Math 4707 Spring 2018 (Darij Grinberg): midterm 1
due date: Wednesday 7 March 2018 at the beginning of class, or before that by email or moodle
Please solve **at most 4** of the 7 exercises!

Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

See [Fall2017-HW1s, solution to Exercise 8] for an example of how a counting proof can be written.

0.1. More on the Sierpinski triangle in Pascal's triangle

Exercise 1. Let $n \in \mathbb{N}$.

(a) Prove that the integer $\dbinom{2^n - 1}{b}$ is odd for each $b \in \{0, 1, \ldots, 2^n - 1\}$.

(b) Prove that the integer $\dbinom{2^n}{b}$ is even for each $b \in \{1, 2, \ldots, 2^n - 1\}$.

[Here, the set $\{0, 1, \ldots, 2^n - 1\}$ means the set of all integers k with $0 \leq k \leq 2^n - 1$, and the set $\{1, 2, \ldots, 2^n - 1\}$ means the set of all integers k with $1 \leq k \leq 2^n - 1$.]

0.2. Counting by symmetry

Recall that if $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$. If $n \in \mathbb{N}$, then S_n shall mean the set of all permutations of the set $[n]$. The number of these permutations is $|S_n| = n!$. (We shall prove this in class soon.) Note that S_n is called the *n-th symmetric group*.

Proposition 0.1. Let $n \geq 4$ be an integer. Then, the number of all permutations $\sigma \in S_n$ satisfying $\sigma(3) > \sigma(4)$ is $n!/2$.

Proof of Proposition 0.1. I say that a permutation $\sigma \in S_n$ is

- *green* if it satisfies $\sigma(3) > \sigma(4)$;
- *red* if it satisfies $\sigma(3) < \sigma(4)$.

Every permutation $\sigma \in S_n$ is either green or red (indeed, every permutation $\sigma \in S_n$ is injective, and thus satisfies $\sigma(3) \neq \sigma(4)$, so that it must satisfy either $\sigma(3) > \sigma(4)$ or $\sigma(3) < \sigma(4)$), but no permutation $\sigma \in S_n$ can be both green and red at the same time (since $\sigma(3) > \sigma(4)$ would contradict $\sigma(3) < \sigma(4)$). Hence, the set S_n is the union of its two disjoint subsets $\{\text{green permutations } \sigma \in S_n\}$ and $\{\text{red permutations } \sigma \in S_n\}$. Thus,

$$|S_n| = |\{\text{green permutations } \sigma \in S_n\}| + |\{\text{red permutations } \sigma \in S_n\}|. \quad (1)$$

On the other hand, I claim that "the colors are equidistributed", i.e., the number of green permutations $\sigma \in S_n$ equals the number of red permutations $\sigma \in S_n$.

To prove this, I will construct a bijection from $\{\text{green permutations } \sigma \in S_n\}$ to $\{\text{red permutations } \sigma \in S_n\}$.

Indeed, let s_3 be the permutation of $[n]$ that swaps the numbers 3 and 4 while leaving all other numbers unchanged. That is, s_3 is given by

$$s_3(i) = \begin{cases} 4, & \text{if } i = 3; \\ 3, & \text{if } i = 4; \\ i, & \text{if } i \notin \{3,4\} \end{cases} \qquad \text{for all } i \in [n].$$

(In one-line notation, s_3 is represented as $(1,2,4,3,5,6,\ldots,n)$, where only the two numbers 3 and 4 are out of order.)

Notice that $s_3 \circ s_3 = \text{id}$. (Visually speaking, this is clear: If we swap 3 and 4, and then swap 3 and 4 again, then all numbers return to their old places.)

If α and β are two permutations of $[n]$, then their composition $\alpha \circ \beta$ is a permutation of $[n]$ as well[1]. Hence, for every permutation $\sigma \in S_n$, the map $\sigma \circ s_3$ is also a permutation of $[n]$.

We now claim that

$$\text{if } \sigma \in S_n \text{ is green, then } \sigma \circ s_3 \in S_n \text{ is red.} \tag{2}$$

[*Proof of (2):* Assume that $\sigma \in S_n$ is green. Thus, $\sigma(3) > \sigma(4)$ (by the definition of "green").

We know $\sigma \circ s_3$ is a permutation of $[n]$. In other words, $\sigma \circ s_3 \in S_n$. We must prove that $\sigma \circ s_3$ is red. In other words, we must prove that $(\sigma \circ s_3)(3) < (\sigma \circ s_3)(4)$ (because this is what it means for $\sigma \circ s_3$ to be red).

But the definition of s_3 shows that $s_3(3) = 4$ and $s_3(4) = 3$. Thus, $(\sigma \circ s_3)(3) = \sigma\left(\underbrace{s_3(3)}_{=4}\right) = \sigma(4)$ and $(\sigma \circ s_3)(4) = \sigma\left(\underbrace{s_3(4)}_{=3}\right) = \sigma(3)$. Hence, $(\sigma \circ s_3)(4) = \sigma(3) > \sigma(4) = (\sigma \circ s_3)(3)$. In other words, $(\sigma \circ s_3)(3) < (\sigma \circ s_3)(4)$. But this is exactly what we wanted to prove. Thus, (2) is proven.]

An analogous argument shows that

$$\text{if } \sigma \in S_n \text{ is red, then } \sigma \circ s_3 \in S_n \text{ is green.} \tag{3}$$

Now, let α be the map

$$\{\text{green permutations } \sigma \in S_n\} \to \{\text{red permutations } \sigma \in S_n\},$$
$$\sigma \mapsto \sigma \circ s_3$$

[1] because permutations of $[n]$ are just bijective maps $[n] \to [n]$, but the composition of two bijective maps is again bijective

(this is well-defined because of (2)). Let β be the map

$$\{\text{red permutations } \sigma \in S_n\} \to \{\text{green permutations } \sigma \in S_n\},$$
$$\sigma \mapsto \sigma \circ s_3$$

(this is well-defined because of (3)). We have $\alpha \circ \beta = \text{id}$ (since every red permutation $\sigma \in S_n$ satisfies

$$\begin{aligned}(\alpha \circ \beta)(\sigma) &= \alpha\left(\underbrace{\beta(\sigma)}_{\substack{=\sigma \circ s_3 \\ \text{(by the definition of } \beta)}}\right) = \alpha(\sigma \circ s_3) \\ &= (\sigma \circ s_3) \circ s_3 \qquad \text{(by the definition of } \alpha) \\ &= \sigma \circ \underbrace{(s_3 \circ s_3)}_{=\text{id}} = \sigma = \text{id}(\sigma)\end{aligned}$$

) and $\beta \circ \alpha = \text{id}$ (by an analogous computation). Thus, the two maps α and β are mutually inverse. Hence, α is a bijection. Thus, we have found a bijection from $\{\text{green permutations } \sigma \in S_n\}$ to $\{\text{red permutations } \sigma \in S_n\}$ (namely, α). Therefore,

$$|\{\text{green permutations } \sigma \in S_n\}| = |\{\text{red permutations } \sigma \in S_n\}|. \tag{4}$$

Now, (1) becomes

$$\begin{aligned}|S_n| &= |\{\text{green permutations } \sigma \in S_n\}| + \underbrace{|\{\text{red permutations } \sigma \in S_n\}|}_{\substack{=|\{\text{green permutations } \sigma \in S_n\}| \\ \text{(by (4))}}} \\ &= |\{\text{green permutations } \sigma \in S_n\}| + |\{\text{green permutations } \sigma \in S_n\}| \\ &= 2 \cdot |\{\text{green permutations } \sigma \in S_n\}|.\end{aligned}$$

Hence,

$$|\{\text{green permutations } \sigma \in S_n\}| = \frac{1}{2} \underbrace{|S_n|}_{=n!} = \frac{1}{2} n! = n!/2.$$

In other words, the number of all green permutations $\sigma \in S_n$ is $n!/2$. In other words, the number of all permutations $\sigma \in S_n$ satisfying $\sigma(3) > \sigma(4)$ is $n!/2$ (because these permutations are precisely the green permutations $\sigma \in S_n$). This proves Proposition 0.1. □

Our above proof was an example of a "counting by symmetry": We did not count the green permutations directly; instead, we showed that they are in bijection with the remaining (i.e., red) permutations $\sigma \in S_n$ (that is, we matched up each green permutation with a red one), from which we concluded that they make up exactly half of the set S_n; and this told us that there are $\frac{1}{2}|S_n| = n!/2$ of them.

Exercise 2. Let $n \geq 4$ be an integer. Prove the following:

(a) The number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2)$ and $\sigma(3) > \sigma(4)$ is $n!/4$.

(b) The number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2) > \sigma(3)$ is $n!/6$.

[**Hint:** You'll need more than 2 colors...]

0.3. More on Fibonacci numbers

Recall that the *Fibonacci sequence* is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 2. \tag{5}$$

Exercise 3. Prove the following:

(a) We have $7f_n = f_{n-4} + f_{n+4}$ for each $n \geq 4$.

(b) We have $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ for each $n \in \mathbb{N}$.

(c) We have $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$ for each $n \in \mathbb{N}$.

(d) We have $f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$ for each $n \in \mathbb{N}$.

(e) We have $f_{m+n+1} = f_{m+1}f_{n+1} + f_m f_n$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

(f) For every $m \in \mathbb{N}$, we have

$$f_{2m+2} = \sum_{\substack{(a,b)\in\mathbb{N}^2;\\ a+b\leq m}} \binom{m-a}{b}\binom{m-b}{a}.$$

[**Hint:** All parts can be proven bijectively; part **(f)** is actually easiest to prove bijectively! (On the other hand, proving part **(a)** bijectively is a challenge; there are much easier ways.) As a reminder: Any exercises from previous problem sets can be used without proof.]

0.4. More lattice path counting

Recall that the set $\mathbb{Z}^2$ is called the *integer lattice*, and its elements $(a, b) \in \mathbb{Z}^2$ are called *points*. We regard these points as points on the Cartesian plane.

A *lattice path* is a path on the integer lattice that uses only two kinds of steps:

- up-steps (U), which have the form $(x, y) \mapsto (x, y+1)$;
- right-steps (R), which have the form $(x, y) \mapsto (x+1, y)$.

Thus, strictly speaking, a *lattice path* is a sequence $(v_0, v_1, \ldots, v_n)$ of points $v_i \in \mathbb{Z}^2$ such that for each $i \in [n]$, the difference vector $v_i - v_{i-1}$ is either $(0, 1)$ or $(1, 0)$.

If $(a, b) \in \mathbb{Z}^2$ and $(c, d) \in \mathbb{Z}^2$ are two points on the integer lattice, then a *lattice path from* (a, b) *to* (c, d) is a lattice path $(v_0, v_1, \ldots, v_n)$ satisfying $v_0 = (a, b)$ and $v_n = (c, d)$.

Exercise 4. **(a)** Given six integers $a_1, b_1, c_1, a_2, b_2, c_2$ satisfying $0 \leq a_1 \leq b_1 \leq c_1$ and $0 \leq a_2 \leq b_2 \leq c_2$. How many lattice paths from $(0,0)$ to (c_1, c_2) pass through none of the points (a_1, a_2) nor (b_1, b_2) ?

(b) Given six integers a, b, c, A, B, C satisfying $0 \leq a \leq b \leq c$ and $0 \leq A \leq B \leq C$. How many c-element subsets S of $[C]$ satisfy $|S \cap [A]| \neq a$ and $|S \cap [B]| \neq b$?

0.5. Zig-zag binary strings

If $n \in \mathbb{N}$, then a *binary n-string* shall mean an n-tuple of elements of $\{0, 1\}$. (For example, $(0, 1, 1, 0, 1)$ is a binary 5-string.)

We say that a binary n-string $(a_1, a_2, \ldots, a_n)$ is *zig-zag* if it satisfies $a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots$ (in other words, $a_i \leq a_{i+1}$ for every odd $i \in [n-1]$, and $a_i \geq a_{i+1}$ for every even $i \in [n-1]$).

For example, $(0, 1, 1, 1, 0, 0, 0, 1)$ is a zig-zag binary 8-string, but $(0, 1, 0, 0, 1)$ is not.

Exercise 5. Find a simple expression (no summation signs, only known functions and sequences) for the number of zig-zag binary n-strings for all $n \in \mathbb{N}$.

0.6. A binomial identity

Exercise 6. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \frac{(-1)^k}{\dbinom{n}{k}} = 2 \cdot \frac{n+1}{n+2} [n \text{ is even}].$$

(Again, we are using the Iverson bracket notation, so $[n \text{ is even}]$ is 1 if n is even and 0 otherwise.)

[Hint: Show that $\dfrac{1}{\dbinom{n}{k}} = \left(\dfrac{1}{\dbinom{n+1}{k}} + \dfrac{1}{\dbinom{n+1}{k+1}} \right) \dfrac{n+1}{n+2}$ for each $k \in \{0, 1, \ldots, n\}$.]

Remark 0.2. The left hand side in Exercise 6 is the alternating sum of the reciprocals of all (nonzero) binomial coefficients in the n-th row of Pascal's triangle. What about the regular (non-alternating) sum? It appears that the simplest known formula merely rewrites it as a different (somewhat simpler) sum:

$$\sum_{k=0}^{n} \frac{1}{\dbinom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

See, e.g., https://math.stackexchange.com/a/481686/ for a proof of this formula (and also of the fact that the sum on the left tends to 2 as $n \to \infty$).

0.7. Splitting integers into binomial coefficients

Exercise 7. Let j be a positive integer. A *j-trail* shall mean a j-tuple $\left(n_1, n_2, \ldots, n_j\right)$ of nonnegative integers satisfying $n_1 < n_2 < \cdots < n_j$.

Let $n \in \mathbb{N}$. Prove that there exists a unique j-trail $\left(n_1, n_2, \ldots, n_j\right)$ satisfying

$$n = \sum_{k=1}^{j} \binom{n_k}{k}.$$

Example 0.3. For $j = 3$, Exercise 7 says the following: For each $n \in \mathbb{N}$, there exists a unique 3-trail (n_1, n_2, n_3) satisfying

$$n = \binom{n_1}{1} + \binom{n_2}{2} + \binom{n_3}{3}.$$

For example, for $n = 0$, this 3-trail is $(0, 1, 2)$; for $n = 1$, this 3-trail is $(0, 1, 3)$; for $n = 5$, this 3-trail is $(0, 2, 4)$ (since $5 = \binom{0}{1} + \binom{2}{2} + \binom{4}{3}$).

References

[Galvin17] David Galvin, *Basic discrete mathematics*, 13 December 2017. http://www.cip.ifi.lmu.de/~grinberg/t/17f/60610lectures2017-Galvin.pdf

[Fall2017-HW1s] Darij Grinberg, *Math 4707 & Math 4990 Fall 2017 (Darij Grinberg): homework set 1 with solutions.* http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw1s.pdf