University of Minnesota, School of Mathematics

Math 4707: Combinatorics, Spring 2018 Homework 3

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Exercise 1

Exercise 0.1. Let $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

Proof. This will be shown by induction on n. For each $n \in \mathbb{N}$, let $\mathcal{A}(n)$ represent the statement "$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$". We will first show that $\mathcal{A}(0)$ is true.

Observe that if $n = 0$, then $\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}$ is the empty sum, which takes on a value of zero, and $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ is also the empty sum, which also takes on a value of zero. Hence, we have equality, so $\mathcal{A}(0)$ is true.

Now we will show that if $\mathcal{A}(n)$ is true for some $n \in \mathbb{N}$, then it follows that $\mathcal{A}(n+1)$ is true. Fix $n \in \mathbb{N}$ and suppose that $\mathcal{A}(n)$ holds. Then, by the induction hypothesis, it holds that $\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.

An identity proven in class says that $\sum_{k=0}^{m} (-1)^{k-1} \binom{m}{k} = [m = 0]$ for all $m \in \mathbb{N}$. Applied to $m = n+1$, this yields $\sum_{k=0}^{n+1} (-1)^{k-1} \binom{n+1}{k} = [n+1 = 0] = 0$.

First, observe that any positive integer k satisfies

$$\binom{n}{k-1} = \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} = \frac{k}{n+1}\frac{(n+1)(n)(n-1)\cdots(n-k+2)}{k!}$$
$$= \frac{k}{n+1}\binom{n+1}{k}. \qquad (1)$$

But the recurrence relation of the binomial coefficients leads to

$$\begin{aligned}
&\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n+1}{k} \\
&= \sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\left(\binom{n}{k}+\binom{n}{k-1}\right) \\
&= \sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n}{k} + \sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n}{k-1} \\
&= \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k} + \frac{(-1)^{n}}{n+1}\binom{n}{n+1} + \sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\frac{k}{n+1}\binom{n+1}{k} \\
&\qquad \left(\begin{array}{c}\text{here, we split off the addend for } k=n+1 \text{ from the first sum,}\\ \text{and rewrote the second sum using (1)}\end{array}\right) \\
&= \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+0+\frac{1}{n+1}\sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k} \\
&\qquad \left(\text{using the induction hypothesis and using } \binom{n}{n+1}=0\right) \\
&= \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\left(\underbrace{\sum_{k=0}^{n+1}(-1)^{k-1}\binom{n+1}{k}}_{=0} - \underbrace{(-1)^{0-1}\binom{n+1}{0}}_{=-1}\right) \\
&= \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}(0-(-1)) \\
&= \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1} = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n+1}.
\end{aligned}$$

This means that $\mathcal{A}(n+1)$ holds.

So we know that $\mathcal{A}(0)$ holds, and if $n \in \mathbb{N}$, and $\mathcal{A}(n)$ holds, then it follows that $\mathcal{A}(n+1)$ holds. Hence, by induction, $\mathcal{A}(n)$ holds for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$, it follows that

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k} = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}.$$

□

Exercise 3

Part A

Exercise 0.2. Let $n \in \mathbb{N}$. Then the number of 2-multijective maps from $[2n]$ to $[n]$ is $\dfrac{(2n)!}{2^n}$.

Proof. Let r represent the number of 2-multijective maps from $[2n]$ to $[n]$. We will first count the number of permutations σ of $[2n]$. On the one hand, there are $(2n)!$ of them. We now go about counting these another way.

To construct a permutation σ of $[2n]$, proceed as follows: First, choose a 2-multijective map m from $[2n]$ to $[n]$. There are r ways to do this. Now, given this m, we define, for each $i \in [n]$, the two values $\sigma(2i-1)$ and $\sigma(2i)$ to be the two preimages of i under m in some order. There are 2^n ways to do this (because for each of the n elements $i \in [n]$, we get to choose the order of the two preimages of i, and there are 2 choices for this order). Thus, in total, the procedure can be performed in $r \cdot 2^n$ many ways. Since it constructs every permutation σ of $[2n]$ exactly once (indeed, the 2-multijection m that was used to construct σ can be reconstructed from σ, because it is the unique map $[2n] \to [n]$ sending both $\sigma(2i-1)$ and $\sigma(2i)$ to i for all $i \in [n]$), we thus see that the number of permutations σ of $[2n]$ is $r \cdot 2^n$.

Comparing our two expressions for this number, we conclude that $r \cdot 2^n = (2n)!$. So $r = \dfrac{(2n)!}{2^n}$. And thus, the number of 2-multijective maps from $[2n]$ to $[n]$ is given by $\dfrac{(2n)!}{2^n}$. □

Part B

Exercise 0.3. Let $n \in \mathbb{N}$. The number of all set compositions $\mathbf{C}$ of $[2n]$ such that each part of $\mathbf{C}$ has size 2 is $\dfrac{(2n)!}{2^n}$.

Proof. Let J be the set of 2-multijective maps from $[2n]$ to $[n]$, and C be the set of all set compositions $\mathbf{C}$ of $[2n]$ such that every part of $\mathbf{C}$ has exactly two elements. We will construct a bijection between these two sets.

First, define a map $\alpha : J \to C$ such that if $j \in J$, then α maps j to a set composition c of $[2n]$ into n parts such that, for each $i \in [n]$, the i-th part of c is the set of the two preimages of i under j. Clearly, this map is well defined, as the number of elements in the image of each $j \in J$ is equal to n, which is also equal to the number of parts of each $c \in C$. And further, each $j \in J$ is a 2-multijection, so that each element of $[n]$ will have exactly 2 preimages under j, which is also equal to the size of each part of each $c \in C$.

Now let $\beta : C \to J$ be defined such that, if $c \in C$, then β maps c to a 2-multijection $j \in J$ such that, for each $i \in [n]$, the two elements of the i-th part of c make up the preimage of i under j. Clearly, this map is well defined, as each $c \in C$ will have n parts, and also each $j \in J$ will have exactly n image elements. And also, each part of each $c \in C$ will have 2 elements, which is the number of preimages of each $i \in [n]$ under each $j \in J$.

We will now show that α and β are mutually inverse. First, let $j \in J$. We will show that $(\beta \circ \alpha)(j) = j$. By definition, α maps j to a set composition c of $[2n]$ such that, for each $i \in [n]$, the i-th part of c comprises the two preimages of i under j. Then β maps this new set composition to a 2-multijective map such that, for each $i \in [n]$, the preimages of i are the elements of the i-th part of c, which are precisely the two preimages of i under j. Hence, this map is precisely j, and $(\beta \circ \alpha)(j) = j$. The proof that, for each $c \in C$, $(\alpha \circ \beta)(c) = c$ is very similar.

Hence, α and β are two functions which are well defined and mutually inverse, and thus they define a bijection between C and J, which means that $|C| = |J|$. In part **(a)**, we showed that $|J| = \dfrac{(2n)!}{2^n}$. So then $|C| = \dfrac{(2n)!}{2^n}$. And thus, the number of set compositions $\mathbf{C}$ of $[2n]$ such that each part of $\mathbf{C}$ has size 2 is equal to $\dfrac{(2n)!}{2^n}$. □

Part C

Exercise 0.4. Let $n \in \mathbb{N}$. The number of perfect matchings of $[2n]$ is equal to $\dfrac{(2n)!}{2^n n!}$.

Proof. Let M be the number of perfect matchings of $[2n]$. We will count the number of all set compositions $\mathbf{C}$ of $[2n]$ such that each part of $\mathbf{C}$ has 2 elements. In part **(b)**, we showed that there are $\dfrac{(2n!)}{2^n}$ such set compositions.

Alternatively, we can construct such a set composition $\mathbf{C}$ of $[2n]$ by first choosing which elements will be together in some part of $\mathbf{C}$, without yet choosing in which part of $\mathbf{C}$ they will be in. In other words, we first choose the **set** of all parts of $\mathbf{C}$, but we don't yet decide which of them will be the first part, which the second, and so on. This is the same as choosing a perfect matching of $[2n]$, of which there are M ways to do. We next select which part of this matching will represent which part of $\mathbf{C}$ by choosing a (total) order on the parts of the matching. Since $|[2n]| = 2n$, our perfect matching of $[2n]$ must have exactly n parts, and hence, there are $n!$ different orderings which these parts can take. Clearly, these steps will create a unique set composition $\mathbf{C}$ of $[2n]$ such that each part of $\mathbf{C}$ has 2 elements. And because there are M ways of selecting the perfect matching of $[2n]$, and $n!$ different ways to order the parts of this perfect matching, there are $Mn!$ of such set compositions.

Hence, it follows that $Mn! = \dfrac{(2n)!}{2^n}$, so $M = \dfrac{(2n)!}{2^n n!}$. Thus, the number of prefect matchings of $[2n]$ is equal to $\dfrac{(2n)!}{2^n n!}$. □

Exercise 5

Exercise 0.5. Let $n \in \mathbb{N}$ and $d \in \mathbb{N}$ with $d \geq 1$. Then the number of 1-even n-tuples in $[d]^n$ is equal to $\dfrac{1}{2}(d^n + (d-2)^n)$.

Proof. Forget that n is fixed. (But d will be fixed.)

For each $n \in \mathbb{N}$, let t_n represent the number of 1-even n-tuples in $[d]^n$.

We will first find a recursive formula for t_n. Fix a positive integer n. We would like to count the number of 1-even n-tuples in $[d]^n$. Let s be such a tuple. Then we consider two possible cases for the final entry of s.

Case 1: The final entry of s is not equal to 1. If s is 1-even, and the last entry of s is not equal to 1, then there must be an even number of entries which are equal to 1 among the first $n-1$ entries of s. This means that, if the first $n-1$ entries are considered as an $(n-1)$-tuple, that tuple would be 1-even. Hence, s is formed by concatenating an element of $[d]$ which is not equal to 1 onto the end of a 1-even $(n-1)$-tuple in $[d]^{n-1}$. Hence, there are $(d-1)t_{n-1}$ ways of forming a 1-even n-tuple whose last entry is not equal to 1.

Case 2: The final entry of s is equal to 1. If s is 1-even, and the last entry of s is equal to 1, then there must be an odd number of entries which are equal to 1 among the first $n-1$ entries of s. This means that, if the first $n-1$ entries are considered as an $(n-1)$-tuple, that tuple would not be 1-even. Hence, s is formed by concatenating 1 onto the end of a non-1-even $(n-1)$-tuple in $[d]^{n-1}$. Since a tuple is either 1-even, or not 1-even, and there are d^{n-1} elements of $[d]^{n-1}$, we can conclude that the number of $(n-1)$-tuples which are not 1-even is equal to $d^{n-1} - t_{n-1}$. Hence, there are $d^{n-1} - t_{n-1}$ ways of forming a 1-even n-tuple whose last entry is equal to 1.

Since a 1-even n-tuple must fall either into case 1, or into case 2, but certainly not both, it follows that

$$t_n = (d-1)t_{n-1} + (d^{n-1} - t_{n-1}) = (d-2)t_{n-1} + d^{n-1}. \tag{2}$$

Now, the claim of the exercise will be proven by induction on n. Unfix n. For each $n \in \mathbb{N}$, let $\mathcal{A}(n)$ be the statement "the number of 1-even tuples of $[d]^n$ is equal to $\frac{1}{2}(d^n + (d-2)^n)$". Observe that if $n = 0$, there is only one element of $[d]^n$, which is the empty tuple. This tuple has no elements, so zero of the entries of this tuple are equal to 1. Hence, this tuple is 1-even, so there is one 1-even tuple of $[d]^n$ (for $n = 0$). And observe that $\frac{1}{2}(d^0 + (d-2)^0) = \frac{1}{2}(1+1) = 1$. Hence, $\mathcal{A}(0)$ holds.

Now suppose that, for some $n \in \mathbb{N}$, we knew that $\mathcal{A}(n)$ is true. We will show that this fact would imply that $\mathcal{A}(n+1)$ is true. By the induction hypothesis, we know that $t_n = \frac{1}{2}(d^n + (d-2)^n)$. Therefore, (2) (applied to $n+1$ instead of n) yields

$$\begin{aligned}
t_{n+1} &= (d-2)t_n + d^n \\
&= (d-2)\frac{1}{2}(d^n + (d-2)^n) + d^n \\
&= \frac{1}{2}d^{n+1} - d^n + \frac{1}{2}(d-2)^{n+1} + d^n \\
&= \frac{1}{2}d^{n+1} + \frac{1}{2}(d-2)^{n+1} \\
&= \frac{1}{2}(d^{n+1} + (d-2)^{n+1}).
\end{aligned}$$

This means that there are $\frac{1}{2}(d^{n+1} + (d-2)^{n+1})$ 1-even $(n+1)$-tuples in $[d]^{n+1}$, so $\mathcal{A}(n+1)$ holds.

We now know that $\mathcal{A}(0)$ holds, and for each $n \in \mathbb{N}$, $\mathcal{A}(n)$ being true would imply that $\mathcal{A}(n+1)$ is true. Hence, by induction, $\mathcal{A}(n)$ is true for all $n \in \mathbb{N}$. In other words, the number of 1-even n-tuples in $[d]^n$ is equal to $\frac{1}{2}(d^n + (d-2)^n)$. □