Math 4707 Spring 2018 (Darij Grinberg): homework set 4
due date: Wednesday 11 April 2018 at the beginning of class, or before that by email or moodle
Please solve **at most 3** of the 6 exercises!

Contents

Recall the following:

- We have $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- If $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$.
- For each $n \in \mathbb{N}$, we let S_n denote the set of all permutations of $[n]$.

0.1. Permutations σ with $\sigma(2) = \sigma(1) + 1$

Exercise 1. Let $n \geq 2$ be an integer. Prove that there are precisely $(n-1)!$ permutations $\sigma \in S_n$ satisfying $\sigma(2) = \sigma(1) + 1$.

0.2. An introduction to rook theory

Definition 0.1. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we let $x^{\underline{n}}$ denote the "n-th lower factorial of x"; this is the real number $x(x-1)\cdots(x-n+1)$. (Thus, $x^{\underline{n}} = n! \cdot \binom{x}{n}$.)

For example, $x^{\underline{0}} = 1$, $x^{\underline{1}} = x$, $x^{\underline{2}} = x(x-1)$, etc.[1].

Exercise 2. Let $n \in \mathbb{N}$. Prove the following:

(a) If $(a_0, a_1, \ldots, a_n)$ is an $(n+1)$-tuple of rational numbers such that each $x \in \{0, 1, \ldots, n\}$ satisfies

$$\sum_{k=0}^{n} a_k x^{\underline{k}} = 0,$$

[1] For all LaTeX users: $x^{\underline{n}}$ is "`x^{\underline{n}}`". Feel free to create a macro for this, e.g., by putting the following line into the header of your TeX file (along with the other "VARIOUS USEFUL COMMANDS"):

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\newcommand{\lf}[2]{{#1}^{\underline{#2}}}
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Then, you can use "`\lf{x}{n}`" to obtain "$x^{\underline{n}}$".

then $(a_0, a_1, \ldots, a_n) = (0, 0, \ldots, 0)$.

(b) If $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ are two $(n+1)$-tuples of rational numbers such that each $x \in \{0, 1, \ldots, n\}$ satisfies

$$\sum_{k=0}^{n} a_k x^{\underline{k}} = \sum_{k=0}^{n} b_k x^{\underline{k}},$$

then $(a_0, a_1, \ldots, a_n) = (b_0, b_1, \ldots, b_n)$.

[Hint: In terms of linear algebra, part **(a)** is saying that the $n+1$ vectors $\left(0^{\underline{k}}, 1^{\underline{k}}, \ldots, n^{\underline{k}}\right)^T \in \mathbb{Q}^{n+1}$ for $k \in \{0, 1, \ldots, n\}$ are linearly independent. You may find this useful or not; the exercise has a fully elementary solution.]

In the following, we will consider each pair $(i, j) \in \mathbb{Z}^2$ of two integers as a square on an (infinite) chessboard; we say that it lies in *row* i and in *column* j. A *rook placement* shall mean a subset X of $\mathbb{Z}^2$ such that any two distinct elements of X lie in different rows and in different columns[2]. (The idea behind this name is that if we place rooks into the squares $(i, j) \in X$, then no two rooks will attack each other.) For example, $\{(1,3), (2,2), (3,7)\}$ is a rook placement, whereas $\{(1,3), (2,2), (7,3)\}$ is not (since the distinct squares $(1,3)$ and $(7,3)$ lie in the same column). If X is a rook placement, then the elements of X are called the *rooks* of X.

Now, fix $n \in \mathbb{N}$. If $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ is an n-tuple of nonnegative integers, then we define the $\mathbf{u}$*-board* $D(\mathbf{u})$ to be the set

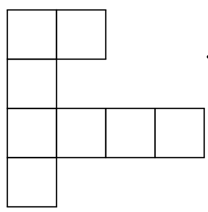
$$\{(i, j) \mid i \in [n] \text{ and } j \in [u_i]\}.$$

We visually represent this set as a "chessboard" consisting of n left-aligned rows[3], where the i-th row consists of u_i boxes (which occupy columns $1, 2, \ldots, u_i$).

For example, if $n = 4$ and $\mathbf{u} = (2, 1, 4, 1)$, then $D(\mathbf{u})$ is the set

$$\{(1,1), (1,2), (2,1), (3,1), (3,2), (3,3), (3,4), (4,1)\},$$

and is visually represented as the "chessboard"



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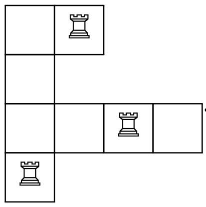
The set (or "chessboard") $D(\mathbf{u})$ is also known as the *Young diagram* (or *Ferrers diagram*) of $\mathbf{u}$; we have briefly seen its use in the proof of Proposition 4.13 on March 21st.

[2] In other words, a rook placement means a subset X of $\mathbb{Z}^2$ such that any two distinct elements (i, j) and (i', j') of X satisfy $i \neq i'$ and $j \neq j'$.

[3] or "ranks", to use the terminology of chess (but we label them $1, 2, \ldots, n$ from top to bottom, not from bottom to top as on an actual chessboard)

If $\mathbf{u} \in \mathbb{N}^n$, then a *rook placement in* $D(\mathbf{u})$ means a subset X of $D(\mathbf{u})$ that is a rook placement. For example, if $n = 4$ and $\mathbf{u} = (2, 1, 4, 1)$, then $\{(1,2), (3,3), (4,1)\}$ is a rook placement in $D(\mathbf{u})$. We represent this rook placement by putting rooks (i.e., ♖ symbols) into the squares that belong to it; so we get



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Note that the empty set $\varnothing$ is always a rook placement in $D(\mathbf{u})$; so is any 1-element subset of $D(\mathbf{u})$.

If $\mathbf{u} \in \mathbb{N}^n$ and $k \in \mathbb{N}$, then we let $R_k(\mathbf{u})$ denote the number of all rook placements in $D(\mathbf{u})$ of size k. (In terms of the picture, "size k" means that it contains exactly k rooks.) For example, if $n = 3$ and $\mathbf{u} = (2, 1, 3)$, then

$$R_0(\mathbf{u}) = 1; \qquad R_1(\mathbf{u}) = 6; \qquad R_2(\mathbf{u}) = 7;$$
$$R_3(\mathbf{u}) = 1; \qquad R_k(\mathbf{u}) = 0 \text{ for all } k \geq 4.$$

It is easy to see that if $\mathbf{u} = \left(\underbrace{u, u, \ldots, u}_{n \text{ times}} \right)$ for some $u \in \mathbb{N}$, then

$$R_k(\mathbf{u}) = \binom{n}{k} \binom{u}{k} k! \qquad \text{for each } k \in \mathbb{N}. \tag{1}$$

(Indeed, in order to place k non-attacking rooks in $D(\mathbf{u})$, we first choose the k rows they will occupy, then the k columns they will occupy, and finally an appropriate permutation of $[k]$ that will determine which column has a rook in which row.) For other n-tuples $\mathbf{u}$, finding $R_k(\mathbf{u})$ is harder.

For example, if you try the same reasoning for $\mathbf{u} = (2n-1, 2n-3, \ldots, 5, 3, 1)$, then you still have $\binom{n}{k}$ choices for the k rows occupied by rooks; but the number of options in the following steps will depend on the specific k rows you have chosen (the lower the rows, the fewer options). This way, you get the formula

$$R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = \sum_{1 \leq s_1 < s_2 < \cdots < s_k \leq n} (s_k - 0)(s_{k-1} - 1)(s_{k-2} - 2) \cdots (s_1 - (k-1)),$$

which is a far cry from the simplicity of (1). But there is a simple formula, which we'll see in Corollary 0.3!

We can restrict ourselves to n-tuples $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ satisfying $u_1 \geq u_2 \geq \cdots \geq u_n$, because switching some of the entries of $\mathbf{u}$ does not change the values of $R_k(\mathbf{u})$.

In general, $R_k(\mathbf{u}) = 0$ for all $k > n$, because each of the n rows $1, 2, \ldots, n$ contains at most one rook. For $k = n$, we have a neat formula:

Proposition 0.2. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ be such that $u_1 \geq u_2 \geq \cdots \geq u_n$. Then,

$$R_n(\mathbf{u}) = \prod_{i=0}^{n-1} (u_{n-i} - i).$$

Proof of Proposition 0.2. Each square in $D(\mathbf{u})$ belongs to one of the n rows $1, 2, \ldots, n$. In a rook placement, each row contains at most one rook. Thus, any rook placement in $D(\mathbf{u})$ contains at most n rooks, and the only way it can contain n rooks is if it contains one rook in each of the n rows $1, 2, \ldots, n$.

Hence, a rook placement in $D(\mathbf{u})$ of size n is the same as a rook placement in $D(\mathbf{u})$ that contains a rook in each of the n rows $1, 2, \ldots, n$. We can construct such a rook placement by the following algorithm:

- First, we place the rook in row n. This rook must lie in one of the u_n columns $1, 2, \ldots, u_n$; so we have u_n options.
- Then, we place the rook in row $n-1$. This rook must lie in one of the u_{n-1} columns $1, 2, \ldots, u_{n-1}$, but **not** in the column that contains the previous rook; so we have $u_{n-1} - 1$ options.
- Then, we place the rook in row $n-2$. This rook must lie in one of the u_{n-2} columns $1, 2, \ldots, u_{n-2}$, but **not** in either of the two columns that contain the previous rooks; so we have $u_{n-2} - 2$ options.
- And so on.

Thus, the total number of choices (and therefore the total number of rook placements in $D(\mathbf{u})$ of size n) is

$$\begin{aligned} &u_n (u_{n-1} - 1)(u_{n-2} - 2) \cdots \left(u_{n-(n-1)} - (n-1)\right) \\ &= (u_n - 0)(u_{n-1} - 1)(u_{n-2} - 2) \cdots (u_1 - (n-1)) = \prod_{i=0}^{n-1} (u_{n-i} - i). \end{aligned}$$

Since the total number of rook placements in $D(\mathbf{u})$ of size n has been denoted by $R_n(\mathbf{u})$, we thus conclude that

$$R_n(\mathbf{u}) = \prod_{i=0}^{n-1} (u_{n-i} - i).$$

This proves Proposition 0.2.

[Are you wondering where we have used the condition $u_1 \geq u_2 \geq \cdots \geq u_n$?

Answer:

Answer: Consider the algorithm above. When we placed the rook in row $n-2$, we argued that we had $u_{n-2}-2$ options, because the rook could be placed in any of the u_{n-2} columns $1,2,\ldots,u_{n-2}$ except for the two columns that contain the previous rooks. This tacitly relied on the fact that the two "forbidden" columns (i.e., the two columns that contain the previous rooks) are among the u_{n-2} columns $1,2,\ldots,u_{n-2}$ (if this was not the case, then we would have more than $u_{n-2}-2$ options for our rook). This fact is true because the first "forbidden" column is among the columns $1,2,\ldots,u_n$ and therefore also among the columns $1,2,\ldots,u_{n-2}$ (since $u_{n-2} \geq u_n$), and similarly for the second "forbidden" column.

] □

Exercise 3. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ be such that $u_1 \geq u_2 \geq \cdots \geq u_n$.

(a) For each $x \in \mathbb{N}$, define an n-tuple $\mathbf{u} + x \in \mathbb{N}^n$ by $\mathbf{u} + x = (u_1 + x, u_2 + x, \ldots, u_n + x)$. Prove that

$$R_n(\mathbf{u} + x) = \sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}}.$$

[**Hint:** When $\mathbf{u}$ is replaced by $\mathbf{u} + x$, the $\mathbf{u}$-board "grows by x extra (full) columns".]

(b) Now, let $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ be such that $v_1 \geq v_2 \geq \cdots \geq v_n$. Assume further that the n numbers $u_1 + 1, u_2 + 2, \ldots, u_n + n$ are the same as the n numbers $v_1 + 1, v_2 + 2, \ldots, v_n + n$, up to order. (In other words, there exists a permutation $\sigma \in S_n$ such that $u_i + i = v_{\sigma(i)} + \sigma(i)$ for all $i \in [n]$.) Prove that

$$R_k(\mathbf{u}) = R_k(\mathbf{v}) \qquad \text{for each } k \in \mathbb{N}.$$

[**Hint:** First, prove that $R_n(\mathbf{u} + x) = R_n(\mathbf{v} + x)$ for each $x \in \mathbb{N}$. Then, use part **(a)** and the previous exercise.]

To illustrate the usefulness of Exercise 3, let me express $R_k(2n-1, 2n-3, \ldots, 5, 3, 1)$ in a much simpler way than before:

Corollary 0.3. Let $k \in \mathbb{N}$. Then,

$$R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = \binom{n}{k}^2 k!.$$

Proof of Corollary 0.3. Define an n-tuple $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ by $(u_i = 2(n-i) + 1 \text{ for each } i \in [n])$. Thus,

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) = (2n-1, 2n-3, \ldots, 5, 3, 1),$$

so that $u_1 \geq u_2 \geq \cdots \geq u_n$.

Define an n-tuple $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ by $(v_i = n \text{ for each } i \in [n])$. Thus,

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = (n, n, \ldots, n),$$

so that $v_1 \geq v_2 \geq \cdots \geq v_n$.

The n numbers $u_1+1, u_2+2, \ldots, u_n+n$ are the same as the n numbers $v_1+1, v_2+2, \ldots, v_n+n$, up to order. (In fact, the former n numbers are $2n, 2n-1, 2n-2, \ldots, n+2, n+1$, whereas the latter n numbers are $n+1, n+2, \ldots, 2n-2, 2n-1, 2n$; these are just two different ways to list all numbers from $n+1$ to $2n$.)

Hence, Exercise 3 **(b)** yields $R_k(\mathbf{u}) = R_k(\mathbf{v})$. In view of $\mathbf{u} = (2n-1, 2n-3, \ldots, 5, 3, 1)$ and $\mathbf{v} = (n, n, \ldots, n)$, this rewrites as $R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = R_k(n, n, \ldots, n)$.

But (1) (applied to $u = n$) yields $R_k(n, n, \ldots, n) = \binom{n}{k}\binom{n}{k}k!$. Hence,

$$R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = R_k(n, n, \ldots, n) = \binom{n}{k}\binom{n}{k}k! = \binom{n}{k}^2 k!.$$

This proves Corollary 0.3. □

Rook theory is the study of rook placements – not only in boards of the form $D(\mathbf{u})$, but also in more general subsets of $\mathbb{Z}^2$. For example, the permutations of $[n]$ can be regarded as rook placements in the board $[n] \times [n]$, whereas the derangements of $[n]$ can be regarded as rook placements in the board $\{(i,j) \in [n] \times [n] \mid i \neq j\}$ (the "$n \times n$-chessboard without the main diagonal"). The theory has been developed in the 70s at the UMN (by Jay Goldman, J. T. Joichi, Victor Reiner and Dennis White).

0.3. The sum of the "widths" of all inversions of σ

In the following, "number" means "real number" or "complex number" or "rational number", as you prefer (this doesn't make a difference in these exercises).

Exercise 4. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. Let $a_1, a_2, \ldots, a_n$ be any n numbers. Prove that

$$\sum_{\substack{1 \leq i < j \leq n; \\ \sigma(i) > \sigma(j)}} (a_j - a_i) = \sum_{i=1}^{n} a_i (i - \sigma(i)).$$

[Here, the symbol "$\sum_{\substack{1 \leq i < j \leq n; \\ \sigma(i) > \sigma(j)}}$" means "sum over all pairs $(i,j) \in [n]^2$ satisfying $i < j$ and $\sigma(i) > \sigma(j)$", that is, "sum over all inversions of σ".]

0.4. A hollowed-out determinant

In the following, matrices are understood to be matrices whose entries are numbers (see above).

Exercise 5. Let $n \in \mathbb{N}$. Let P and Q be two subsets of $[n]$ such that $|P| + |Q| > n$. Let $A = (a_{i,j})_{1 \leq i \leq n,\ 1 \leq j \leq n}$ be an $n \times n$-matrix such that

$$\text{every } i \in P \text{ and } j \in Q \text{ satisfy } a_{i,j} = 0.$$

Then, prove that $\det A = 0$.

Example 0.4. Applying Exercise 5 to $n = 5$, $P = \{1,3,5\}$ and $Q = \{2,3,4\}$, we see that

$$\det \begin{pmatrix} a_1 & 0 & 0 & 0 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & 0 & 0 & 0 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & 0 & 0 & 0 & e_5 \end{pmatrix} = 0 \qquad \text{for any numbers } a_1, a_5, \ldots, e_5.$$

0.5. Arrowhead matrices

Exercise 6. Let n be a positive integer.

(a) A permutation $\sigma \in S_n$ will be called *arrowheaded* if each $i \in [n-1]$ satisfies $\sigma(i) = i$ or $\sigma(i) = n$.

[For example, the permutation in S_5 whose one-line notation is $[1,5,3,4,2]$ is arrowheaded.]

Describe all arrowheaded permutations $\sigma \in S_n$ and find their number.

(b) Given n numbers $a_1, a_2, \ldots, a_n$ as well as $n-1$ numbers $b_1, b_2, \ldots, b_{n-1}$ and $n-1$ further numbers $c_1, c_2, \ldots, c_{n-1}$. Let A be the $n \times n$-matrix

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & c_1 \\ 0 & a_2 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & c_{n-1} \\ b_1 & b_2 & \cdots & b_{n-1} & a_n \end{pmatrix}.$$

(This is the matrix whose (i,j)-th entry is $\begin{cases} a_i, & \text{if } i = j; \\ b_j, & \text{if } i = n \text{ and } j \neq n; \\ c_i, & \text{if } i \neq n \text{ and } j = n; \\ 0, & \text{if } i \neq n \text{ and } j \neq n \text{ and } i \neq j \end{cases}$ for all $i \in [n]$ and $j \in [n]$.) Prove that

$$\det A = a_1 a_2 \cdots a_n - \sum_{i=1}^{n-1} b_i c_i \left(\prod_{\substack{j \in [n-1]; \\ j \neq i}} a_j \right).$$

The matrix A is called a "reverse arrowhead matrix" due to the shape that its nonzero entries form. ("Reverse" because the usual arrowhead matrix has its arrow pointing northwest rather than southeast.)