University of Minnesota, School of Mathematics

Math 4707: Combinatorics, Spring 2018
Homework 4

Nathaniel Gorski (edited by Darij Grinberg)

April 12, 2018

Exercise 1

Exercise 0.1. Let $n \geq 2$ be an integer. Then there are precisely $(n-1)!$ permutations $\sigma \in S_n$ which satisfy $\sigma(2) = \sigma(1) + 1$.

Proof. Let us construct such a permutation σ. We will first select $\sigma(1)$. Since $\sigma(2)$ must be one higher than $\sigma(1)$, it follows that $\sigma(1)$ must be an element of $[n-1]$, of which there are $n-1$. And, $\sigma(2)$ must be equal to $\sigma(1)+1$, for which there is exactly one choice.

We now wish to select the images for each element of $[n]\backslash[2]$. This is the same as selecting a bijection from $[n]\backslash[2]$ to $[n]\backslash\{\sigma(1), \sigma(2)\}$. Since $[2] \subseteq [n]$ and $\{\sigma(1), \sigma(2)\} \subseteq [n]$, it follows that $|[n]\backslash[2]| = |[n]| - |[2]|$, and $|[n]\backslash\{\sigma(1), \sigma(2)\}| = |[n]| - |\{\sigma(1), \sigma(2)\}|$. And clearly, $[n]$ has n elements, $[2]$ has 2 elements, and $\{\sigma(1), \sigma(2)\}$ has 2 elements. Therefore, a bijection between $[n]\backslash[2]$ to $[n]\backslash\{\sigma(1), \sigma(2)\}$ is a bijection between two $(n-2)$-element sets. We know that there are $(n-2)!$ such bijections. Hence, there are $(n-2)!$ ways to select the images of each element of $[n]\backslash[2]$ for σ. This completes the construction.

In this construction, three decisions were made: $\sigma(1)$ was chosen from $n-1$ options, then $\sigma(2)$ was chosen from 1 option, and then the images of the elements of $[n]\backslash[2]$ were chosen from $(n-2)!$ options. So, by the multiplication principle, there are $(n-1)(1)(n-2)! = (n-1)!$ total ways of constructing σ. $\square$

Exercise 4

Exercise 0.2. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. And let $a_1, a_2, \ldots, a_n$ be any n numbers. Then

$$\sum_{\substack{1 \le i < j \le n; \\ \sigma(i) > \sigma(j)}} (a_j - a_i) = \sum_{i=1}^{n} a_i (i - \sigma(i)).$$

Proof. Let $U : [n] \to [n]$ be the map sending each $i \in [n]$ to the number of distinct elements $j \in [n]$ such that $i > j$ but $\sigma(i) < \sigma(j)$.

Let $L : [n] \to [n]$ be the map sending each $i \in [n]$ to the number of distinct elements $j \in [n]$ such that $i < j$ but $\sigma(i) > \sigma(j)$.

Let $M : [n] \to [n]$ be the map sending each $i \in [n]$ to the number of distinct elements $j \in [n]$ such that $i > j$ and also $\sigma(i) > \sigma(j)$.

Fix $i \in [n]$.

There are exactly $i - 1$ elements j of $[n]$ such that $i > j$. Because σ is a bijection, each element $j \in [n]$ with $j < i$ must satisfy either $\sigma(j) < \sigma(i)$ or $\sigma(j) > \sigma(i)$. But the number of $j < i$ where $\sigma(j) < \sigma(i)$ is counted by $M(i)$, whereas the number of $j < i$ where $\sigma(j) > \sigma(i)$ is counted by $U(i)$. Hence, it follows that $i - 1 = M(i) + U(i)$. Equivalently, $i = M(i) + U(i) + 1$.

There are exactly $\sigma(i) - 1$ elements k of $[n]$ such that $\sigma(i) > k$. We can substitute $\sigma(j)$ for k in this statement (since σ is a bijection $[n] \to [n]$), and thus obtain the following: There are exactly $\sigma(i) - 1$ elements j of $[n]$ such that $\sigma(i) > \sigma(j)$. Each such j satisfies either $i > j$ or $i < j$ (since otherwise, $i = j$ would contradict $\sigma(i) > \sigma(j)$). The number of j such that $\sigma(i) > \sigma(j)$ but $i < j$ is counted by $L(i)$, whereas the number of j such that $\sigma(i) > \sigma(j)$ and $i > j$ is counted by $M(i)$. So we obtain $\sigma(i) - 1 = L(i) + M(i)$. So $\sigma(i) = L(i) + M(i) + 1$.

Subtracting the equality $\sigma(i) = L(i) + M(i) + 1$ from the equality $i = M(i) + U(i) + 1$, we obtain

$$i - \sigma(i) = (M(i) + U(i) + 1) - (L(i) + M(i) + 1) = U(i) - L(i). \tag{1}$$

Now, forget that we fixed i. Hence, (1) is proven for each $i \in [n]$.

But the sum $\sum_{\substack{1 \le j < i \le n; \\ \sigma(j) > \sigma(i)}} a_i$ contains each a_i exactly $U(i)$ times (by the definition of $U(i)$). Hence,

$$\sum_{\substack{1 \le j < i \le n; \\ \sigma(j) > \sigma(i)}} a_i = \sum_{i=1}^{n} a_i U(i). \tag{2}$$

Similarly,

$$\sum_{\substack{1 \le i < j \le n; \\ \sigma(i) > \sigma(j)}} a_i = \sum_{i=1}^{n} a_i L(i). \tag{3}$$

Now

$$\begin{aligned}
\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}} (a_j - a_i) &= \sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}} a_j - \sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}} a_i \\
&= \sum_{\substack{1\leq j<i\leq n;\\ \sigma(j)>\sigma(i)}} a_i - \sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}} a_i \\
&\qquad \text{(here, we have renamed } (i,j) \text{ as } (j,i) \text{ in the first sum)} \\
&= \sum_{i=1}^{n} a_i U(i) - \sum_{i=1}^{n} a_i L(i) \qquad \text{(by (2) and (3))} \\
&= \sum_{i=1}^{n} a_i \underbrace{(U(i) - L(i))}_{\substack{=i-\sigma(i)\\ \text{(by (1))}}} = \sum_{i=1}^{n} a_i (i - \sigma(i)).
\end{aligned}$$

This solves the exercise. □