, \\ \sigma \mapsto \sigma \circ s_3.$$

Likewise, we can define a map

$$\{\text{green-red permutations } \sigma \in S_n\} \to \{\text{green-green permutations } \sigma \in S_n\}, \\ \sigma \mapsto \sigma \circ s_3.$$

These two maps are mutually inverse[4]. Thus, we have found a bijection from $\{\text{green-green permutations } \sigma \in S_n\}$ to $\{\text{green-red permutations } \sigma \in S_n\}$. Hence,

$$|\{\text{green-green permutations } \sigma \in S_n\}| \\ = |\{\text{green-red permutations } \sigma \in S_n\}|. \tag{11}$$

Similarly,

$$|\{\text{red-green permutations } \sigma \in S_n\}| \\ = |\{\text{red-red permutations } \sigma \in S_n\}|. \tag{12}$$

But we also have

$$|\{\text{green-green permutations } \sigma \in S_n\}| \\ = |\{\text{red-green permutations } \sigma \in S_n\}|. \tag{13}$$

(The proof of this is analogous to the proof of (11), but we now need to use s_1 instead of s_3. Thus, the mutually inverse bijections between $\{\text{green-green permutations } \sigma \in S_n\}$ and $\{\text{red-green permutations } \sigma \in S_n\}$ no longer send a permutation σ to $\sigma \circ s_3$, but instead send a permutation σ to $\sigma \circ s_1$.)

Combining the three equalities (11), (12) and (13), we see that the four numbers

$$|\{\text{green-green permutations } \sigma \in S_n\}|, \qquad |\{\text{green-red permutations } \sigma \in S_n\}|, \\ |\{\text{red-green permutations } \sigma \in S_n\}|, \qquad |\{\text{red-red permutations } \sigma \in S_n\}|$$

are all the same. Hence, all four addends on the right hand side of (10) equal $|\{\text{green-green permutations } \sigma \in S_n\}|$. Thus, (10) simplifies to

$$|S_n| = 4 \cdot |\{\text{green-green permutations } \sigma \in S_n\}|.$$

[3] The proof of this is analogous to the proof of (7) above, except that we now also need to check that $(\sigma \circ s_3)(1) > (\sigma \circ s_3)(2)$ (but this is obvious: we have $(\sigma \circ s_3)(1) = \sigma\left(\underbrace{s_3(1)}_{=1}\right) = \sigma(1)$ and similarly $(\sigma \circ s_3)(2) = \sigma(2)$).

[4] This can be proven in the same way as we showed that α and β are mutually inverse in the proof of Proposition 0.5 above.

Hence,

$$|\{\text{green-green permutations } \sigma \in S_n\}| = \frac{1}{4} \underbrace{|S_n|}_{=n!} = \frac{1}{4} n! = n!/4.$$

In other words, the number of all green-green permutations $\sigma \in S_n$ is $n!/4$. In other words, the number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2)$ and $\sigma(3) > \sigma(4)$ is $n!/4$ (because these permutations are precisely the green-green permutations $\sigma \in S_n$). This solves Exercise 2 **(a)**.

(b) I say that a permutation $\sigma \in S_n$ is

- *color-123* if it satisfies $\sigma(1) > \sigma(2) > \sigma(3)$;
- *color-132* if it satisfies $\sigma(1) > \sigma(3) > \sigma(2)$;
- *color-213* if it satisfies $\sigma(2) > \sigma(1) > \sigma(3)$;
- *color-231* if it satisfies $\sigma(2) > \sigma(3) > \sigma(1)$;
- *color-312* if it satisfies $\sigma(3) > \sigma(1) > \sigma(2)$;
- *color-321* if it satisfies $\sigma(3) > \sigma(2) > \sigma(1)$.

(By now we have dropped all pretense that our adjectives are real colors.)

Each permutation $\sigma \in S_n$ has exactly one of the six properties "color-123", "color-132", "color-213", "color-231", "color-312" and "color-321" (because these properties classify permutations $\sigma \in S_n$ according to the relative order between their first three values $\sigma(1), \sigma(2), \sigma(3)$). Hence,

$$\begin{aligned} |S_n| &= |\{\text{color-123 permutations } \sigma \in S_n\}| + |\{\text{color-132 permutations } \sigma \in S_n\}| \\ &\quad + |\{\text{color-213 permutations } \sigma \in S_n\}| + |\{\text{color-231 permutations } \sigma \in S_n\}| \\ &\quad + |\{\text{color-312 permutations } \sigma \in S_n\}| + |\{\text{color-321 permutations } \sigma \in S_n\}| . \end{aligned} \tag{14}$$

We shall now, as before, prove that these six properties ("colors") are "equidistributed". The arguments we use will again be similar to those used in the proof of Proposition 0.5, so we restrict ourselves to a sketch:

- It is easy to see that if $\sigma \in S_n$ is color-123, then $\sigma \circ s_1 \in S_n$ is color-213. This lets us define a map

  $$\{\text{color-123 permutations } \sigma \in S_n\} \to \{\text{color-213 permutations } \sigma \in S_n\},$$
  $$\sigma \mapsto \sigma \circ s_1.$$

  Likewise, we can define a map

  $$\{\text{color-213 permutations } \sigma \in S_n\} \to \{\text{color-123 permutations } \sigma \in S_n\},$$
  $$\sigma \mapsto \sigma \circ s_1.$$

These two maps are mutually inverse. Thus, we have found a bijection from $\{\text{color-123 permutations } \sigma \in S_n\}$ to $\{\text{color-213 permutations } \sigma \in S_n\}$. Hence,

$$\begin{aligned} & |\{\text{color-123 permutations } \sigma \in S_n\}| \\ & = |\{\text{color-213 permutations } \sigma \in S_n\}| . \end{aligned} \tag{15}$$

- Similarly,

$$\begin{aligned} & |\{\text{color-132 permutations } \sigma \in S_n\}| \\ & = |\{\text{color-231 permutations } \sigma \in S_n\}| \end{aligned} \tag{16}$$

  and

$$\begin{aligned} & |\{\text{color-312 permutations } \sigma \in S_n\}| \\ & = |\{\text{color-321 permutations } \sigma \in S_n\}| . \end{aligned} \tag{17}$$

- A similar argument using s_2 instead of s_1 shows that

$$\begin{aligned} & |\{\text{color-213 permutations } \sigma \in S_n\}| \\ & = |\{\text{color-312 permutations } \sigma \in S_n\}| \end{aligned} \tag{18}$$

  and

$$\begin{aligned} & |\{\text{color-321 permutations } \sigma \in S_n\}| \\ & = |\{\text{color-231 permutations } \sigma \in S_n\}| . \end{aligned} \tag{19}$$

Combining the five equalities (15), (16), (17), (18) and (19), we conclude that the six numbers

$$\begin{array}{ll} |\{\text{color-123 permutations } \sigma \in S_n\}| , & |\{\text{color-132 permutations } \sigma \in S_n\}| \\ |\{\text{color-213 permutations } \sigma \in S_n\}| , & |\{\text{color-231 permutations } \sigma \in S_n\}| \\ |\{\text{color-312 permutations } \sigma \in S_n\}| , & |\{\text{color-321 permutations } \sigma \in S_n\}| \end{array}$$

are all the same. Hence, all six addends on the right hand side of (14) equal $|\{\text{color-123 permutations } \sigma \in S_n\}|$. Thus, (14) simplifies to

$$|S_n| = 6 \cdot |\{\text{color-123 permutations } \sigma \in S_n\}| .$$

Hence,

$$|\{\text{color-123 permutations } \sigma \in S_n\}| = \frac{1}{6} \underbrace{|S_n|}_{=n!} = \frac{1}{6} n! = n!/6.$$

In other words, the number of all color-123 permutations $\sigma \in S_n$ is $n!/6$. In other words, the number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2) > \sigma(3)$ is $n!/6$ (because these permutations are precisely the color-123 permutations $\sigma \in S_n$). This solves Exercise 2 **(b)**. □

Remark 0.6. Let $n \in \mathbb{N}$. The simple transpositions $s_1, s_2, \ldots, s_{n-1}$ introduced in the above solution of Exercise 2 are rather significant. Here are three more of their properties:

(a) If $k \in [n-1]$ and $l \in [n-1]$ satisfy $|k-l| > 1$, then $s_k \circ s_l = s_l \circ s_k$. (This is often called a "locality principle"; essentially, it says that swapping k and $k+1$ doesn't interact with swapping l and $l+1$ when $|k-l| > 1$.)

(b) For each $k \in [n-2]$, we have $s_k \circ s_{k+1} \circ s_k = s_{k+1} \circ s_k \circ s_{k+1}$. (This is called the "braid relation", and is easily verified by hand; in fact, both $s_k \circ s_{k+1} \circ s_k$ and $s_{k+1} \circ s_k \circ s_{k+1}$ turn out to be the permutation of $[n]$ that swaps k with $k+2$ while leaving all other numbers unchanged.)

(c) Each permutation in S_n can be written as a composition of some of the $s_1, s_2, \ldots, s_{n-1}$. (Note that the composition may be empty – in which case it is understood to mean the trivial permutation id – and that any of the $s_1, s_2, \ldots, s_{n-1}$ can appear many times in the composition.)

For an example, the permutation of $[5]$ that sends $1,2,3,4,5$ to $5,3,1,2,4$ can be written as $s_4 \circ s_3 \circ s_2 \circ s_1 \circ s_3 \circ s_2$.

(We will prove this later this semester.)

Remark 0.7. Exercise 2 **(b)** can be generalized: If $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$, then the number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2) > \cdots > \sigma(k)$ is $n!/k!$. This can be proven in the same way as we solved Exercise 2 **(b)**, except that we now need to be more systematic about the colors (there are $k!$ colors now) and justify their equidistribution abstractly – which is more difficult.

There is a simpler proof, though. [**Hint:** In order to construct a permutation $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2) > \cdots > \sigma(k)$, you can first choose the set $\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}$ (this should be a k-element subset of $[n]$, so there are $\dbinom{n}{k}$ choices for it), which automatically determines the values $\sigma(1), \sigma(2), \ldots, \sigma(k)$ (namely, they must be the k elements of this set in decreasing order); then choose the remaining $n-k$ values $\sigma(k+1), \sigma(k+2), \ldots, \sigma(n)$ (these are just $n-k$ distinct values chosen from an $(n-k)$-element set, so there are $(n-k)!$ choices for them). Thus, the total number of options is $\dbinom{n}{k}(n-k)! = n!/k!$.]

0.3. More on Fibonacci numbers

Recall that the *Fibonacci sequence* is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 2. \tag{20}$$

Exercise 3. Prove the following:

(a) We have $7f_n = f_{n-4} + f_{n+4}$ for each $n \geq 4$.

(b) We have $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ for each $n \in \mathbb{N}$.

(c) We have $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$ for each $n \in \mathbb{N}$.

(d) We have $f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$ for each $n \in \mathbb{N}$.

(e) We have $f_{m+n+1} = f_{m+1}f_{n+1} + f_m f_n$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

(f) For every $m \in \mathbb{N}$, we have

$$f_{2m+2} = \sum_{\substack{(a,b)\in\mathbb{N}^2; \\ a+b\leq m}} \binom{m-a}{b}\binom{m-b}{a}.$$

[**Hint:** All parts can be proven bijectively; part **(f)** is actually easiest to prove bijectively! (On the other hand, proving part **(a)** bijectively is a challenge; there are much easier ways.) As a reminder: Any exercises from previous problem sets can be used without proof.]

Solution to Exercise 3 (sketched). We shall use the symbol "$\overset{(20)}{=}$" for "equals, because of the recurrence equation (20)". For example, $f_5 \overset{(20)}{=} f_4 + f_3$ and $f_3 + f_2 \overset{(20)}{=} f_4$ and $f_{k+5} \overset{(20)}{=} f_{k+4} + f_{k+3}$ for every $k \in \mathbb{N}$.

(a) Let $n \geq 4$. Then,

$$\begin{aligned}
f_{n-4} + \underbrace{f_{n+4}}_{\overset{(20)}{=} f_{n+3}+f_{n+2}} &= f_{n-4} + \underbrace{f_{n+3}}_{\overset{(20)}{=} f_{n+2}+f_{n+1}} + \underbrace{f_{n+2}}_{\overset{(20)}{=} f_{n+1}+f_n} \\
&= f_{n-4} + f_{n+2} + f_{n+1} + f_{n+1} + f_n = f_{n-4} + \underbrace{f_{n+2}}_{\overset{(20)}{=} f_{n+1}+f_n} + 2f_{n+1} + f_n \\
&= f_{n-4} + f_{n+1} + f_n + 2f_{n+1} + f_n = f_{n-4} + 2f_n + 3\underbrace{f_{n+1}}_{\overset{(20)}{=} f_n+f_{n-1}} \\
&= f_{n-4} + 2f_n + 3\left(f_n + f_{n-1}\right) = f_{n-4} + 5f_n + 3\underbrace{f_{n-1}}_{\overset{(20)}{=} f_{n-2}+f_{n-3}} \\
&= f_{n-4} + 5f_n + 3\left(f_{n-2} + f_{n-3}\right) = \underbrace{f_{n-4} + f_{n-3}}_{\overset{(20)}{=} f_{n-2}} + 5f_n + 3f_{n-2} + 2f_{n-3} \\
&= f_{n-2} + 5f_n + 3f_{n-2} + 2f_{n-3} = 5f_n + 2f_{n-2} + 2\underbrace{\left(f_{n-2} + f_{n-3}\right)}_{\overset{(20)}{=} f_{n-1}} \\
&= 5f_n + 2f_{n-2} + 2f_{n-1} = 5f_n + 2\underbrace{\left(f_{n-1} + f_{n-2}\right)}_{\overset{(20)}{=} f_n} \\
&= 5f_n + 2f_n = 7f_n.
\end{aligned}$$

This solves Exercise 3 **(a)**.

(b) We shall solve Exercise 3 **(b)** by induction on n:

Induction base: Comparing $f_1 + f_2 + \cdots + f_0 = (\text{empty sum}) = 0$ with $f_{0+2} - 1 = \underbrace{f_2}_{=1} - 1 = 1 - 1 = 0$, we obtain $f_1 + f_2 + \cdots + f_0 = f_{0+2} - 1$. In other words, Exercise 3 **(b)** holds for $n = 0$. This completes the induction base.

Induction step: Let $k \in \mathbb{N}$. Assume that Exercise 3 **(b)** holds for $n = k$. We must prove that Exercise 3 **(b)** holds for $n = k + 1$.

We have $f_1 + f_2 + \cdots + f_k = f_{k+2} - 1$ (since Exercise 3 **(b)** holds for $n = k$). Now,

$$f_1 + f_2 + \cdots + f_{k+1} = \underbrace{(f_1 + f_2 + \cdots + f_k)}_{=f_{k+2}-1} + f_{k+1} = f_{k+2} - 1 + f_{k+1} = \underbrace{f_{k+2} + f_{k+1}}_{\overset{(20)}{=} f_{k+3} = f_{(k+1)+2}} - 1$$
$$= f_{(k+1)+2} - 1.$$

In other words, Exercise 3 **(b)** holds for $n = k + 1$. This completes the induction step. Thus, Exercise 3 **(b)** is solved.

(c) We shall solve Exercise 3 **(c)** by induction on n:

Induction base: Comparing $f_1 + f_3 + f_5 + \cdots + f_{2\cdot 0-1} = (\text{empty sum}) = 0$ with $f_{2\cdot 0} = f_0 = 0$, we obtain $f_1 + f_3 + f_5 + \cdots + f_{2\cdot 0-1} = f_{2\cdot 0}$. In other words, Exercise 3 **(c)** holds for $n = 0$. This completes the induction base.

Induction step: Let $k \in \mathbb{N}$. Assume that Exercise 3 **(c)** holds for $n = k$. We must prove that Exercise 3 **(c)** holds for $n = k + 1$.

We have $f_1 + f_3 + f_5 + \cdots + f_{2k-1} = f_{2k}$ (since Exercise 3 **(c)** holds for $n = k$). Now,

$$f_1 + f_3 + f_5 + \cdots + f_{2(k+1)-1} = \underbrace{(f_1 + f_3 + f_5 + \cdots + f_{2k-1})}_{=f_{2k}} + \underbrace{f_{2(k+1)-1}}_{=f_{2k+1}} = f_{2k} + f_{2k+1}$$
$$= f_{2k+1} + f_{2k} \overset{(20)}{=} f_{2k+2} = f_{2(k+1)}.$$

In other words, Exercise 3 **(c)** holds for $n = k + 1$. This completes the induction step. Thus, Exercise 3 **(c)** is solved.

(d) We shall solve Exercise 3 **(d)** by induction on n:

Induction base: Comparing $f_2 + f_4 + f_6 + \cdots + f_{2\cdot 0} = (\text{empty sum}) = 0$ with $f_{2\cdot 0+1} - 1 = \underbrace{f_1}_{=1} - 1 = 1 - 1 = 0$, we obtain $f_2 + f_4 + f_6 + \cdots + f_{2\cdot 0} = f_{2\cdot 0+1} - 1$. In other words, Exercise 3 **(d)** holds for $n = 0$. This completes the induction base.

Induction step: Let $k \in \mathbb{N}$. Assume that Exercise 3 **(d)** holds for $n = k$. We must prove that Exercise 3 **(d)** holds for $n = k + 1$.

We have $f_2 + f_4 + f_6 + \cdots + f_{2k} = f_{2k+1} - 1$ (since Exercise 3 **(d)** holds for $n = k$).

Now,

$$f_2 + f_4 + f_6 + \cdots + f_{2(k+1)} = \underbrace{(f_2 + f_4 + f_6 + \cdots + f_{2k})}_{= f_{2k+1} - 1} + \underbrace{f_{2(k+1)}}_{= f_{2k+2}} = f_{2k+1} - 1 + f_{2k+2}$$
$$= \underbrace{f_{2k+2} + f_{2k+1}}_{\overset{(20)}{=} f_{2k+3} = f_{2(k+1)+1}} - 1 = f_{2(k+1)+1} - 1.$$

In other words, Exercise 3 **(d)** holds for $n = k+1$. This completes the induction step. Thus, Exercise 3 **(d)** is solved.

(e) We shall solve Exercise 3 **(e)** by induction on n:

Induction base: For all $m \in \mathbb{N}$, we have $f_{m+0+1} = f_{m+1} f_{0+1} + f_m f_0$ [5]. In other words, Exercise 3 **(e)** holds for $n = 0$. This completes the induction base.

Induction step: Let $k \in \mathbb{N}$. Assume that Exercise 3 **(e)** holds for $n = k$. We must prove that Exercise 3 **(e)** holds for $n = k+1$.

We have assumed that Exercise 3 **(e)** holds for $n = k$. In other words, we have

$$f_{m+k+1} = f_{m+1} f_{k+1} + f_m f_k \qquad \text{for all } m \in \mathbb{N}. \tag{21}$$

Now, let $m \in \mathbb{N}$. Then, we can apply (21) to $m+1$ instead of m. Thus, we obtain

$$f_{(m+1)+k+1} = \underbrace{f_{(m+1)+1}}_{= f_{m+2} \overset{(20)}{=} f_{m+1} + f_m} f_{k+1} + f_{m+1} f_k$$
$$= (f_{m+1} + f_m) f_{k+1} + f_{m+1} f_k = f_{m+1} f_{k+1} + f_m f_{k+1} + f_{m+1} f_k$$
$$= f_{m+1} \underbrace{(f_{k+1} + f_k)}_{\overset{(20)}{=} f_{k+2} = f_{(k+1)+1}} + f_m f_{k+1} = f_{m+1} f_{(k+1)+1} + f_m f_{k+1}.$$

Now, forget that we fixed m. We thus have shown that $f_{(m+1)+k+1} = f_{m+1} f_{(k+1)+1} + f_m f_{k+1}$ for all $m \in \mathbb{N}$. In other words, Exercise 3 **(e)** holds for $n = k+1$. This completes the induction step. Thus, Exercise 3 **(e)** is solved.

(f) Recall the definition of a *lacunar* subset of $\mathbb{Z}$. (We defined this in Math 4707 Homework set 2.)

For any $n \in \mathbb{N}$, we have

$$(\text{the number of all lacunar subsets of } [n]) = f_{n+2}. \tag{22}$$

(For a proof of (22), see Exercise 4 **(c)** on Fall 2017 Math 4707 Homework set 1, in which the number of all lacunar subsets of $[n]$ was denoted by $g(n)$.)

[5] *Proof.* Let $m \in \mathbb{N}$. Then, $f_{m+0+1} = f_{m+1}$. Comparing this with $f_{m+1} \underbrace{f_{0+1}}_{= f_1 = 1} + f_m \underbrace{f_0}_{= 0} = f_{m+1} 1 + f_m 0 = f_{m+1}$, we obtain $f_{m+0+1} = f_{m+1} f_{0+1} + f_m f_0$, qed.

For any $n \in \mathbb{N}$, $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, we let $N(n, a, b)$ denote the number of all lacunar subsets of $[n]$ that contain exactly a even and exactly b odd elements. Then, Exercise 3 **(a)** from Math 4707 Homework set 2 states that

$$N(2m, a, b) = [a \leq m][b \leq m] \binom{m-a}{b} \binom{m-b}{a} \qquad (23)$$

for all $m \in \mathbb{N}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

Now, let $m \in \mathbb{N}$. If a and b are two elements of $\{0, 1, \ldots, m\}$ satisfying $a + b > m$, then

$$\binom{m-a}{b} = 0. \qquad (24)$$

[*Proof of (24):* Let a and b be two elements of $\{0, 1, \ldots, m\}$ satisfying $a + b > m$. From $a + b > m$, we obtain $b > m - a$, so that $m - a < b$. Also, $a \leq m$ (since $a \in \{0, 1, \ldots, m\}$) and thus $m - a \geq 0$, so that $m - a \in \mathbb{N}$. Hence, (1) (applied to $m - a$ and b instead of m and n) shows that $\binom{m-a}{b} = 0$. This proves (24).]

Any subset of $[2m]$ has at most m even elements (because the whole set $[2m]$ has only m even elements) and at most m odd elements (similarly). Thus, we can classify the lacunar subsets of $[2m]$ according to their number of even elements (which is an integer in $\{0, 1, \ldots, m\}$) and their number of odd elements (which is an integer in $\{0,1,\ldots,m\}$). Hence,

$$\begin{aligned}
&(\text{the number of all lacunar subsets of } [2m]) \\
&= \sum_{a\in\{0,1,\ldots,m\}} \sum_{b\in\{0,1,\ldots,m\}} \underbrace{\left(\begin{array}{c}\text{the number of all lacunar subsets of } [2m] \text{ that contain} \\ \text{exactly } a \text{ even and exactly } b \text{ odd elements}\end{array}\right)}_{\substack{=N(2m,a,b) \\ (\text{by the definition of } N(2m,a,b))}} \\
&= \sum_{a\in\{0,1,\ldots,m\}} \sum_{b\in\{0,1,\ldots,m\}} \underbrace{N(2m,a,b)}_{\substack{=[a\leq m][b\leq m]\dbinom{m-a}{b}\dbinom{m-b}{a} \\ (\text{by (23)})}} \\
&= \sum_{a\in\{0,1,\ldots,m\}} \sum_{b\in\{0,1,\ldots,m\}} \underbrace{[a\leq m]}_{\substack{=1 \\ (\text{since } a\in\{0,1,\ldots,m\})}} \underbrace{[b\leq m]}_{\substack{=1 \\ (\text{since } b\in\{0,1,\ldots,m\})}} \dbinom{m-a}{b}\dbinom{m-b}{a} \\
&= \underbrace{\sum_{a\in\{0,1,\ldots,m\}} \sum_{b\in\{0,1,\ldots,m\}}}_{=\sum_{(a,b)\in\{0,1,\ldots,m\}^2}} \dbinom{m-a}{b}\dbinom{m-b}{a} = \sum_{(a,b)\in\{0,1,\ldots,m\}^2} \dbinom{m-a}{b}\dbinom{m-b}{a} \\
&= \underbrace{\sum_{\substack{(a,b)\in\{0,1,\ldots,m\}^2; \\ a+b\leq m}}}_{\substack{=\sum_{\substack{(a,b)\in\mathbb{N}^2; \\ a+b\leq m}} \\ (\text{since any two nonnegative} \\ \text{integers } a \text{ and } b \\ \text{satisfying } a+b\leq m \text{ must} \\ \text{satisfy } (a,b)\in\{0,1,\ldots,m\}^2)}} \dbinom{m-a}{b}\dbinom{m-b}{a} + \sum_{\substack{(a,b)\in\{0,1,\ldots,m\}^2; \\ a+b>m}} \underbrace{\dbinom{m-a}{b}}_{\substack{=0 \\ (\text{by (24)})}} \dbinom{m-b}{a} \\
&= \sum_{\substack{(a,b)\in\mathbb{N}^2; \\ a+b\leq m}} \dbinom{m-a}{b}\dbinom{m-b}{a} + \underbrace{\sum_{\substack{(a,b)\in\{0,1,\ldots,m\}^2; \\ a+b>m}} 0\dbinom{m-b}{a}}_{=0} = \sum_{\substack{(a,b)\in\mathbb{N}^2; \\ a+b\leq m}} \dbinom{m-a}{b}\dbinom{m-b}{a}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
&(\text{the number of all lacunar subsets of } [2m]) \\
&= f_{2m+2} \qquad (\text{by (22), applied to } n=2m),
\end{aligned}$$

we obtain $f_{2m+2} = \sum_{\substack{(a,b)\in\mathbb{N}^2; \\ a+b\leq m}} \dbinom{m-a}{b}\dbinom{m-b}{a}$. This solves Exercise 3 **(f)**. □

Remark 0.8. Exercise 3 **(a)** is one of the so-called *Zeckendorf family identities*. Here are the first seven of them:

$$1f_n = f_n \text{ for all } n \geq 0;$$
$$2f_n = f_{n-2} + f_{n+1} \text{ for all } n \geq 2;$$
$$3f_n = f_{n-2} + f_{n+2} \text{ for all } n \geq 2;$$
$$4f_n = f_{n-2} + f_n + f_{n+2} \text{ for all } n \geq 2;$$
$$5f_n = f_{n-4} + f_{n-1} + f_{n+3} \text{ for all } n \geq 4;$$
$$6f_n = f_{n-4} + f_{n+1} + f_{n+3} \text{ for all } n \geq 4;$$
$$7f_n = f_{n-4} + f_{n+4} \text{ for all } n \geq 4.$$

See [Grinbe10] for more about them (particularly, about how to construct them all).

Of course, parts **(a)**, **(b)**, **(c)**, **(d)**, **(e)** of Exercise 3 can also be proven using Binet's formula (and the formula for the sum of a geometric series); the computations aren't particularly fun but can be done.

Alternative solutions abound. For example, part **(d)** can be easily derived from parts **(b)** and **(c)**.

Exercise 3 **(e)** also is a particular case of [Grinbe16, Theorem 2.26 **(a)**] (obtained by setting $a = 1$ and $b = 1$, so that the sequence $(x_0, x_1, x_2, \ldots)$ defined in [Grinbe16, Theorem 2.26 **(a)**] becomes precisely the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$).

Perhaps more interesting is the question of solving Exercise 3 bijectively (when f_n is understood, e.g., as the number of all lacunar subsets of $[n-2]$, or as the number of domino tilings of a $2 \times (n-1)$-rectangle). This, too, can be done. A bijective proof for part **(a)** can, in principle, be obtained from the bijective proof of (20) (but the bijection will involve myriad cases). More generally, each of the Zeckendorf family identities can be proven bijectively; see [WooZei09, §3.7] for details.

For a bijective proof of part **(b)**, see [BenQui03, Identity 1] or [BenQui04, Identity 1]. (Notice that the notations in [BenQui03, Identity 1] or [BenQui04, Identity 1] are different from ours; their f_n is our f_{n+1}.)

For a bijective proof of part **(c)**, see [BenQui03, Identity 2]. A similar argument can be used to prove part **(d)**.

For a bijective proof of part **(e)**, see [BenQui03, Identity 3] or [BenQui04, Identity 2].

For a bijective proof of part **(f)** (different from the one we have given), see [BenQui04, Identity 5] (but beware that Benjamin and Quinn are cavalier about the summation bounds: their double sum $\sum\limits_{i \geq 0} \sum\limits_{j \geq 0}$ should be $\sum\limits_{i \in \{0,1,\ldots,n\}} \sum\limits_{j \in \{0,1,\ldots,n\}}$).

0.4. More lattice path counting

Recall that the set $\mathbb{Z}^2$ is called the *integer lattice*, and its elements $(a, b) \in \mathbb{Z}^2$ are called *points*. We regard these points as points on the Cartesian plane.

A *lattice path* is a path on the integer lattice that uses only two kinds of steps:

- up-steps (U), which have the form $(x, y) \mapsto (x, y+1)$;
- right-steps (R), which have the form $(x, y) \mapsto (x+1, y)$.

Thus, strictly speaking, a *lattice path* is a sequence $(v_0, v_1, \ldots, v_n)$ of points $v_i \in \mathbb{Z}^2$ such that for each $i \in [n]$, the difference vector $v_i - v_{i-1}$ is either $(0, 1)$ or $(1, 0)$.

If $(a, b) \in \mathbb{Z}^2$ and $(c, d) \in \mathbb{Z}^2$ are two points on the integer lattice, then a *lattice path from (a, b) to (c, d)* is a lattice path $(v_0, v_1, \ldots, v_n)$ satisfying $v_0 = (a, b)$ and $v_n = (c, d)$.

Exercise 4. (a) Given six integers $a_1, b_1, c_1, a_2, b_2, c_2$ satisfying $0 \leq a_1 \leq b_1 \leq c_1$ and $0 \leq a_2 \leq b_2 \leq c_2$. How many lattice paths from $(0, 0)$ to (c_1, c_2) pass through none of the points (a_1, a_2) nor (b_1, b_2) ?

(b) Given six integers a, b, c, A, B, C satisfying $0 \leq a \leq b \leq c$ and $0 \leq A \leq B \leq C$. How many c-element subsets S of $[C]$ satisfy $|S \cap [A]| \neq a$ and $|S \cap [B]| \neq b$?

Solution to Exercise 4 (sketched). **(b)** Observe that $A \leq B \leq C$ and thus $[A] \subseteq [B] \subseteq [C]$.

We define the following three sets:

$$\begin{aligned} U &= \{c\text{-element subsets } S \text{ of } [C]\}; \\ X &= \{c\text{-element subsets } S \text{ of } [C] \text{ satisfying } |S \cap [A]| = a\}; \\ Y &= \{c\text{-element subsets } S \text{ of } [C] \text{ satisfying } |S \cap [B]| = b\}. \end{aligned}$$

Then,

$$\begin{aligned} & U \setminus (X \cup Y) \\ & = \{c\text{-element subsets } S \text{ of } [C] \text{ satisfying neither } |S \cap [A]| = a \text{ nor } |S \cap [B]| = b\} \\ & = \{c\text{-element subsets } S \text{ of } [C] \text{ satisfying } |S \cap [A]| \neq a \text{ and } |S \cap [B]| \neq b\}. \end{aligned}$$

Hence, $|U \setminus (X \cup Y)|$ is the number of all c-element subsets S of $[C]$ satisfying $|S \cap [A]| \neq a$ and $|S \cap [B]| \neq b$. This is the number that we need to compute.

But $X \cup Y$ is clearly a subset of U. Thus,

$$\begin{aligned} |U \setminus (X \cup Y)| = |U| - \underbrace{|X \cup Y|}_{=|X|+|Y|-|X \cap Y|} &= |U| - (|X| + |Y| - |X \cap Y|) \\ &= |U| - |X| - |Y| + |X \cap Y|. \end{aligned} \tag{25}$$

Hence, we need to compute $|U|$, $|X|$, $|Y|$ and $|X \cap Y|$.

Computing $|U|$ is easy: The definition of U yields

$$\begin{aligned}|U| &= |\{c\text{-element subsets } S \text{ of } [C]\}| \\ &= (\text{the number of all } c\text{-element subsets } S \text{ of } [C]) \\ &= \binom{C}{c} \end{aligned} \qquad (26)$$

(by the combinatorial interpretation of binomial coefficients).

Let us next compute $|X|$. Recall that

$$X = \{c\text{-element subsets } S \text{ of } [C] \text{ satisfying } |S \cap [A]| = a\}.$$

Thus, $|X|$ is the number of all c-element subsets S of $[C]$ satisfying $|S \cap [A]| = a$. Such a subset must always contain exactly a elements from $[A]$ (because it must satisfy $|S \cap [A]| = a$) and exactly $c - a$ elements from $[C] \setminus [A]$ (because it must have c elements in total). Thus, we can construct such a subset as follows:

- First, we decide which a elements of $[A]$ shall belong to S. This can be done in $\binom{A}{a}$ many ways (since we are just choosing an a-element subset of $[A]$).
- Next, we decide which $c - a$ elements of $[C] \setminus [A]$ shall belong to S. This can be done in $\binom{C-A}{c-a}$ many ways (since $|[C] \setminus [A]| = C - A$).

Thus, altogether, the number of options is $\binom{A}{a}\binom{C-A}{c-a}$. We thus have proven that

$$|X| = \binom{A}{a}\binom{C-A}{c-a}. \qquad (27)$$

Similarly,

$$|Y| = \binom{B}{b}\binom{C-B}{c-b}. \qquad (28)$$

It remains to compute $|X \cap Y|$. The definitions of X and Y yield

$$X \cap Y = \{c\text{-element subsets } S \text{ of } [C] \text{ satisfying both } |S \cap [A]| = a \text{ and } |S \cap [B]| = b\}.$$

Thus, $|X \cap Y|$ is the number of all c-element subsets S of $[C]$ satisfying both $|S \cap [A]| = a$ and $|S \cap [B]| = b$. Such a subset must always contain exactly a elements from $[A]$ (because it must satisfy $|S \cap [A]| = a$), exactly $b - a$ elements from $[B] \setminus [A]$ (because it must satisfy $|S \cap [B]| = b$ and $|S \cap [A]| = a$, so that

$$\begin{aligned}
\left| \underbrace{S \cap ([B] \setminus [A])}_{=(S\cap[B])\setminus(S\cap[A])} \right| &= |(S \cap [B]) \setminus (S \cap [A])| \\
&= \underbrace{|S \cap [B]|}_{=b} - \underbrace{|S \cap [A]|}_{=a} \qquad \left(\text{since } S \cap \underbrace{[A]}_{\subseteq [B]} \subseteq S \cap [B] \right) \\
&= b - a
\end{aligned}$$

), and exactly $c - b$ elements from $[C] \setminus [B]$ (because it must have c elements in total, but $|S \cap [B]| = b$). Thus, we can construct such a subset as follows:

- First, we decide which a elements of $[A]$ shall belong to S. This can be done in $\binom{A}{a}$ many ways (since we are just choosing an a-element subset of $[A]$).
- Next, we decide which $b - a$ elements of $[B] \setminus [A]$ shall belong to S. This can be done in $\binom{B-A}{b-a}$ many ways (since $|[B] \setminus [A]| = B - A$).
- Next, we decide which $c - b$ elements of $[C] \setminus [B]$ shall belong to S. This can be done in $\binom{C-B}{c-b}$ many ways (since $|[C] \setminus [B]| = C - B$).

Thus, altogether, the number of options is $\binom{A}{a}\binom{B-A}{b-a}\binom{C-B}{c-b}$. We thus have proven that

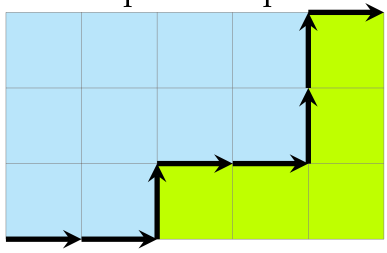
$$|X \cap Y| = \binom{A}{a}\binom{B-A}{b-a}\binom{C-B}{c-b}. \tag{29}$$

Now, (25) becomes

$$\begin{aligned}
&|U \setminus (X \cup Y)| \\
&= \underbrace{|U|}_{\substack{=\binom{C}{c} \\ \text{(by (26))}}} - \underbrace{|X|}_{\substack{=\binom{A}{a}\binom{C-A}{c-a} \\ \text{(by (27))}}} - \underbrace{|Y|}_{\substack{=\binom{B}{b}\binom{C-B}{c-b} \\ \text{(by (28))}}} + \underbrace{|X \cap Y|}_{\substack{=\binom{A}{a}\binom{B-A}{b-a}\binom{C-B}{c-b} \\ \text{(by (29))}}} \\
&= \binom{C}{c} - \binom{A}{a}\binom{C-A}{c-a} - \binom{B}{b}\binom{C-B}{c-b} + \binom{A}{a}\binom{B-A}{b-a}\binom{C-B}{c-b}.
\end{aligned}$$

So this is the number of all c-element subsets S of $[C]$ satisfy $|S \cap [A]| \neq a$ and $|S \cap [B]| \neq b$.

(As far as I know, this answer cannot be simplified any further.)

(a) If $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ is a lattice path, then the *up-set* of $\mathbf{v}$ shall mean the set of all $i \in [n]$ such that the difference vector $v_i - v_{i-1}$ equals $(0,1)$ (in other words, such that the step from v_{i-1} to v_i is an up-step). Roughly speaking, the up-set of a lattice path $\mathbf{v}$ tells us which steps of the path $\mathbf{v}$ are up-steps. For example, the up-set of the lattice path  is $\{3,6,7\}$, since the up-steps of this path are its 3-rd, 6-th and 7-th steps.

If $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ is a lattice path from $(0,0)$ to (c_1, c_2), then we necessarily have $n = c_1 + c_2$ (indeed, we need precisely $c_1 + c_2$ steps to get from $(0,0)$ to (c_1, c_2), because each step increases the sum of the coordinates of the point by exactly 1), and therefore the up-set of $\mathbf{v}$ is a subset of $[n] = [c_1 + c_2]$. Moreover, this up-set is a c_2-element set (indeed, its size is the number of all up-steps of $\mathbf{v}$, but this number must be c_1 because $\mathbf{v}$ goes from a point with y-coordinate 0 to a point with y-coordinate c_2), and thus is a c_2-element subset of $[c_1 + c_2]$.

Let $\mathcal{L}$ be the set of all lattice paths from $(0,0)$ to (c_1, c_2). Thus, what we have just showed is the following: For any $\mathbf{v} \in \mathcal{L}$, the up-set of $\mathbf{v}$ is a c_2-element subset of $[c_1 + c_2]$. Thus, we can define a map

$$\begin{aligned} \operatorname{Ups} : \mathcal{L} &\to \{c_2\text{-element subsets of } [c_1 + c_2]\}, \\ \mathbf{v} &\mapsto (\text{the up-set of } \mathbf{v}). \end{aligned}$$

It is easy to see that this map Ups is injective (since any lattice path $\mathbf{v}$ from $(0,0)$ to (c_1, c_2) is uniquely determined by its up-set) and surjective (because for any c_2-element subset S of $[c_1 + c_2]$, we can easily construct a lattice path from $(0,0)$ to (c_1, c_2) whose up-set is S). Thus, Ups is a bijection.

Moreover, it is easy to check the following two facts:

- A lattice path $\mathbf{v} \in \mathcal{L}$ passes through the point (a_1, a_2) if and only if its up-set $\operatorname{Ups} \mathbf{v}$ satisfies $|(\operatorname{Ups} \mathbf{v}) \cap [a_1 + a_2]| = a_2$. (Indeed, for the path $\mathbf{v}$ to pass through (a_1, a_2), it must spend precisely a_2 of its first $a_1 + a_2$ steps moving upwards; i.e., its up-set $\operatorname{Ups} \mathbf{v}$ should contain exactly a_2 elements of $[a_1 + a_2]$, but this is equivalent to saying that $|(\operatorname{Ups} \mathbf{v}) \cap [a_1 + a_2]| = a_2$.)
- A lattice path $\mathbf{v} \in \mathcal{L}$ passes through the point (b_1, b_2) if and only if its up-set $\operatorname{Ups} \mathbf{v}$ satisfies $|(\operatorname{Ups} \mathbf{v}) \cap [b_1 + b_2]| = b_2$. (The reason for this is analogous.)

Hence, a lattice path $\mathbf{v} \in \mathcal{L}$ passes through none of the points (a_1, a_2) and (b_1, b_2) if and only if it satisfies neither $|(\operatorname{Ups} \mathbf{v}) \cap [a_1 + a_2]| = a_2$ nor $|(\operatorname{Ups} \mathbf{v}) \cap [b_1 + b_2]| =$

b_2. Thus,

$$\begin{aligned}
&\left(\begin{array}{c}\text{the number of lattice paths } \mathbf{v}\in\mathcal{L} \text{ passing through}\\ \text{none of the points } (a_1,a_2) \text{ and } (b_1,b_2)\end{array}\right)\\
&=\left(\begin{array}{c}\text{the number of lattice paths } \mathbf{v}\in\mathcal{L} \text{ satisfying}\\ \text{neither } |(\operatorname{Ups}\mathbf{v})\cap[a_1+a_2]|=a_2 \text{ nor } |(\operatorname{Ups}\mathbf{v})\cap[b_1+b_2]|=b_2\end{array}\right)\\
&=\left(\begin{array}{c}\text{the number of } S\in\{c_2\text{-element subsets of } [c_1+c_2]\} \text{ satisfying}\\ \text{neither } |S\cap[a_1+a_2]|=a_2 \text{ nor } |S\cap[b_1+b_2]|=b_2\end{array}\right)\\
&\qquad\left(\begin{array}{c}\text{here, we have substituted } S \text{ for } \operatorname{Ups}\mathbf{v}\text{, since}\\ \operatorname{Ups}:\mathcal{L}\to\{c_2\text{-element subsets of } [c_1+c_2]\} \text{ is a bijection}\end{array}\right)\\
&=\left(\begin{array}{c}\text{the number of } c_2\text{-element subsets } S \text{ of } [c_1+c_2] \text{ satisfying}\\ \text{neither } |S\cap[a_1+a_2]|=a_2 \text{ nor } |S\cap[b_1+b_2]|=b_2\end{array}\right)\\
&=\left(\begin{array}{c}\text{the number of } c_2\text{-element subsets } S \text{ of } [c_1+c_2] \text{ satisfying}\\ |S\cap[a_1+a_2]|\neq a_2 \text{ and } |S\cap[b_1+b_2]|\neq b_2\end{array}\right). \qquad (30)
\end{aligned}$$

Now, define six integers a,b,c,A,B,C by $a=a_2$, $b=b_2$, $c=c_2$, $A=a_1+a_2$, $B=b_1+b_2$ and $C=c_1+c_2$. Clearly, $0\leq a\leq b\leq c$ (since $0\leq a_2\leq b_2\leq c_2$) and $0\leq A\leq B\leq C$ (since $0\leq a_1\leq b_1\leq c_1$ and $0\leq a_2\leq b_2\leq c_2$). Thus, our answer to Exercise 4 **(b)** yields

$$\begin{aligned}
&\left(\begin{array}{c}\text{the number of } c\text{-element subsets } S \text{ of } [C] \text{ satisfying}\\ |S\cap[A]|\neq a \text{ and } |S\cap[B]|\neq b\end{array}\right)\\
&=\binom{C}{c}-\binom{A}{a}\binom{C-A}{c-a}-\binom{B}{b}\binom{C-B}{c-b}+\binom{A}{a}\binom{B-A}{b-a}\binom{C-B}{c-b}.
\end{aligned}$$

In view of $a=a_2$, $b=b_2$, $c=c_2$, $A=a_1+a_2$, $B=b_1+b_2$ and $C=c_1+c_2$, this rewrites as

$$\begin{aligned}
&\left(\begin{array}{c}\text{the number of } c_2\text{-element subsets } S \text{ of } [c_1+c_2] \text{ satisfying}\\ |S\cap[a_1+a_2]|\neq a_2 \text{ and } |S\cap[b_1+b_2]|\neq b_2\end{array}\right)\\
&=\binom{c_1+c_2}{c_2}-\binom{a_1+a_2}{a_2}\binom{(c_1+c_2)-(a_1+a_2)}{c_2-a_2}-\binom{b_1+b_2}{b_2}\binom{(c_1+c_2)-(b_1+b_2)}{c_2-b_2}\\
&\qquad+\binom{a_1+a_2}{a_2}\binom{(b_1+b_2)-(a_1+a_2)}{b_2-a_2}\binom{(c_1+c_2)-(b_1+b_2)}{c_2-b_2}. \qquad (31)
\end{aligned}$$

Recall that the lattice paths from $(0,0)$ to (c_1,c_2) are precisely the lattice paths

$\mathbf{v} \in \mathcal{L}$ (because of how we defined $\mathcal{L}$). Hence,

$$\begin{aligned}
&\left(\begin{array}{c}\text{the number of lattice paths from } (0,0) \text{ to } (c_1,c_2) \text{ passing}\\ \text{through none of the points } (a_1,a_2) \text{ and } (b_1,b_2)\end{array}\right)\\
&=\left(\begin{array}{c}\text{the number of lattice paths } \mathbf{v}\in\mathcal{L} \text{ passing through}\\ \text{none of the points } (a_1,a_2) \text{ and } (b_1,b_2)\end{array}\right)\\
&=\left(\begin{array}{c}\text{the number of } c_2\text{-element subsets } S \text{ of } [c_1+c_2] \text{ satisfying}\\ |S\cap[a_1+a_2]|\neq a_2 \text{ and } |S\cap[b_1+b_2]|\neq b_2\end{array}\right)\\
&\qquad (\text{by (30)})\\
&=\binom{c_1+c_2}{c_2}-\binom{a_1+a_2}{a_2}\binom{(c_1+c_2)-(a_1+a_2)}{c_2-a_2}-\binom{b_1+b_2}{b_2}\binom{(c_1+c_2)-(b_1+b_2)}{c_2-b_2}\\
&\qquad+\binom{a_1+a_2}{a_2}\binom{(b_1+b_2)-(a_1+a_2)}{b_2-a_2}\binom{(c_1+c_2)-(b_1+b_2)}{c_2-b_2}\\
&\qquad (\text{by (31)}).
\end{aligned}$$

□

0.5. Zig-zag binary strings

If $n \in \mathbb{N}$, then a *binary n-string* shall mean an n-tuple of elements of $\{0,1\}$. (For example, $(0,1,1,0,1)$ is a binary 5-string.)

We say that a binary n-string $(a_1, a_2, \ldots, a_n)$ is *zig-zag* if it satisfies $a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots$ (in other words, $a_i \leq a_{i+1}$ for every odd $i \in [n-1]$, and $a_i \geq a_{i+1}$ for every even $i \in [n-1]$).

For example, $(0,1,1,1,0,0,0,1)$ is a zig-zag binary 8-string, but $(0,1,0,0,1)$ is not.

Exercise 5. Find a simple expression (no summation signs, only known functions and sequences) for the number of zig-zag binary n-strings for all $n \in \mathbb{N}$.

Solution to Exercise 5 (sketched). For each $n \in \mathbb{N}$, let z_n be the number of zig-zag binary n-strings. We must then find a simple expression for z_n.

We claim that

$$z_n = f_{n+2} \qquad \text{for each } n \in \mathbb{N}. \tag{32}$$

In order to prove this, we will first show that

$$z_n = z_{n-1} + z_{n-2} \qquad \text{for each } n > 1. \tag{33}$$

But before we do this, let us find the first few values of z_n. The only binary 0-string is the empty 0-tuple $()$, and it is zig-zag. Thus, there exists exactly 1 zig-zag binary 0-string; in other words, $z_0 = 1$. Likewise, $z_1 = 2$, because there are exactly 2 zig-zag binary 1-strings (namely, (0) and (1)). There are 3 zig-zag binary 2-strings (namely, $(0,0)$, $(0,1)$ and $(1,1)$); hence, $z_2 = 3$.

Now, we introduce a "mirror" version of zig-zag strings: We shall say that a binary n-string $(a_1, a_2, \ldots, a_n)$ is *zag-zig* if it satisfies $a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots$ (in other words, $a_i \geq a_{i+1}$ for every odd $i \in [n-1]$, and $a_i \leq a_{i+1}$ for every even $i \in [n-1]$). Clearly, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} & (\text{the number of zag-zig binary } n\text{-strings}) \\ & = (\text{the number of zig-zag binary } n\text{-strings}) . \end{aligned} \qquad (34)$$

(Indeed, there is a bijection $\{\text{zag-zig binary } n\text{-strings}\} \to \{\text{zig-zag binary } n\text{-strings}\}$, which sends every zag-zig binary n-string $(a_1, a_2, \ldots, a_n)$ to the zig-zag binary string $(1-a_1, 1-a_2, \ldots, 1-a_n)$.)

Let us now prove (33):

[*Proof of (33):* Let $n > 1$ be an integer. There are two kinds of zig-zag binary n-strings: those that start with a 0, and those that start with a 1. Let us count them separately.

A zig-zag binary n-string that starts with a 0 must have the form $(0, a_2, a_3, \ldots, a_n)$, where $0 \leq a_2 \geq a_3 \leq a_4 \geq a_5 \leq \cdots$. The inequality $0 \leq a_2$ is automatically satisfied, and thus can be omitted; hence, the requirement "$0 \leq a_2 \geq a_3 \leq a_4 \geq a_5 \leq \cdots$" becomes "$a_2 \geq a_3 \leq a_4 \geq a_5 \leq \cdots$". In other words, this requirement says that the binary $(n-1)$-string $(a_2, a_3, \ldots, a_n)$ is zag-zig. Hence, the map

$$\{\text{zig-zag binary } n\text{-strings starting with a } 0\} \to \{\text{zag-zig binary } (n-1)\text{-strings}\},$$
$$(a_1, a_2, \ldots, a_n) \mapsto (a_2, a_3, \ldots, a_n)$$

is well-defined. This map is furthermore a bijection (as can be easily seen: the inverse map sends each $(a_2, a_3, \ldots, a_n)$ to $(0, a_2, a_3, \ldots, a_n)$). Thus,

$$\begin{aligned} & |\{\text{zig-zag binary } n\text{-strings starting with a } 0\}| \\ & = |\{\text{zag-zig binary } (n-1)\text{-strings}\}| \\ & = (\text{the number of zag-zig binary } (n-1)\text{-strings}) \\ & = (\text{the number of zig-zag binary } (n-1)\text{-strings}) \\ & \qquad (\text{by (34) (applied to } n-1 \text{ instead of } n)) \\ & = z_{n-1} \end{aligned} \qquad (35)$$

(since z_{n-1} is defined as the number of zig-zag binary $(n-1)$-strings).

On the other hand, a zig-zag binary n-string that starts with a 1 must have the form $(1, a_2, a_3, \ldots, a_n)$, where $1 \leq a_2 \geq a_3 \leq a_4 \geq a_5 \leq \cdots$. The inequality $1 \leq a_2$ is equivalent to $a_2 = 1$ (because $a_2 \in \{0, 1\}$), and thus a zig-zag binary n-string that starts with a 1 must have the form $(1, 1, a_3, a_4, \ldots, a_n)$, where $1 \leq 1 \geq a_3 \leq a_4 \geq a_5 \leq \cdots$. The inequalities $1 \leq 1 \geq a_3$ are automatically satisfied (since $a_3 \in \{0, 1\}$), and thus can be omitted; hence, the requirement "$1 \leq 1 \geq a_3 \leq a_4 \geq a_5 \leq \cdots$" becomes "$a_3 \leq a_4 \geq a_5 \leq a_6 \geq \cdots$". In other words, this requirement says that the binary $(n-2)$-string $(a_3, a_4, \ldots, a_n)$ is zig-zag. Hence, the map

$$\{\text{zig-zag binary } n\text{-strings starting with a } 1\} \to \{\text{zig-zag binary } (n-2)\text{-strings}\},$$
$$(a_1, a_2, \ldots, a_n) \mapsto (a_3, a_4, \ldots, a_n)$$

is well-defined. This map is furthermore a bijection (as can be easily seen: the inverse map sends each $(a_3, a_4, \ldots, a_n)$ to $(1, 1, a_3, a_4, \ldots, a_n)$). Thus,

$$\begin{aligned} &\left|\{\text{zig-zag binary } n\text{-strings starting with a } 1\}\right| \\ &= \left|\{\text{zig-zag binary } (n-2)\text{-strings}\}\right| \\ &= (\text{the number of zig-zag binary } (n-2)\text{-strings}) \\ &= z_{n-2} \end{aligned} \tag{36}$$

(since z_{n-2} is defined as the number of zig-zag binary $(n-2)$-strings).

Now, each zig-zag binary n-string either starts with a 0 or starts with a 1 (since its first entry is in $\{0, 1\}$). Hence,

$$\begin{aligned} &(\text{the number of zig-zag binary } n\text{-strings}) \\ &= \underbrace{(\text{the number of zig-zag binary } n\text{-strings starting with a } 0)}_{\substack{=\left|\{\text{zig-zag binary } n\text{-strings starting with a } 0\}\right| \\ =z_{n-1} \\ (\text{by } (35))}} \\ &\qquad + \underbrace{(\text{the number of zig-zag binary } n\text{-strings starting with a } 1)}_{\substack{=\left|\{\text{zig-zag binary } n\text{-strings starting with a } 1\}\right| \\ =z_{n-2} \\ (\text{by } (36))}} \\ &= z_{n-1} + z_{n-2}. \end{aligned}$$

Now, recall that z_n is defined as the number of zig-zag binary n-strings. Thus,

$$z_n = (\text{the number of zig-zag binary } n\text{-strings}) = z_{n-1} + z_{n-2}.$$

This proves (33).]

Now, it is straightforward to prove (32) by strong induction on n. The induction step relies on the recurrence equation (33) (which is exactly the same as the recurrence equation for the Fibonacci sequence) and on the starting values $z_0 = 1$ and $z_1 = 2$. We leave the details to the reader. Thus, Exercise 5 is solved. □

Remark 0.9. Exercise 5 is [Stan11, Exercise 1.35 **(e)**]; see the same source for other combinatorial properties of Fibonacci numbers.

There is also a bijective proof of (32), relying on (22). The key step is to show that the map

$$\begin{aligned} \{\text{zig-zag binary } n\text{-strings}\} &\to \{\text{lacunar subsets of } [n]\}, \\ (a_1, a_2, \ldots, a_n) &\mapsto \{i \in [n] \mid a_i \equiv i \bmod 2\} \end{aligned}$$

is a bijection. The inverse map sends a lacunar subset S of $[n]$ to the n-tuple $(a_1, a_2, \ldots, a_n)$, where a_i is the remainder of $i + [i \notin S]$ upon division by 2. Can you check that these maps are well-defined and mutually inverse?

0.6. A binomial identity

Exercise 6. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \frac{(-1)^k}{\dbinom{n}{k}} = 2 \cdot \frac{n+1}{n+2} [n \text{ is even}] .$$

(Again, we are using the Iverson bracket notation, so $[n \text{ is even}]$ is 1 if n is even and 0 otherwise.)

[**Hint:** Show that $\dfrac{1}{\dbinom{n}{k}} = \left(\dfrac{1}{\dbinom{n+1}{k}} + \dfrac{1}{\dbinom{n+1}{k+1}} \right) \dfrac{n+1}{n+2}$ for each $k \in \{0, 1, \ldots, n\}$.]

Remark 0.10. The left hand side in Exercise 6 is the alternating sum of the reciprocals of all (nonzero) binomial coefficients in the n-th row of Pascal's triangle. What about the regular (non-alternating) sum? It appears that the simplest known formula merely rewrites it as a different (somewhat simpler) sum:

$$\sum_{k=0}^{n} \frac{1}{\dbinom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

See, e.g., `https://math.stackexchange.com/a/481686/` for a proof of this formula (and also of the fact that the sum on the left tends to 2 as $n \to \infty$). (This formula is also proven in [Grinbe16, solution to Exercise 3.20 **(b)**].)

Let us give a sketch of a solution to Exercise 6. (The solution in all its details can be found in [Grinbe16, solution to Exercise 3.20 **(a)**]; but I believe almost all of it is routine for you at this point.)

Following the hint, we begin by proving the following lemma:

Lemma 0.11. Let $n \in \mathbb{N}$. Let $k \in \{0, 1, \ldots, n\}$. Then,

$$\frac{1}{\dbinom{n}{k}} = \left(\frac{1}{\dbinom{n+1}{k}} + \frac{1}{\dbinom{n+1}{k+1}} \right) \frac{n+1}{n+2}. \tag{37}$$

(In particular, all the fractions in this equality are well-defined.)

Proof of Lemma 0.11 (sketched). Because of $k \in \{0, 1, \ldots, n\}$, we have

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}. \tag{38}$$

In particular, $\dbinom{n}{k} \neq 0$, so that the fraction $\dfrac{1}{\dbinom{n}{k}}$ in (37) is well-defined. But $k \in \{0, 1, \ldots, n\}$ also entails that both k and $k+1$ belong to $\{0, 1, \ldots, n+1\}$. Therefore,

$$\binom{n+1}{k} = \frac{(n+1)!}{k!\,((n+1)-k)!} = \frac{(n+1)!}{k!\,(n-k+1)!} \tag{39}$$

and

$$\binom{n+1}{k+1} = \frac{(n+1)!}{(k+1)!\,((n+1)-(k+1))!} = \frac{(n+1)!}{(k+1)!\,(n-k)!}. \tag{40}$$

These two numbers are nonzero; thus, the fractions $\dfrac{1}{\dbinom{n+1}{k}}$ and $\dfrac{1}{\dbinom{n+1}{k+1}}$ in (37) are also well-defined. (Of course, so is the fraction $\dfrac{n+1}{n+2}$, since $n+2 \geq 2 > 0$.)

It remains to verify the equality (37) itself. But this is easy: The equalities (38), (39) and (40) allow us to rewrite (37) as

$$\frac{1}{\left(\dfrac{n!}{k!\,(n-k)!}\right)} = \left(\frac{1}{\left(\dfrac{(n+1)!}{k!\,(n-k+1)!}\right)} + \frac{1}{\left(\dfrac{(n+1)!}{(k+1)!\,(n-k)!}\right)}\right)\frac{n+1}{n+2}.$$

In view of $(n+1)! = (n+1)\cdot n!$, $(n-k+1)! = (n-k+1)\cdot(n-k)!$ and $(k+1)! = (k+1)\cdot k!$, this can be further rewritten as

$$\frac{1}{\left(\dfrac{n!}{k!\,(n-k)!}\right)} = \left(\frac{1}{\left(\dfrac{(n+1)\cdot n!}{k!\,(n-k+1)\cdot(n-k)!}\right)} + \frac{1}{\left(\dfrac{(n+1)\cdot n!}{(k+1)\cdot k!\,(n-k)!}\right)}\right)\frac{n+1}{n+2}.$$

If we multiply both sides of this equality by $\dfrac{n!}{k!\,(n-k)!}$, we obtain

$$1 = \left(\frac{1}{\left(\dfrac{n+1}{n-k+1}\right)} + \frac{1}{\left(\dfrac{n+1}{k+1}\right)}\right)\frac{n+1}{n+2}.$$

But this follows from straightforward computations. Thus, Lemma 0.11 is proven. □

Solution to Exercise 6. Let $\lambda = \dfrac{n+1}{n+2}$. Every $k \in \{0, 1, \ldots, n\}$ satisfies

$$\frac{(-1)^k}{\dbinom{n}{k}} = (-1)^k \underbrace{\frac{1}{\dbinom{n}{k}}}_{\substack{=\left(\frac{1}{\binom{n+1}{k}}+\frac{1}{\binom{n+1}{k+1}}\right)\frac{n+1}{n+2} \\ \text{(by (37))}}}$$

$$= (-1)^k \left(\frac{1}{\dbinom{n+1}{k}} + \frac{1}{\dbinom{n+1}{k+1}} \right) \underbrace{\frac{n+1}{n+2}}_{=\lambda} = (-1)^k \left(\frac{1}{\dbinom{n+1}{k}} + \frac{1}{\dbinom{n+1}{k+1}} \right) \lambda$$

$$= \frac{(-1)^k}{\dbinom{n+1}{k}} \lambda + \underbrace{\frac{(-1)^k}{\dbinom{n+1}{k+1}}}_{\substack{=\frac{-(-1)^{k+1}}{\binom{n+1}{k+1}} \\ \text{(since } (-1)^k = -(-1)^{k+1}\text{)}}} \lambda$$

$$= \frac{(-1)^k}{\dbinom{n+1}{k}} \lambda + \frac{-(-1)^{k+1}}{\dbinom{n+1}{k+1}} \lambda = \frac{(-1)^k}{\dbinom{n+1}{k}} \lambda - \frac{(-1)^{k+1}}{\dbinom{n+1}{k+1}} \lambda.$$

Adding up this equality for all $k \in \{0, 1, \ldots, n\}$, we obtain

$$\begin{aligned}\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} &= \sum_{k=0}^{n} \left(\frac{(-1)^k}{\binom{n+1}{k}} \lambda - \frac{(-1)^{k+1}}{\binom{n+1}{k+1}} \lambda \right) = \frac{(-1)^0}{\binom{n+1}{0}} \lambda - \frac{(-1)^{n+1}}{\binom{n+1}{n+1}} \lambda \\ &\qquad \text{(by the telescope principle)} \\ &= \frac{1}{1} \lambda - \frac{(-1)^{n+1}}{1} \lambda \\ &\qquad \left(\text{since } (-1)^0 = 1 \text{ and } \binom{n+1}{0} = 1 \text{ and } \binom{n+1}{n+1} = 1 \right) \\ &= \lambda - \underbrace{(-1)^{n+1}}_{= -(-1)^n} \lambda = \lambda + (-1)^n \lambda = \underbrace{\left(1 + (-1)^n\right)}_{\substack{=2[n \text{ is even}] \\ \text{(because } 1+(-1)^n \\ \text{equals 2 when } n \text{ is even,} \\ \text{and equals 0 when } n \text{ is odd)}}} \lambda \\ &= 2 [n \text{ is even}] \underbrace{\lambda}_{= \frac{n+1}{n+2}} = 2 \cdot \frac{n+1}{n+2} [n \text{ is even}].\end{aligned}$$

This solves Exercise 6. □

0.7. Splitting integers into binomial coefficients

Exercise 7. Let j be a positive integer. A *j-trail* shall mean a j-tuple $(n_1, n_2, \ldots, n_j)$ of nonnegative integers satisfying $n_1 < n_2 < \cdots < n_j$.

Let $n \in \mathbb{N}$. Prove that there exists a unique j-trail $(n_1, n_2, \ldots, n_j)$ satisfying

$$n = \sum_{k=1}^{j} \binom{n_k}{k}.$$

Example 0.12. For $j = 3$, Exercise 7 says the following: For each $n \in \mathbb{N}$, there exists a unique 3-trail (n_1, n_2, n_3) satisfying

$$n = \binom{n_1}{1} + \binom{n_2}{2} + \binom{n_3}{3}.$$

For example, for $n = 0$, this 3-trail is $(0, 1, 2)$; for $n = 1$, this 3-trail is $(0, 1, 3)$; for $n = 5$, this 3-trail is $(0, 2, 4)$ (since $5 = \binom{0}{1} + \binom{2}{2} + \binom{4}{3}$).

Exercise 7 is a result of Macaulay, and appears in various texts on commutative algebra (e.g., [SwaHun06, Lemma A.5.1]).

The following solution essentially follows [SwaHun06, proof of Lemma A.5.1]. We begin with proving lemmas:

Lemma 0.13. Every $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ satisfy

$$\sum_{r=0}^{n} \binom{r+q}{r} = \binom{n+q+1}{n}.$$

Lemma 0.13 is one of the forms of the hockey-stick identity. It is easily proven by induction on n (see [Grinbe16, Exercise 3.3 **(a)**] for the details).

From now on, we shall be tacitly using the following fact: If $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, then

$$\binom{m}{n} \geq 0. \tag{41}$$

(This follows, e.g., from the fact that $\binom{m}{n}$ is the number of all n-element subsets of $[m]$.)

Lemma 0.14. Let $k \in \mathbb{N}$.

(a) If k is positive, then $\binom{k-1}{k} < \binom{k}{k} < \binom{k+1}{k} < \binom{k+2}{k} < \cdots$.

(b) We have $\binom{0}{k} \leq \binom{1}{k} \leq \binom{2}{k} \leq \cdots$.

(c) Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be such that $\binom{a}{k} > \binom{b}{k}$. Then, $a > b$.

Proof of Lemma 0.14 (sketched). **(a)** Assume that k is positive. We must prove that $\binom{k-1}{k} < \binom{k}{k} < \binom{k+1}{k} < \binom{k+2}{k} < \cdots$. In other words, we must show that $\binom{a}{k} < \binom{a+1}{k}$ for each $a \in \{k-1, k, k+1, \ldots\}$.

So let $a \in \{k-1, k, k+1, \ldots\}$. Then, $k-1 \in \mathbb{N}$ (since k is a positive integer), so that $k-1 \geq 0$. Also, $a \in \{k-1, k, k+1, \ldots\}$, so that $a \geq k-1$ and therefore $k-1 \in \{0, 1, \ldots, a\}$ (since $k-1 \geq 0$). Hence, $\binom{a}{k-1} = \dfrac{a!}{(k-1)!\,(a-(k-1))!} > 0$. Now, the recurrence relation of the binomial coefficients yields

$$\binom{a+1}{k} = \underbrace{\binom{a}{k-1}}_{>0} + \binom{a}{k} > \binom{a}{k}.$$

In other words, $\binom{a}{k} < \binom{a+1}{k}$. This is precisely what we wanted to prove. Thus, Lemma 0.14 **(a)** is proven.

(b) We must show that $\dbinom{0}{k} \leq \dbinom{1}{k} \leq \dbinom{2}{k} \leq \cdots$. In other words, we must show that $\dbinom{a}{k} \leq \dbinom{a+1}{k}$ for each $a \in \mathbb{N}$.

So let $a \in \mathbb{N}$. We must show that $\dbinom{a}{k} \leq \dbinom{a+1}{k}$.

This is obvious when $k = 0$ (because in this case, both $\dbinom{a}{k}$ and $\dbinom{a+1}{k}$ equal 1). Hence, we WLOG assume that $k \neq 0$. Thus, $k \geq 1$ (since $k \in \mathbb{N}$), so that $k-1 \in \mathbb{N}$. Also, $a \in \mathbb{N}$. Hence, $\dbinom{a}{k-1} \geq 0$ (by (41)). Now, the recurrence relation of the binomial coefficients yields

$$\binom{a+1}{k} = \underbrace{\binom{a}{k-1}}_{\geq 0} + \binom{a}{k} \geq \binom{a}{k}.$$

In other words, $\dbinom{a}{k} \leq \dbinom{a+1}{k}$. This is precisely what we wanted to prove. Thus, Lemma 0.14 **(b)** is proven.

(c) Assume the contrary. Thus, $a \leq b$.

Lemma 0.14 **(b)** yields $\dbinom{0}{k} \leq \dbinom{1}{k} \leq \dbinom{2}{k} \leq \cdots$. In other words, for any $u \in \mathbb{N}$ and $v \in \mathbb{N}$ satisfying $u \leq v$, we have $\dbinom{u}{k} \leq \dbinom{v}{k}$. Applying this to $u = a$ and $v = b$, we obtain $\dbinom{a}{k} \leq \dbinom{b}{k}$ (since $a \leq b$). But this contradicts $\dbinom{a}{k} > \dbinom{b}{k}$. This contradiction shows that our assumption was wrong. Thus, Lemma 0.14 **(c)** is proven. □

Lemma 0.15. Let g be a positive integer. Let $n_1, n_2, \ldots, n_g$ be nonnegative integers satisfying $n_1 < n_2 < \cdots < n_g$. Then:

(a) We have $n_i \leq n_g - g + i$ for each $i \in [g]$.

(b) We have

$$\binom{n_g}{g-1} > \sum_{k=1}^{g-1} \binom{n_k}{k}.$$

(c) Let m_g be an integer such that $n_g < m_g$. Then,

$$\binom{m_g}{g} > \sum_{k=1}^{g} \binom{n_k}{k}.$$

Proof of Lemma 0.15 (sketched). **(a)** We have $n_i - i \leq n_{i+1} - (i+1)$ for each $i \in [g-1]$ [6]. In other words,

$$n_1 - 1 \leq n_2 - 2 \leq \cdots \leq n_g - g. \tag{42}$$

Now, let $i \in [g]$. Thus, $1 \leq i \leq g$, so that $g \geq 1$ and thus $1 \in [g]$. Also, $i \leq g$; therefore, (42) yields $n_i - i \leq n_g - g$. Hence, $n_i \leq n_g - g + i$. This proves Lemma 0.15 **(a)**.

[6] *Proof.* Let $i \in [g-1]$. Thus, $n_i < n_{i+1}$ (since $n_1 < n_2 < \cdots < n_g$), so that $n_i \leq n_{i+1} - 1$ (since n_i and n_{i+1} are integers). Subtracting i from both sides of this inequality, we obtain $n_i - i \leq n_{i+1} - 1 - i = n_{i+1} - (i+1)$. Qed.

(b) Let $k \in [g-1]$. Thus, Lemma 0.15 **(a)** (applied to $i = k$) yields $n_k \leq n_g - g + k$. Thus, $n_g - g + k \geq n_k \geq 0$ (since n_k is nonnegative), so that $n_g - g + k \in \mathbb{N}$. Also, $n_k \in \mathbb{N}$ (since n_k is a nonnegative integer). But Lemma 0.14 **(b)** yields $\binom{0}{k} \leq \binom{1}{k} \leq \binom{2}{k} \leq \cdots$. Hence, $\binom{u}{k} \leq \binom{v}{k}$ for any $u \in \mathbb{N}$ and $v \in \mathbb{N}$ satisfying $u \leq v$. Applying this to $u = n_k$ and $v = n_g - g + k$, we obtain $\binom{n_k}{k} \leq \binom{n_g - g + k}{k}$ (since $n_k \leq n_g - g + k$). In other words,

$$\binom{n_g - g + k}{k} \geq \binom{n_k}{k}. \tag{43}$$

Now, forget that we fixed k. We thus have proven the inequality (43) for each $k \in [g-1]$.

But $g - 1 \in \mathbb{N}$ (since g is a positive integer). Hence, Lemma 0.13 (applied to $g-1$ and $n_g - g$ instead of n and q) yields

$$\sum_{r=0}^{g-1} \binom{r + n_g - g}{r} = \binom{(n_g - g) + (g-1) + 1}{g-1} = \binom{n_g}{g-1}$$

(since $(n_g - g) + (g-1) + 1 = n_g$). Thus,

$$\begin{aligned}
\binom{n_g}{g-1} &= \sum_{r=0}^{g-1} \binom{r + n_g - g}{r} = \sum_{k=0}^{g-1} \binom{k + n_g - g}{k} \\
&\qquad \text{(here, we have renamed the summation index } r \text{ as } k\text{)} \\
&= \underbrace{\binom{0 + n_g - g}{0}}_{=1} + \sum_{k=1}^{g-1} \underbrace{\binom{k + n_g - g}{k}}_{\substack{= \binom{n_g - g + k}{k} \geq \binom{n_k}{k} \\ \text{(by (43))}}} \geq 1 + \sum_{k=1}^{g-1} \binom{n_k}{k} > \sum_{k=1}^{g-1} \binom{n_k}{k}.
\end{aligned}$$

This proves Lemma 0.15 **(b)**.

(c) We have $n_g < m_g$, so that $n_g \leq m_g - 1$ (since n_g and m_g are integers). Thus, $n_g + 1 \leq m_g$.

Lemma 0.14 **(b)** (applied to $k = g$) yields $\binom{0}{g} \leq \binom{1}{g} \leq \binom{2}{g} \leq \cdots$. Hence, $\binom{u}{g} \leq \binom{v}{g}$ for any $u \in \mathbb{N}$ and $v \in \mathbb{N}$ satisfying $u \leq v$. Applying this to $u = n_g + 1$

and $v = m_g$, we obtain $\dbinom{n_g+1}{g} \leq \dbinom{m_g}{g}$ (since $n_g + 1 \leq m_g$). Hence,

$$\binom{m_g}{g} \geq \binom{n_g+1}{g} = \underbrace{\binom{n_g}{g-1}}_{\substack{> \sum_{k=1}^{g-1} \binom{n_k}{k} \\ \text{(by Lemma 0.15 (b))}}} + \binom{n_g}{g}$$

(by the recurrence relation of the binomial coefficients)

$$> \sum_{k=1}^{g-1} \binom{n_k}{k} + \binom{n_g}{g} = \sum_{k=1}^{g} \binom{n_k}{k}.$$

This proves Lemma 0.15 **(c)**. □

We are now close to the uniqueness part of Exercise 7:

Lemma 0.16. Let j be a positive integer. Let $n \in \mathbb{N}$. Then, there exists **at most one** j-trail $(n_1, n_2, \ldots, n_j)$ satisfying

$$n = \sum_{k=1}^{j} \binom{n_k}{k}. \tag{44}$$

Proof of Lemma 0.16 (sketched). We need to show that any two j-trails $(n_1, n_2, \ldots, n_j)$ satisfying (44) are equal.

So let $(a_1, a_2, \ldots, a_j)$ and $(b_1, b_2, \ldots, b_j)$ be two such j-trails. We must then prove that $(a_1, a_2, \ldots, a_j)$ and $(b_1, b_2, \ldots, b_j)$ are equal.

Assume the contrary. Thus, $(a_1, a_2, \ldots, a_j) \neq (b_1, b_2, \ldots, b_j)$. Hence, there exists some $k \in [j]$ satisfying $a_k \neq b_k$. Let g be the **largest** such k. Thus, g is an element of $[j]$ satisfying $a_g \neq b_g$, but

$$\text{every } k \in [j] \text{ that is larger than } g \text{ must satisfy } a_k = b_k. \tag{45}$$

We can rewrite (45) as follows: Every $k \in \{g+1, g+2, \ldots, j\}$ must satisfy $a_k = b_k$. Thus,

$$\text{every } k \in \{g+1, g+2, \ldots, j\} \text{ must satisfy } \binom{a_k}{k} = \binom{b_k}{k}. \tag{46}$$

We know that $(a_1, a_2, \ldots, a_j)$ is a j-trail. In other words, $(a_1, a_2, \ldots, a_j)$ is a j-tuple of nonnegative integers satisfying $a_1 < a_2 < \cdots < a_j$ (because this is how a j-trail was defined). Similarly, $(b_1, b_2, \ldots, b_j)$ is a j-tuple of nonnegative integers satisfying $b_1 < b_2 < \cdots < b_j$.

Also, $(a_1, a_2, \ldots, a_j)$ is a j-trail $(n_1, n_2, \ldots, n_j)$ satisfying (44). In other words, $(a_1, a_2, \ldots, a_j)$ is a j-trail and satisfies

$$n = \sum_{k=1}^{j} \binom{a_k}{k}. \tag{47}$$

Similarly, $(b_1, b_2, \ldots, b_j)$ is a j-trail and satisfies

$$n = \sum_{k=1}^{j} \binom{b_k}{k}. \tag{48}$$

We can WLOG assume that $a_g \leq b_g$ (since otherwise, we can simply switch the roles of $(a_1, a_2, \ldots, a_j)$ and $(b_1, b_2, \ldots, b_j)$). Thus, $a_g < b_g$ (since $a_g \neq b_g$). Now, $a_1, a_2, \ldots, a_g$ are nonnegative integers satisfying $a_1 < a_2 < \cdots < a_g$ (since $a_1 < a_2 < \cdots < a_j$). Hence, Lemma 0.15 **(c)** (applied to a_i and b_g instead of n_i and m_g) yields

$$\binom{b_g}{g} > \sum_{k=1}^{g} \binom{a_k}{k}. \tag{49}$$

Now, (47) yields

$$\begin{aligned} n &= \sum_{k=1}^{j} \binom{a_k}{k} = \sum_{k=1}^{g} \binom{a_k}{k} + \sum_{k=g+1}^{j} \underbrace{\binom{a_k}{k}}_{\substack{=\binom{b_k}{k} \\ \text{(by (46))}}} \\ &= \sum_{k=1}^{g} \binom{a_k}{k} + \sum_{k=g+1}^{j} \binom{b_k}{k}. \end{aligned} \tag{50}$$

On the other hand, (48) yields

$$\begin{aligned} n &= \sum_{k=1}^{j} \binom{b_k}{k} = \sum_{k=1}^{g-1} \underbrace{\binom{b_k}{k}}_{\substack{\geq 0 \\ \text{(since } b_k \text{ and } k \text{ are} \\ \text{nonnegative integers)}}} + \sum_{k=g}^{j} \binom{b_k}{k} \geq \underbrace{\sum_{k=1}^{g-1} 0}_{=0} + \sum_{k=g}^{j} \binom{b_k}{k} \\ &= \sum_{k=g}^{j} \binom{b_k}{k} = \underbrace{\binom{b_g}{g}}_{\substack{> \sum_{k=1}^{g} \binom{a_k}{k} \\ \text{(by (49))}}} + \sum_{k=g+1}^{j} \binom{b_k}{k} \\ &> \sum_{k=1}^{g} \binom{a_k}{k} + \sum_{k=g+1}^{j} \binom{b_k}{k} = n \qquad \text{(by (50))}. \end{aligned}$$

This is clearly absurd. Thus, we have found a contradiction. This shows that our assumption was wrong. Hence, $(a_1, a_2, \ldots, a_j)$ and $(b_1, b_2, \ldots, b_j)$ are equal. This proves Lemma 0.16. $\square$

Next, we need another simple lemma:

Lemma 0.17. Let k be a positive integer. Let $n \in \mathbb{N}$. Then, there exists an $h \in \{k-1, k, k+1, \ldots\}$ such that $\binom{h}{k} \leq n < \binom{h+1}{k}$.

Notice that the h in Lemma 0.17 is also unique; but we will not need this.

Proof of Lemma 0.17 (sketched). The gist of the proof is the following: The sequence

$$\left(\binom{k-1}{k}, \binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \ldots\right)$$

of nonnegative integers begins with $\binom{k-1}{k} = 0$, and is strictly increasing (by Lemma 0.14 **(a)**); therefore, it eventually "outgrows" the number n. In other words, $\binom{g}{k} > n$ for large enough g. Now, if we pick the smallest such g, then $g-1$ will be an $h \in \{k-1, k, k+1, \ldots\}$ such that $\binom{h}{k} \leq n < \binom{h+1}{k}$; this proves Lemma 0.17.

For the sake of completeness, let me show a more formalized version of this argument.

We have $k-1 \in \mathbb{N}$ (since k is a positive integer) and $k-1 < k$. Thus, (1) (applied to $k-1$ and k instead of m and n) yields $\binom{k-1}{k} = 0$.

Lemma 0.14 **(a)** yields that

$$\binom{k-1}{k} < \binom{k}{k} < \binom{k+1}{k} < \binom{k+2}{k} < \cdots .$$

Hence, the elements $\binom{k-1}{k}, \binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \ldots$ are distinct. Thus, in particular, the elements $\binom{k-1}{k}, \binom{k}{k}, \binom{k+1}{k}, \ldots, \binom{k+n}{k}$ are distinct. In other words, the elements $\binom{g}{k}$ for $g \in \{k-1, k, k+1, \ldots, k+n\}$ are distinct. Hence,

$$\left|\left\{\binom{g}{k} \mid g \in \{k-1, k, k+1, \ldots, k+n\}\right\}\right| = |\{k-1, k, k+1, \ldots, k+n\}| = n+2 > n+1. \tag{51}$$

Thus, there exists some $g \in \{k-1, k, k+1, \ldots, k+n\}$ such that $\binom{g}{k} \notin \{0, 1, \ldots, n\}$ [7]. Let p be the **smallest** such g. Thus, p is an element of $\{k-1, k, k+1, \ldots, k+n\}$ such that $\binom{p}{k} \notin \{0, 1, \ldots, n\}$, but each $g \in \{k-1, k, k+1, \ldots, k+n\}$ that is smaller than p must satisfy

$$\binom{g}{k} \in \{0, 1, \ldots, n\}. \tag{52}$$

[7] *Proof.* Assume the contrary. Thus, there exists no such g. In other words, each $g \in$

If we had $p = k-1$, then we would have $\binom{p}{k} = \binom{k-1}{k} = 0 \in \{0,1,\ldots,n\}$ (since $n \in \mathbb{N}$), which would contradict $\binom{p}{k} \notin \{0,1,\ldots,n\}$. Hence, we cannot have $p = k-1$. Thus, $p \neq k-1$. Combining this with $p \in \{k-1,k,k+1,\ldots,k+n\}$, we obtain

$$p \in \{k-1,k,k+1,\ldots,k+n\} \setminus \{k-1\} = \{k,k+1,\ldots,k+n\},$$

so that $p-1 \in \{k-1,k,k+1,\ldots,k+n-1\} \subseteq \{k-1,k,k+1,\ldots,k+n\}$. Since $p-1$ is smaller than p, we can thus apply (52) to $g = p-1$. As a result, we obtain $\binom{p-1}{k} \in \{0,1,\ldots,n\}$. Hence, $\binom{p-1}{k} \leq n$.

Also, $p \in \{k-1,k,k+1,\ldots,k+n\} \subseteq \mathbb{N}$ (since $k-1 \in \mathbb{N}$). Hence, p is a nonnegative integer. Now, $\binom{p}{k} \in \mathbb{N}$ (since both p and k are nonnegative integers). Combining this with $\binom{p}{k} \notin \{0,1,\ldots,n\}$, we obtain $\binom{p}{k} \in \mathbb{N} \setminus \{0,1,\ldots,n\} = \{n+1,n+2,n+3,\ldots\}$, so that $\binom{p}{k} \geq n+1 > n$. Thus, $n < \binom{p}{k} = \binom{(p-1)+1}{k}$ (since $p = (p-1)+1$).

Altogether, we now know that $p-1 \in \{k-1,k,k+1,\ldots,k+n-1\} \subseteq \{k-1,k,k+1,\ldots\}$ and $\binom{p-1}{k} \leq n < \binom{(p-1)+1}{k}$. Hence, there exists an $h \in \{k-1,k,k+1,\ldots\}$ such that $\binom{h}{k} \leq n < \binom{h+1}{k}$ (namely, $h = p-1$). This proves Lemma 0.17. □

We can now show the existence part of Exercise 7:

Lemma 0.18. Let j be a positive integer. Let $n \in \mathbb{N}$. Then, there exists **at least one** j-trail $(n_1, n_2, \ldots, n_j)$ satisfying

$$n = \sum_{k=1}^{j} \binom{n_k}{k}. \tag{53}$$

Proof of Lemma 0.18 (sketched). We shall prove Lemma 0.18 by induction on j:

Induction base: For each $n \in \mathbb{N}$, there exists at least one 1-trail $(n_1, n_2, \ldots, n_1)$ satisfying $n = \sum_{k=1}^{1} \binom{n_k}{k}$ (namely, the 1-trail (n)). [8] In other words, Lemma 0.18 holds for $j = 1$. This completes the induction base.

$\{k-1,k,k+1,\ldots,k+n\}$ satisfies $\binom{g}{k} \in \{0,1,\ldots,n\}$. In other words,

$$\left\{ \binom{g}{k} \mid g \in \{k-1,k,k+1,\ldots,k+n\} \right\} \subseteq \{0,1,\ldots,n\}.$$

Hence,

$$\left| \left\{ \binom{g}{k} \mid g \in \{k-1,k,k+1,\ldots,k+n\} \right\} \right| \leq |\{0,1,\ldots,n\}| = n+1.$$

But this contradicts (51). This contradiction shows that our assumption was wrong, qed.

[8] Indeed, a 1-trail $(n_1, n_2, \ldots, n_1)$ is the same as a 1-tuple (n_1) consisting of a single nonnega-

Induction step: Let i be a positive integer. Assume that Lemma 0.18 holds for $j = i$. We must prove that Lemma 0.18 holds for $j = i + 1$.

Let $n \in \mathbb{N}$. We are going to show that there is at least one $(i+1)$-trail $(n_1, n_2, \ldots, n_{i+1})$ satisfying $n = \sum_{k=1}^{i+1} \binom{n_k}{k}$.

Lemma 0.17 (applied to $k = i + 1$) shows that there exists an $h \in \{(i+1) - 1, i+1, (i+1)+1, \ldots\}$ such that $\binom{h}{i+1} \leq n < \binom{h+1}{i+1}$. Consider this h.

We have

$$h \in \{(i+1) - 1, i+1, (i+1)+1, \ldots\} = \{i, i+1, i+2, \ldots\},$$

so that $h \in \mathbb{N}$ and $h \geq i$.

From $\binom{h}{i+1} \leq n$, we conclude that $n - \binom{h}{i+1} \geq 0$, so that $n - \binom{h}{i+1} \in \mathbb{N}$. Thus, we can define an $m \in \mathbb{N}$ by $m = n - \binom{h}{i+1}$.

Now, recall that we assumed that Lemma 0.18 holds for $j = i$. Hence, we can apply Lemma 0.18 to i and m instead of j and n. We thus conclude that there exists **at least one** i-trail $(n_1, n_2, \ldots, n_i)$ satisfying

$$m = \sum_{k=1}^{i} \binom{n_k}{k}. \tag{54}$$

Consider this i-trail $(n_1, n_2, \ldots, n_i)$.

We know that $(n_1, n_2, \ldots, n_i)$ is an i-trail. In other words, $(n_1, n_2, \ldots, n_i)$ is an i-tuple of nonnegative integers satisfying $n_1 < n_2 < \cdots < n_i$ (because this is how an i-trail was defined).

We have $m = n - \binom{h}{i+1}$, so that

$$m + \binom{h}{i+1} = n < \binom{h+1}{i+1} = \binom{h}{i} + \binom{h}{i+1}$$

(by the recurrence relation of the binomial coefficients). Subtracting $\binom{h}{i+1}$ from

tive integer n_1; and this 1-trail satisfies $n = \sum_{k=1}^{1} \binom{n_k}{k}$ if and only if we have $n = n_1$ (because $\sum_{k=1}^{1} \binom{n_k}{k} = \binom{n_1}{1} = n_1$).

this inequality, we obtain $m < \binom{h}{i}$. Hence,

$$\binom{h}{i} > m = \sum_{k=1}^{i} \binom{n_k}{k} = \sum_{k=1}^{i-1} \underbrace{\binom{n_k}{k}}_{\substack{\geq 0 \\ \text{(since } n_k \text{ and } k \\ \text{are nonnegative)}}} + \binom{n_i}{i} \geq \underbrace{\sum_{k=1}^{i-1} 0}_{=0} + \binom{n_i}{i} = \binom{n_i}{i}.$$

Hence, Lemma 0.14 **(c)** (applied to i, h and n_i instead of k, a and b) yields $h > n_i$. In other words, $n_i < h$.

Now, let us extend the i-tuple $(n_1, n_2, \ldots, n_i)$ to an $(i+1)$-tuple $(n_1, n_2, \ldots, n_{i+1})$ by setting $n_{i+1} = h$. Thus, $n_i < h = n_{i+1}$. Combining $n_1 < n_2 < \cdots < n_i$ with $n_i < n_{i+1}$, we obtain $n_1 < n_2 < \cdots < n_{i+1}$. Also, $n_{i+1} = h \in \mathbb{N}$, so that n_{i+1} is a nonnegative integer.

Also, $n_1, n_2, \ldots, n_{i+1}$ are nonnegative integers (since $n_1, n_2, \ldots, n_i$ are nonnegative integers, and since n_{i+1} is a nonnegative integer). Thus, $(n_1, n_2, \ldots, n_{i+1})$ is an $(i+1)$-tuple of nonnegative integers satisfying $n_1 < n_2 < \cdots < n_{i+1}$. In other words, $(n_1, n_2, \ldots, n_{i+1})$ is an $(i+1)$-trail (by the definition of an $(i+1)$-trail). Moreover, we have

$$\sum_{k=1}^{i+1} \binom{n_k}{k} = \underbrace{\sum_{k=1}^{i} \binom{n_k}{k}}_{\substack{=m \\ \text{(by (54))}}} + \underbrace{\binom{n_{i+1}}{i+1}}_{\substack{=\binom{h}{i+1} \\ \text{(since } n_{i+1}=h)}} = m + \binom{h}{i+1} = n.$$

In other words, $n = \sum_{k=1}^{i+1} \binom{n_k}{k}$.

Thus, we have constructed an $(i+1)$-trail $(n_1, n_2, \ldots, n_{i+1})$ satisfying $n = \sum_{k=1}^{i+1} \binom{n_k}{k}$. Hence, there exists **at least one** such $(i+1)$-trail.

Now, forget that we fixed n. We thus have shown that for each $n \in \mathbb{N}$, there exists **at least one** $(i+1)$-trail $(n_1, n_2, \ldots, n_{i+1})$ satisfying $n = \sum_{k=1}^{i+1} \binom{n_k}{k}$. In other words, Lemma 0.18 holds for $j = i+1$. This completes the induction step. Thus, Lemma 0.18 is proven by induction. □

Solution to Exercise 7 (sketched). Lemma 0.18 shows that there exists **at least one** j-trail $(n_1, n_2, \ldots, n_j)$ satisfying

$$n = \sum_{k=1}^{j} \binom{n_k}{k}.$$

Moreover, this j-trail is unique, according to Lemma 0.16. Hence, Exercise 7 is solved. □

References

[BenQui03] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, The Mathematical Association of America, 2003.

[BenQui04] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Magic of Fibonacci Numbers and More*, Mathematical Adventures for Students and Amateurs, (David F. Hayes and Tatiana Shubin, editors), Spectrum Series of MAA, pp. 83–98, 2004.

[Galvin17] David Galvin, *Basic discrete mathematics*, 13 December 2017. `http://www.cip.ifi.lmu.de/~grinberg/t/17f/60610lectures2017-Galvin.pdf`

[Grinbe10] Darij Grinberg, *Zeckendorf family identities generalized*, version 24 February 2018.
`http://www.cip.ifi.lmu.de/~grinberg/zeckendorfBRIEF.pdf`
(Detailed version: `http://www.cip.ifi.lmu.de/~grinberg/zeckendorfLONG.pdf` .)

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

[Fall2017-HW1s] Darij Grinberg, *Math 4707 & Math 4990 Fall 2017 (Darij Grinberg): homework set 1 with solutions.*
`http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw1s.pdf`

[Stan11] Richard Stanley, *Enumerative Combinatorics, volume 1*, Second edition, version of 15 July 2011. Available at `http://math.mit.edu/~rstan/ec/` .

[SwaHun06] Irena Swanson and Craig Huneke, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society Lecture Note Series #336, Cambridge University Press, Cambridge, 2006.
`http://people.reed.edu/~iswanson/book/SwansonHuneke.pdf`

[WooZei09] Philip Matchett Wood, Doron Zeilberger, *A translation method for finding combinatorial bijections*, Annals of Combinatorics **13** (2009), pp. 383–402.

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trans-method.html