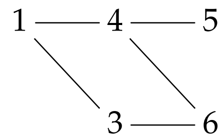
Math 4707 Spring 2018 (Darij Grinberg): homework set 5 [corrected version]
due date: Wednesday 25 April 2018 at the beginning of class, or before that by email or moodle
Please solve **at most 4** of the 7 exercises!

Contents

For the notations that we will use, we refer to Spring 2017 Math 5707 Homework set #2 and to classwork. The word "graph" means "multigraph" unless it appears as part of "simple graph". Here are the notations that I did not introduce in class:

- A two-element set $\{u, v\}$ will be denoted by uv when no confusion can arise. This will be mostly used for two-element sets that appear as edges in simple graphs (or as images of edges in multigraphs). For example, the simple graph



  has edges $13, 14, 36, 45, 46$.

- The set of all vertices of a graph G is called the *vertex set* of G, and is denoted by $\operatorname{V}(G)$.

  The set of all edges of a graph G is called the *edge set* of G, and is denoted by $\operatorname{E}(G)$.

- If v is a vertex and e is an edge of a graph (V, E, φ), then we say that v *belongs to* e (or, equivalently, e *contains* v) if v is an endpoint of e (that is, $v \in \varphi(e)$).

0.1. Perfect matchings of a $2 \times n$ grid

Definition 0.1. Let $n \in \mathbb{N}$. Then, the *path graph* P_n is defined to be the simple graph whose vertices are the n numbers $1, 2, \ldots, n$, and whose edges are $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}$. Here is how it looks like:

$$1 \text{ —— } 2 \text{ —— } \cdots \text{ —— } n$$

Definition 0.2. Let G and H be two simple graphs. The *Cartesian product* of G and H is a new simple graph, denoted $G \times H$, which is defined as follows:

- The vertex set $\operatorname{V}(G \times H)$ of $G \times H$ is the Cartesian product $\operatorname{V}(G) \times \operatorname{V}(H)$. (So the vertices of $G \times H$ are all pairs of the form (v, w), where v is a vertex of G and w is a vertex of H.)
- A vertex (g, h) of $G \times H$ is adjacent to a vertex (g', h') of $G \times H$ if and only if we have

  $$\textbf{either } \left(g = g' \text{ and } hh' \in \operatorname{E}(H)\right) \textbf{ or } \left(h = h' \text{ and } gg' \in \operatorname{E}(G)\right).$$

  (In particular, exactly one of the two equalities $g = g'$ and $h = h'$ has to hold when (g, h) is adjacent to (g', h').)

Definition 0.3. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. The *grid graph* $G_{n,m}$ is defined to be the Cartesian product $P_n \times P_m$.

Here is how the grid graph $G_{3,4}$ looks like:

$$\begin{array}{ccccccc}
(1,1) & \text{——} & (1,2) & \text{——} & (1,3) & \text{——} & (1,4) \\
| & & | & & | & & | \\
(2,1) & \text{——} & (2,2) & \text{——} & (2,3) & \text{——} & (2,4) \\
| & & | & & | & & | \\
(3,1) & \text{——} & (3,2) & \text{——} & (3,3) & \text{——} & (3,4)
\end{array}$$

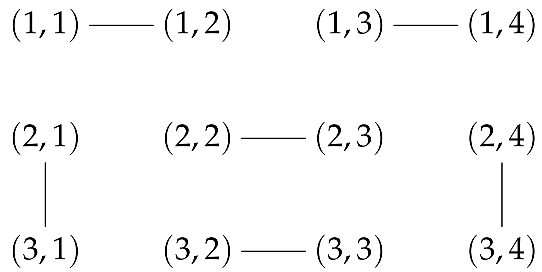
(Check that you understand how the definition of a Cartesian product of two graphs causes it to look like this.) For arbitrary $n, m \in \mathbb{N}$, the grid graph $G_{n,m}$ is the simple graph whose vertex set is $[n] \times [m]$, and whose edges have the form

$$\begin{array}{ll}
(i,j)\,(i+1,j) & \text{for } i \in [n-1] \text{ and } j \in [m], \text{ and} \\
(i,j)\,(i,j+1) & \text{for } i \in [n] \text{ and } j \in [m-1].
\end{array}$$

Recall that a *matching* of a graph G means a set M of disjoint edges of G. A *perfect matching* of a graph G means a matching M of G such that each vertex of G belongs to exactly one edge in M. For example,

$$\{(1,1)\,(1,2),\,(1,3)\,(1,4),\,(2,1)\,(3,1),\,(2,2)\,(2,3),\,(3,2)\,(3,3),\,(2,4)\,(3,4)\}$$

is a perfect matching of the grid graph $G_{3,4}$ shown above; let me visualize this matching by drawing only the edges of this matching (omitting all the other edges):

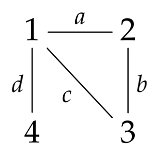


Exercise 1. Let $n \in \mathbb{N}$. How many perfect matchings does the grid graph $G_{2,n}$ have?

[Hint: This is something you know in disguise.]

0.2. Eulerian circuits of a windmill

The concept of a circuit in a graph is somewhat ambiguous: In the graph



, (1)

do you consider $(1, a, 2, b, 3, c, 1)$ and $(2, b, 3, c, 1, a, 2)$ as the same circuit? What about $(1, a, 2, b, 3, c, 1)$ and $(1, c, 3, b, 2, a, 1)$? According to our definition of a circuit (we defined it as a specific kind of walk), the answer is "no" in both cases:

$$(2, b, 3, c, 1, a, 2) \neq (1, a, 2, b, 3, c, 1) \neq (1, c, 3, b, 2, a, 1).$$

But most people would like to equate $(1, a, 2, b, 3, c, 1)$ with $(2, b, 3, c, 1, a, 2)$, at the very least, since these are "the same circuit with different starting points". So they say "circuit" but really mean "equivalence class of circuits with respect to cyclic rotation (and perhaps mirror reflection)". This is all irrelevant as long as we just discuss the **existence** of circuits; but when we start **counting** circuits, it becomes important. Depending on how circuits are defined, the graph (1) has either 6 or 3 or 2 or 1 cycles (which, as you remember, are circuits satisfying some conditions). According to our definition (which I don't want to change), it has 6 cycles.

Definition 0.4. Let G be a graph.

(a) A walk $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k)$ of G is said to be *Eulerian* if each edge of G appears exactly once among the k edges $e_1, e_2, \ldots, e_k$.

(b) Let $\mathbf{w} = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k)$ be a walk of G. Then, k is called the *length* of $\mathbf{w}$. If $k > 0$, then e_1 is called the *starting edge* of $\mathbf{w}$.

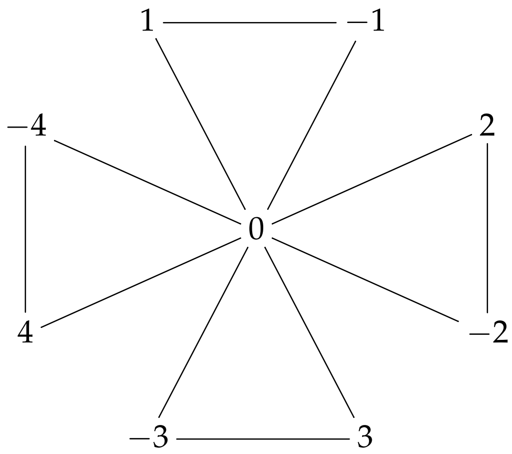
Counting all Eulerian circuits of a graph is usually difficult. For example, the number of Eulerian circuits in a complete graph K_n grows very fast with n and doesn't have a known expression (see sequence A007082 in the OEIS for the values when n is odd; of course, the values when n is even are 0).

Exercise 2. Let g be a positive integer. Let G be the simple graph whose vertices are the $2g+1$ integers $-g, -g+1, \ldots, g-1, g$, and whose edges are

$$\begin{aligned} &\{0, i\} && \text{for all } i \in \{1, 2, \ldots, g\}; \\ &\{0, -i\} && \text{for all } i \in \{1, 2, \ldots, g\}; \\ &\{i, -i\} && \text{for all } i \in \{1, 2, \ldots, g\} \end{aligned}$$

(these are $3g$ edges in total).

[Here is how G looks like in the case when $g = 4$:



(2)

]

(a) Find the number of Eulerian circuits of G whose starting point is 0 and whose starting edge is $\{0, 1\}$.

(b) Find the number of Eulerian circuits of G whose starting point is 0.

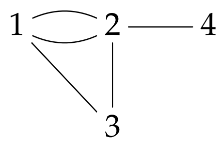
0.3. Counting walks in a graph

A graph always has a finite number of paths (since a path can never have more vertices than the graph has), but usually has an infinite number of walks (indeed, if the graph has a cycle, then you can build arbitrarily long walks by walking along this cycle over and over). Nevertheless, walks are much easier to count than paths. The next exercise states a formula for the number of walks of a given length between two given vertices in terms of the *adjacency matrix* of a graph. This matrix is an important representation of a graph.

Definition 0.5. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$-matrix. Let $i \in [n]$ and $j \in [m]$. Then, $A_{i,j}$ will denote the (i, j)-th entry of A.

Definition 0.6. Let $G = (V, E, \varphi)$ be a graph. Assume that $V = [n]$ for some $n \in \mathbb{N}$. Then, the *adjacency matrix* of G is defined as the $n \times n$-matrix whose (i, j)-th entry (for each $i \in [n]$ and $j \in [n]$) is the number of edges whose endpoints are i and j.

For example, the graph



(3)

has adjacency matrix

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Clearly, the adjacency matrix of a graph $G = (V, E, \varphi)$ with $V = [n]$ is symmetric. Furthermore, this adjacency matrix "encodes" the whole structure of G apart from the identities of the edges.

Exercise 3. Let $G = (V, E, \varphi)$ be a graph. Assume that $V = [n]$ for some $n \in \mathbb{N}$. Let A be the adjacency matrix of G. Let $i \in [n]$ and $j \in [n]$ and $k \in \mathbb{N}$. Prove that $\left(A^k\right)_{i,j}$ is the number of walks from i to j that have length k.

0.4. Your friends have more friends than you

If G is a graph, and if v is a vertex of G, then $\deg v$ denotes the degree of v (that is, the number of edges of G that contain v).

Exercise 4. Let $G = (V, E, \varphi)$ be a graph.

If $v \in V$ and $e \in E$ are such that $v \in \varphi(e)$ (that is, the edge e contains the vertex v), then we let e/v denote the endpoint of e distinct from v. For each $v \in V$, we define a rational number q_v by

$$q_v = \sum_{\substack{e \in E; \\ v \in \varphi(e)}} \frac{\deg(e/v)}{\deg v}.$$

(Note that the denominator $\deg v$ on the right hand side is nonzero whenever the sum is nonempty!)

[Roughly speaking, q_v is the average degree of the neighbors of v. But to be more precise, this is an average over all **edges** containing v, not just over all **neighbors** of v; the degree of a neighbor of v will factor in the stronger the more edges join this neighbor to v. When v is an isolated vertex – i.e., when $\deg v = 0$ –, the number q_v is 0.]

Prove that

$$\sum_{v \in V} q_v \geq \sum_{v \in V} \deg v. \tag{4}$$

[**Hint:** It helps to use the inequality $\frac{x}{y}+\frac{y}{x}\geq 2$, which holds for any two positive reals x and y (it is a consequence of $\frac{x}{y}+\frac{y}{x}-2=\frac{(x-y)^2}{xy}\geq 0$).]

If the graph G is a social network (vertices being people, and edges being friendships), then the inequality (4) (when divided by $|V|$) can be construed as saying "the average person is unpopular", where being "unpopular" means that your average friend has at least as many friends as you do. This is a slippery statement (involving an average within an average) and needs to be interpreted correctly: For example, in the graph (2), the vertex 0 has degree 8, while all other vertices have degree 2; the corresponding numbers q_v are $q_0=2$ and $q_v=5$ (for $v\neq 0$), respectively. Thus, (4) says that

$$2+5+5+5+5+5+5+5+5\geq 8+2+2+2+2+2+2+2+2,$$

which indeed holds (fairly strongly). The vertex 0, of course, is popular (having $\deg 0=8$ friends, whereas its average friend has $q_0=2$ friends), but this is balanced out by the unpopularity of all the other vertices.

Note that (4) does **not** mean that most vertices are unpopular. For example, if G is the simple graph with 5 vertices $1,2,3,4,5$ and all the $\binom{5}{2}=10$ possible edges between them except for the edge $\{4,5\}$, then the vertices $1,2,3$ of G are popular (a majority), while the vertices $4,5$ are unpopular. Nevertheless, (4) holds here, since the popularity of $1,2,3$ "outweighs" the unpopularity of $4,5$ when the appropriate averages are added.

0.5. When do transpositions generate all permutations?

Exercise 5. Let $G=(V,E,\varphi)$ be a connected graph.

For each $e=\{u,v\}\in\mathcal{P}_2(V)$, we let t_e be the permutation of V that swaps u with v while leaving all other elements of V unchanged.

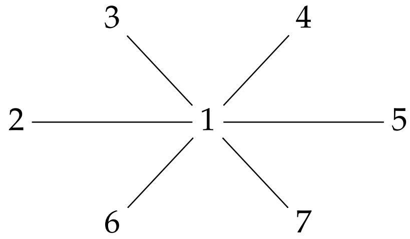
An *E-transposition* shall mean a permutation of the form t_e for some $e\in\varphi(E)$.

Prove that every permutation of V can be written as a composition of some E-transpositions.

Remark 0.7. In Exercise 5, we can WLOG assume (by relabeling the vertices) that $V=[n]$ for some $n\in\mathbb{N}$. Thus, Exercise 5 makes a statement about permutations of $[n]$.

For instance, if we apply Exercise 5 to the connected simple graph $P_n=([n],\{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\})$ (for some $n>0$), then we obtain the well-known fact that every permutation of $[n]$ can be written as a composition of some simple transpositions (because the E-transpositions in this case are precisely the simple transpositions $s_1,s_2,\ldots,s_{n-1}$).

For another example, we can apply Exercise 5 to the connected simple graph $([n], \{\{1,2\}, \{1,3\}, \ldots, \{1,n\}\})$ (for some $n > 0$); this graph is called a "star", because here is how it looks like for $n = 7$:



Thus, Exercise 5 shows that every permutation of $[n]$ can be written as a composition of some transpositions, each of which swaps 1 with one of the numbers $2, 3, \ldots, n$. (This fact was Exercise 3 on Fall 2017 Math 4990 homework set #7.)

Exercise 5 also has a converse: If $G = (V, E, \varphi)$ is a graph such that every permutation of V can be written as a composition of some E-transpositions, then G is connected or V is empty. This is not hard to check.

0.6. Latin rectangles and squares

Definition 0.8. Let $n \in \mathbb{N}$ and $r \in \mathbb{N}$. A *Latin $r \times n$-rectangle* is an $r \times n$-matrix with the following properties:

- Each row contains the integers $1, 2, \ldots, n$ in some order.
- No number appears more than once in a column.

For example, $\begin{pmatrix} 1 & 4 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ is a Latin 2×4-rectangle, and $\begin{pmatrix} 4 & 3 & 1 & 5 & 2 \\ 1 & 2 & 5 & 4 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$ is a Latin 3×5-rectangle, whereas $\begin{pmatrix} 1 & 4 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ is not a Latin 2×4-rectangle (as the number 1 appears twice in the first column) and $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is not a Latin 3×3-rectangle (since the second row is $2, 2, 3$, which is not a rearrangement of $1, 2, 3$).

Clearly, a Latin $r \times n$-rectangle can only exist if $r \leq n$.

Definition 0.9. Let $n \in \mathbb{N}$. A *Latin square of size n* means a Latin $n \times n$-rectangle.

Exercise 6. Let $r \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $r \leq n$. Let A be a Latin $r \times n$-rectangle. Show that A can be extended to a Latin square of size n by appending $n - r$ extra rows.

[**Hint:** By induction, it suffices to show that, as long as $r < n$, you can extend A to a Latin $(r+1) \times n$-rectangle by appending one extra row. You can use Hall's marriage theorem without proof here, even though we have not shown it in class.]

Latin squares are another classical combinatorial object whose number has not been expressed to a reasonable standard; see the Wikipedia page for what is known and why people care.

0.7. Latin squares and signs

Exercise 7. Let $n \in \mathbb{N}$. Let A be a Latin square of size n. Recall Definition 0.5.

For each $i \in [n]$, let r_i be the permutation of $[n]$ whose one-line notation is the i-th row of A (that is, which satisfies $r_i(j) = A_{i,j}$ for each $j \in [n]$).

For each $j \in [n]$, let c_j be the permutation of $[n]$ whose one-line notation is the j-th column of A (that is, which satisfies $c_j(i) = A_{i,j}$ for each $i \in [n]$).

For each $k \in [n]$, let z_k be the permutation of $[n]$ such that for each $i \in [n]$, we have $A_{i,z_k(i)} = k$. (Thus, the permutation z_k sends each $i \in [n]$ to the position of the entry k in the i-th row of A. This is indeed a permutation, as follows easily from the definition of a Latin square and from the pigeonhole principle.)

Prove that

$$\left(\prod_{i=1}^{n}(-1)^{r_i}\right)\left(\prod_{j=1}^{n}(-1)^{c_j}\right)\left(\prod_{k=1}^{n}(-1)^{z_k}\right) = (-1)^{n(n-1)/2}.$$

Example 0.10. For this example, let $n = 4$ and $A = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 2 & 1 & 3 & 4 \end{pmatrix}$. Then, A is a Latin square of size 4. The permutations r_i, c_j, z_k of Exercise 7 then look as follows in one-line notation:

$$\begin{aligned}
&r_1 = [4,3,1,2], && r_2 = [1,2,4,3], && r_3 = [3,4,2,1], && r_4 = [2,1,3,4]; \\
&c_1 = [4,1,3,2], && c_2 = [3,2,4,1], && c_3 = [1,4,2,3], && c_4 = [2,3,1,4]; \\
&z_1 = [3,1,4,2], && z_2 = [4,2,3,1], && z_3 = [2,4,1,3], && z_4 = [1,3,2,4].
\end{aligned}$$

[**Hint to Exercise 7:** Let S be the set of all triples $(i, j, A_{i,j}) \in [n]^3$ for $i \in [n]$ and $j \in [n]$. Notice that you can visualize the set $[n]^3$ as an $n \times n \times n$-cube built out of

$1 \times 1 \times 1$-blocks – like a Rubik's cube –, and then S is a set of n^2 blocks of this cube such that each strip parallel to one of the three coordinate axes contains exactly one block from S. In other words, if you fix two entries of a triple $(i,j,k) \in [n]^3$, then there exists exactly one value for the third entry that causes the triple to belong to S. (Notice that this rule characterizes sets $S \subseteq [n]^3$ that come from Latin squares; thus, it can be viewed as a more symmetric definition of a Latin square.)

Now, whenever α, β and γ are three binary relations on the set $[n]$ (for example, α can be any of the relations $=$, $<$, $>$ and $\neq$, and so can be β and γ), we define the set $N(\alpha,\beta,\gamma)$ by

$$N(\alpha,\beta,\gamma) = \left\{ \left((x,y,z), (x',y',z') \right) \in S \times S \mid x\alpha x' \text{ and } y\beta y' \text{ and } z\gamma z' \right\}.$$

For example,

$$N(<,>,=) = \left\{ \left((x,y,z), (x',y',z') \right) \in S \times S \mid x < x' \text{ and } y > y' \text{ and } z = z' \right\}.$$

Now, you can follow the following roadmap:

(a) Prove that

$$\begin{aligned} |N(<,<,>)| &= |N(>,>,<)| && \text{and} \\ |N(<,>,<)| &= |N(>,<,>)| && \text{and} \\ |N(>,<,<)| &= |N(<,>,>)|. \end{aligned}$$

(b) Let $*$ be the binary relation on $[n]^2$ such that **every** pair (i,j) satisfies $a * b$. (The "joker relation".) Show that

$$\begin{aligned} |N(*,<,>)| &= (n(n-1)/2)^2 && \text{and} \\ |N(<,>,*)| &= (n(n-1)/2)^2 && \text{and} \\ |N(>,*,<)| &= (n(n-1)/2)^2. \end{aligned}$$

(c) Prove that

$$|N(=,<,>)| + |N(<,<,>)| + |N(>,<,>)| = |N(*,<,>)|.$$

Add this equality with two similar ones, and derive that

$$|N(=,<,>)| + |N(<,>,=)| + |N(>,=,<)| \equiv 3(n(n-1)/2)^2 \equiv n(n-1)/2 \bmod 2.$$

(d) Prove that

$$\begin{aligned} \prod_{i=1}^{n} (-1)^{r_i} &= (-1)^{|N(=,<,>)|} && \text{and} \\ \prod_{j=1}^{n} (-1)^{c_j} &= (-1)^{|N(>,=,<)|} && \text{and} \\ \prod_{k=1}^{n} (-1)^{z_k} &= (-1)^{|N(<,>,=)|}. \end{aligned}$$

(e) Derive the claim of Exercise 7.]