

Math 4707 Spring 2018 (Darij Grinberg): homework set 3 (corrected version 20 Mar 2018)

due date: Wednesday 21 March 2018 at the beginning of class, or before that by email or moodle

Please solve **at most 3** of the 6 exercises!

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Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

Recall the following:

- If $n \in \mathbb{N}$, then $[n]$ denotes the n -element set $\{1, 2, \dots, n\}$.
- We use the Iverson bracket notation.
- If $a \in \mathbb{N}$ and $b \in \mathbb{N}$, then $\text{sur}(a, b)$ denotes the number of surjective maps from $[a]$ to $[b]$.

0.1. Another binomial identity

Exercise 1. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n}{k} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}.$$

[Hint: How does the left hand side grow when n is replaced by $n + 1$?]

0.2. More on inclusion/exclusion

Exercise 2. Let A , B and C be three finite sets such that $C \subseteq B$. Let $a = |A|$, $b = |B|$ and $c = |C|$.

(a) Prove that the number of maps $f : A \rightarrow B$ satisfying $C \subseteq f(A)$ is

$$\sum_{k=0}^c (-1)^k \binom{c}{k} (b - k)^a.$$

(b) Prove that the number of surjective maps $f : A \rightarrow B$ is

$$\text{sur}(a, b) = \sum_{k=0}^b (-1)^k \binom{b}{k} (b-k)^a = \sum_{k=0}^b (-1)^{b-k} \binom{b}{k} k^a.$$

[This is a formula I mentioned but did not prove in class; of course, you cannot use it without proof.]

(c) Prove that

$$\sum_{k=0}^c (-1)^k \binom{c}{k} (b-k)^a = 0$$

whenever $c > a$.

(d) Prove that

$$\sum_{k=0}^a (-1)^k \binom{a}{k} (b-k)^a = a!.$$

[**Hint:** For part (a), notice that a map $f : A \rightarrow B$ satisfies $C \subseteq f(A)$ if and only if the image of f misses none of the c elements of C . Parts (b), (c) and (d) should follow from (a).]

0.3. Multijections, set compositions and set partitions

Let us introduce set partitions and set compositions. These are important concepts in combinatorics, and we will see more of them.

Definition 0.1. Let X be a finite set.

(a) A *set composition* of X means a tuple (S_1, S_2, \dots, S_k) of disjoint nonempty subsets of X such that $X = S_1 \cup S_2 \cup \dots \cup S_k$.

(b) A *set partition* of X means a set $\{S_1, S_2, \dots, S_k\}$ of disjoint nonempty subsets of X (written in such a way that S_1, S_2, \dots, S_k are distinct) such that $X = S_1 \cup S_2 \cup \dots \cup S_k$.

(c) The *parts* of a set composition (S_1, S_2, \dots, S_k) are the sets S_1, S_2, \dots, S_k .

(d) The *parts* of a set partition $\{S_1, S_2, \dots, S_k\}$ are the sets S_1, S_2, \dots, S_k .

Example 0.2. For this example, let $X = \{1, 2, 3\}$.

(a) The 2-tuple $(\{1, 3\}, \{2\})$ is a set composition of X , since $\{1, 3\}$ and $\{2\}$ are disjoint nonempty subsets of X satisfying $X = \{1, 3\} \cup \{2\}$.

(b) The 2-tuple $(\{1, 3\}, \{2, 3\})$ is **not** a set composition of X , since $\{1, 3\}$ and $\{2, 3\}$ are not disjoint.

(c) The 3-tuple $(\{1, 3\}, \{\}, \{2\})$ is **not** a set composition of X , since $\{\}$ is not nonempty.

(d) The 2-tuple $(\{1\}, \{3\})$ is **not** a set composition of X , since $X \neq \{1\} \cup \{3\}$.

(e) The set $\{\{1, 3\}, \{2\}\}$ is a set partition of X , since $\{1, 3\}$ and $\{2\}$ are disjoint nonempty subsets of X satisfying $X = \{1, 3\} \cup \{2\}$.

(f) If (S_1, S_2, \dots, S_k) is a set composition of X , then $\{S_1, S_2, \dots, S_k\}$ is a set partition of X . The converse also holds if you assume S_1, S_2, \dots, S_k to be distinct.

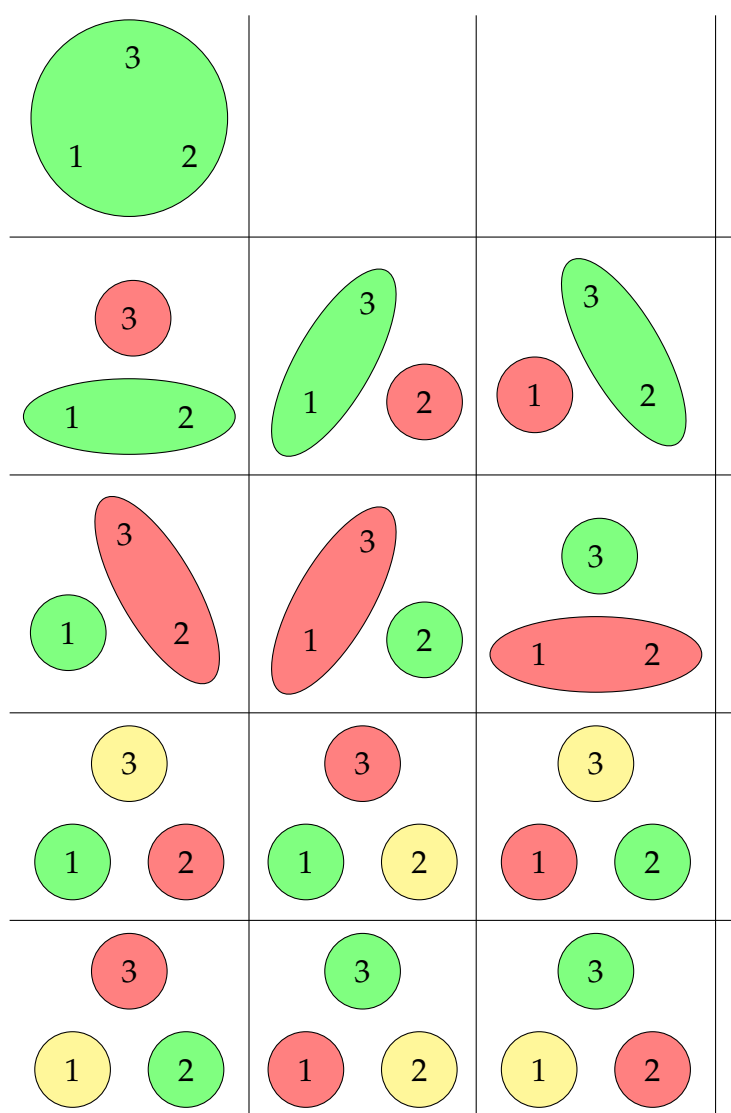
(g) The set **compositions** $(\{1,3\}, \{2\})$ and $(\{2\}, \{1,3\})$ of X are distinct, but the set **partitions** $\{\{1,3\}, \{2\}\}$ and $\{\{2\}, \{1,3\}\}$ are identical.

This illustrates the difference between set compositions and set partitions: The former come with an ordering of their parts, while the latter don't. This is why set compositions are often called *ordered set partitions*.

Example 0.3. (a) Here are all set compositions of the set $X = \{1,2,3\}$:

$(\{1,2,3\}),$
 $(\{1,2\}, \{3\}),$ $(\{1,3\}, \{2\}),$ $(\{2,3\}, \{1\}),$
 $(\{1\}, \{2,3\}),$ $(\{2\}, \{1,3\}),$ $(\{3\}, \{1,2\}),$
 $(\{1\}, \{2\}, \{3\}),$ $(\{1\}, \{3\}, \{2\}),$ $(\{2\}, \{1\}, \{3\}),$
 $(\{2\}, \{3\}, \{1\}),$ $(\{3\}, \{1\}, \{2\}),$ $(\{3\}, \{2\}, \{1\}).$

And here are the same set compositions, drawn as pictures (each part of the set composition corresponds to a colored blob):

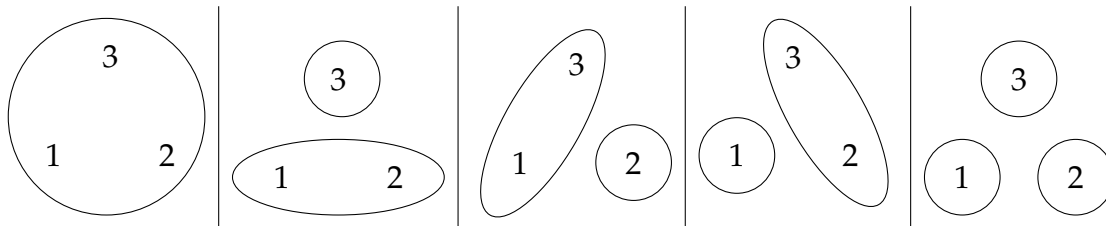


Here, the colors have been chosen as follows: The green-colored blob is the first part of the set composition; the red-colored blob is the second; the yellow-colored blob is the third.

(b) Here are all set partitions of the set $X = \{1, 2, 3\}$:

$$\{\{1, 2, 3\}\}, \quad \{\{1, 2\}, \{3\}\}, \quad \{\{1, 3\}, \{2\}\}, \quad \{\{2, 3\}, \{1\}\}, \\ \{\{1\}, \{2\}, \{3\}\}.$$

And here are the same set partitions, drawn as pictures (each part of the set partition corresponds to a blob):



Definition 0.4. Let X be a set. Let $k \in \mathbb{N}$.

(a) A *set composition of X into k parts* is a set composition of X having exactly k parts.

(b) A *set partition of X into k parts* is a set partition of X having exactly k parts.

For example, $(\{1, 5\}, \{2\}, \{3, 4, 6, 7\})$ is a set composition of $[7]$ into 3 parts.

There are several natural things to be counted now:

- set compositions of a given set X ;
- set partitions of a given set X ;
- set compositions of a given set X into k parts for a given $k \in \mathbb{N}$;
- set partitions of a given set X into k parts for a given $k \in \mathbb{N}$.

Let's only briefly comment on the first two questions, and then discuss the last two.

- If X is an n -element set, then the number of set compositions of X is the n -th *ordered Bell number* $\tilde{B}(n)$. Here is a list of the first values of $\tilde{B}(n)$ (see A000670 at OEIS for more):

$$\begin{aligned} \tilde{B}(0) &= 1, & \tilde{B}(1) &= 1, & \tilde{B}(2) &= 3, & \tilde{B}(3) &= 13, \\ \tilde{B}(4) &= 75, & \tilde{B}(5) &= 541, & \tilde{B}(6) &= 4683, & \tilde{B}(7) &= 47293. \end{aligned}$$

No explicit formulas for $\tilde{B}(n)$ are known. A quick way to compute $\tilde{B}(n)$ for arbitrary $n \in \mathbb{N}$ is using the recursive equation

$$\tilde{B}(n) = \sum_{i=0}^{n-1} \binom{n}{i} \tilde{B}(i) \quad \text{for all } n > 0.$$

(The proof is easy: Classify all set compositions of $[n]$ according to the size of their last part, and treat all the remaining parts as a set composition of a smaller set.)

- If X is an n -element set, then the number of set partitions of X is the n -th Bell number $B(n)$. Here is a list of the first values of $B(n)$ (see A000110 at OEIS for more):

$$\begin{array}{llll} B(0) = 1, & B(1) = 1, & B(2) = 2, & B(3) = 5, \\ B(4) = 15, & B(5) = 52, & B(6) = 203, & B(7) = 877. \end{array}$$

No explicit formulas for $B(n)$ are known. A quick way to compute $B(n)$ for arbitrary $n \in \mathbb{N}$ is using the recursive equation

$$B(n+1) = \sum_{i=0}^n \binom{n}{i} B(i) \quad \text{for all } n \in \mathbb{N}.$$

(The proof is easy: Classify all set partitions of $[n+1]$ according to how many elements lie in the same part as $n+1$, and treat all the remaining parts as a set partition of a smaller set.)

Now, let us study the other two questions.

Definition 0.5. Let X be a set. Let $k \in \mathbb{N}$.

- (a) We let $\text{SC}_k(X)$ denote the set of all set compositions of X into k parts.
- (b) We let $\text{SP}_k(X)$ denote the set of all set partitions of X into k parts.

Proposition 0.6. Let X be a finite set. Let $k \in \mathbb{N}$. Then, $|\text{SC}_k(X)| = \text{sur}(|X|, k)$.

Proof of Proposition 0.6 (sketched). For any sets A and B , we let $\text{Sur}(A, B)$ be the set of all surjections from A to B . If A and B are finite sets, then

$$|\text{Sur}(A, B)| = (\text{the number of surjections from } A \text{ to } B) = \text{sur}(|A|, |B|)$$

(as we have shown in class). Applying this to $A = X$ and $B = [k]$, we obtain

$$|\text{Sur}(X, [k])| = \text{sur}\left(|X|, \underbrace{|[k]|}_{=k}\right) = \text{sur}(|X|, k).$$

Here is an outline of the remainder of the proof: We want to find $|\text{SC}_k(X)|$; that is, we want to count all set compositions (S_1, S_2, \dots, S_k) of X into k parts. We can construct such a set composition by choosing, for each $x \in X$, which part S_i it shall belong to¹. This information can be encoded as a map $f : X \rightarrow [k]$ (which sends each $x \in X$ to the $i \in [k]$ satisfying $x \in S_i$); this map f has to be surjective (since the parts S_i should be nonempty, so each S_i must have at least one $x \in X$ in it), but otherwise is subject to no constraints. Hence, the number of set compositions (S_1, S_2, \dots, S_k) of X into k parts equals the number of surjective maps $X \rightarrow [k]$; in other words, it equals $|\text{Sur}(X, [k])| = \text{sur}(|X|, k)$. This proves Proposition 0.6.

Here is a more rigorous way to make this argument (without vague terms such as “information” and “encoded”). We are going to construct a bijection $\text{SC}_k(X) \rightarrow \text{Sur}(X, [k])$ (that is, a bijection between the set compositions of X into k parts and the surjections from X to $[k]$). Here is how:

¹Each $x \in X$ must belong to **exactly one** part of (S_1, S_2, \dots, S_k) (because (S_1, S_2, \dots, S_k) is a set composition of X).

- We define a map $\Phi : \text{SC}_k(X) \rightarrow \text{Sur}(X, [k])$ to be the map that sends any set composition $(S_1, S_2, \dots, S_k) \in \text{SC}_k(X)$ to the map

$$f : X \rightarrow [k], \\ x \mapsto (\text{the unique } i \in [k] \text{ such that } x \in S_i).$$

It is easy to see that Φ is well-defined (indeed, for any set composition (S_1, S_2, \dots, S_k) , the resulting map $f : X \rightarrow [k]$ is surjective, because the sets S_1, S_2, \dots, S_k are nonempty).

- We define a map $\Psi : \text{Sur}(X, [k]) \rightarrow \text{SC}_k(X)$ to be the map that sends any surjective map $f : X \rightarrow [k]$ to the set composition $(S_1, S_2, \dots, S_k) \in \text{SC}_k(X)$, where

$$S_i = f^{-1}(\{i\}) = \{x \in X \mid f(x) = i\} \quad \text{for each } i \in [k].$$

Again, it is easily shown that this map Ψ is well-defined.

It is easy to check that the maps Φ and Ψ are mutually inverse, and thus are bijections. Hence, we have found a bijection $\text{SC}_k(X) \rightarrow \text{Sur}(X, [k])$. This lets us conclude that

$$|\text{SC}_k(X)| = |\text{Sur}(X, [k])| = \text{sur}(|X|, k).$$

Thus, Proposition 0.6 is proven. □

Next, let us count set partitions of a given finite set X into a given number of parts:

Proposition 0.7. Let X be a finite set. Let $k \in \mathbb{N}$. Then, $|\text{SP}_k(X)| = \frac{\text{sur}(|X|, k)}{k!}$.

Let us give an informal proof of this proposition. A formalization of it (omitting only really straightforward details) is given in the Appendix below.

Outline of an informal proof of Proposition 0.7. Let us count the set compositions (S_1, S_2, \dots, S_k) of X into k parts in two different ways:

- On the one hand, this number is $|\text{SC}_k(X)| = \text{sur}(|X|, k)$ (by Proposition 0.6).
- On the other hand, we can construct a set composition (S_1, S_2, \dots, S_k) of X into k parts by first choosing a set partition of X into k parts (there are $|\text{SP}_k(X)|$ ways to do this), and then choosing how to order it² (there are $k!$ ways to do this, since it has k distinct parts). Thus, the number of set compositions (S_1, S_2, \dots, S_k) of X into k parts equals $|\text{SP}_k(X)| \cdot k!$.

Comparing these two results, we conclude that $\text{sur}(|X|, k) = |\text{SP}_k(X)| \cdot k!$. Thus, $\frac{\text{sur}(|X|, k)}{k!} = |\text{SP}_k(X)|$. This proves Proposition 0.7. □

²i.e., in what order to list its parts

The above proof outline glances over technicalities such as what “ordering” a set partition means, and why there are $k!$ ways to do this; this is the reason why I am calling it informal and give a more complete version in the Appendix below.

The method we used to prove Proposition 0.7 is another example of the “shepherd’s principle” (“To count sheep, you count the legs and then you divide by 4”). Our sheep here were the set partitions in $\text{SP}_k(X)$, whereas their legs were the set compositions in $\text{SC}_k(X)$. A leg $(S_1, S_2, \dots, S_k) \in \text{SC}_k(X)$ belongs to the sheep $\{S_1, S_2, \dots, S_k\} \in \text{SP}_k(X)$. This principle (also known as “proof by multijection”) is useful whenever the legs are easier to count than the sheep. (For example, in the above case, the legs were in bijection with the surjections $X \rightarrow [k]$, whereas the sheep were not in a visible bijection with anything.)

Definition 0.8. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. The (n, k) -th Stirling number of the 2nd kind is defined to be $\frac{\text{sur}(n, k)}{k!}$; it is commonly denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ or by $S(n, k)$.

Proposition 0.7 shows that $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is the number of set partitions of a given n -element set into k parts. Thus, in particular, $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is a nonnegative integer.

The recurrence relation $\text{sur}(n, k) = k \cdot (\text{sur}(n-1, k) + \text{sur}(n-1, k-1))$ for arbitrary $n > 0$ and $k > 0$ (see, for example, Proposition 3.12 in classwork from 21 February 2018) now leads to the following recurrence relations for Stirling numbers of the 2nd kind:

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}.$$

Meanwhile, Exercise 2 (b) leads to

$$\left\{ \begin{smallmatrix} a \\ b \end{smallmatrix} \right\} = \frac{1}{b!} \sum_{k=0}^b (-1)^{b-k} \binom{b}{k} k^a.$$

Without knowing about set partitions, would you have guessed that the right hand side is a nonnegative integer?

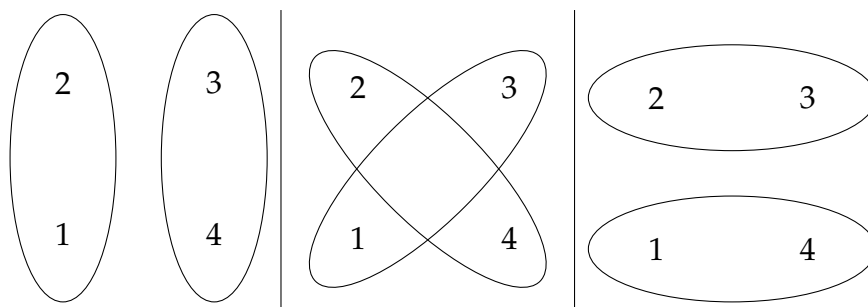
Now, I want you to count a special kind of set partitions:

Definition 0.9. A *perfect matching* of a set X means a set partition \mathbf{P} of X such that each part of \mathbf{P} has size 2.

Example 0.10. (a) The perfect matchings of the set $[4]$ are

$$\{\{1, 2\}, \{3, 4\}\}, \quad \{\{1, 3\}, \{2, 4\}\}, \quad \{\{1, 4\}, \{2, 3\}\}.$$

Here are the same 3 perfect matchings, drawn as blobs:



(b) The set $[6]$ has 15 perfect matchings; three of them are

$$\{\{1,2\}, \{3,4\}, \{5,6\}\}, \quad \{\{1,4\}, \{2,6\}, \{3,5\}\}, \quad \{\{1,6\}, \{2,5\}, \{3,4\}\}.$$

Clearly, a perfect matching of a finite set X must have precisely $|X|/2$ parts (since $|X|$ is the sum of the sizes of all parts, but these sizes all equal 2). Thus, a perfect matching of a finite set X can only exist when $|X|$ is even.

Definition 0.11. Let A and B be two sets. Let $j \in \mathbb{N}$. A map $f : A \rightarrow B$ will be called *j-multijjective* if for each $b \in B$, we have

$$(\text{the number of } a \in A \text{ satisfying } f(a) = b) = j.$$

In the classical picture illustrating a map, a map $f : A \rightarrow B$ is *j-multijjective* if and only if each element of B is hit by exactly j arrows.

For example, a 1-multijjective map is the same as a bijective map (make sure you understand why).

For another example, Proposition 0.13 below says that the map π in that proposition is $k!$ -multijjective.

Exercise 3. Let $n \in \mathbb{N}$. Prove the following:

(a) The number of all 2-multijjective maps from $[2n]$ to $[n]$ is $\frac{(2n)!}{2^n}$.

(b) The number of all set compositions \mathbf{C} of $[2n]$ such that each part of \mathbf{C} has size 2 is $\frac{(2n)!}{2^n}$.

(c) The number of all perfect matchings of $[2n]$ is $\frac{(2n)!}{2^n n!}$.

[Hint: You don't need to imitate the level of detail that is given in the Appendix.]

0.4. "Image-injective maps"

If S is a set, then a map $f : S \rightarrow S$ is said to be *image-injective*³ if and only if its restriction $f|_{f(S)}$ is injective. For example:

- The map $[4] \rightarrow [4]$ sending 1, 2, 3, 4 to 4, 1, 4, 1 (respectively) is image-injective (since its image is $\{1, 4\}$, and its restriction to $\{1, 4\}$ is injective).
- The map $[6] \rightarrow [6]$ sending 1, 2, 3, 4, 5, 6 to 2, 4, 4, 6, 6, 2 (respectively) is image-injective (since its image is $\{2, 4, 6\}$, and its restriction to $\{2, 4, 6\}$ is injective).

³This is my terminology; don't expect to see it in the literature.

- Any injective map $f : S \rightarrow S$ is image-injective. So is any constant map (i.e., any map $f : S \rightarrow S$ such that all values of f are equal).
- The map $[3] \rightarrow [3]$ sending $1, 2, 3$ to $2, 2, 1$ (respectively) is not image-injective (since its restriction to its image $\{1, 2\}$ is not injective).

As usual, if S is a set, and $f : S \rightarrow S$ is a map, then f^2 means the map $f \circ f : S \rightarrow S$. (More generally, f^k means the map $\underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$ whenever $k \in \mathbb{N}$.)

Exercise 4. Let $n \in \mathbb{N}$.

(a) Prove that a map $f : [n] \rightarrow [n]$ is image-injective if and only if it has the following property: For any $a \in [n]$ and $b \in [n]$ satisfying $f^2(a) = f^2(b)$, we must have $f(a) = f(b)$.

(b) Prove that the number of image-injective maps $[n] \rightarrow [n]$ is $\sum_{k=0}^n \binom{n}{k} k! k^{n-k}$.

(c) Prove that the number of image-injective maps $[n] \rightarrow [n]$ is divisible by n whenever n is positive.

[Hint: This is rather similar to [Fall2017-HW6os, Exercise 1]; feel free to imitate the solution of the latter exercise.]

0.5. Counting certain tuples

Exercise 5. Let $n \in \mathbb{N}$, and let d be a positive integer.

An n -tuple $(x_1, x_2, \dots, x_n) \in [d]^n$ will be called *1-even* if the number 1 occurs in it an even number of times (i.e., the number of $i \in [n]$ satisfying $x_i = 1$ is even). (For example, the 3-tuples $(1, 5, 1)$ and $(3, 2, 6)$ are 1-even (yes, 0 is an even number), while the 3-tuple $(2, 1, 4)$ is not.)

Prove that the number of 1-even n -tuples in $[d]^n$ is $\frac{1}{2}(d^n + (d-2)^n)$.

[Hint: Set $e = d - 1$; then, $(d-2)^n = (e-1)^n$ and $d^n = (e+1)^n$. There might also be a bijective proof – after multiplying by 2 –, but I don't know it.]

0.6. And more binomial identities

Exercise 6. (a) Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Prove that every $j \in \{0, 1, \dots, n\}$ satisfies

$$\sum_{k=0}^n \binom{m+k}{k} \binom{n-k}{j} = \binom{n+m+1}{m+j+1}.$$

(b) Let x and y be two real numbers. Let $z = x + y$. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Prove that

$$x^{m+1} \sum_{k=0}^n \binom{m+k}{k} y^k z^{n-k} = \sum_{i=m+1}^{n+m+1} \binom{n+m+1}{i} x^i y^{(n+m+1)-i}$$

and

$$y^{n+1} \sum_{k=0}^m \binom{n+k}{k} x^k z^{m-k} = \sum_{i=0}^m \binom{n+m+1}{i} x^i y^{(n+m+1)-i}$$

and

$$x^{m+1} \sum_{k=0}^n \binom{m+k}{k} y^k z^{n-k} + y^{n+1} \sum_{k=0}^m \binom{n+k}{k} x^k z^{m-k} = z^{n+m+1}.$$

(c) Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^n \binom{n+k}{k} \frac{1}{2^k} = 2^n.$$

[Hint: Part (a) is a restatement of something proven in class. Derive (b) from (a), and (c) from (b).]

0.7. Appendix: Rigorous proof of Proposition 0.7

Let us now give a proof of Proposition 0.7 that keeps to the standards of rigor in most parts of mathematics. First, we need a lemma from the “isn’t this obvious?” department:

Lemma 0.12. Let $k \in \mathbb{N}$. Let x_1, x_2, \dots, x_k be k distinct objects. Let y_1, y_2, \dots, y_k be k objects such that $\{x_1, x_2, \dots, x_k\} = \{y_1, y_2, \dots, y_k\}$. Then, there exists a permutation σ of $[k]$ such that every $i \in [k]$ satisfies $y_i = x_{\sigma(i)}$.

Roughly speaking, Lemma 0.12 says that if k distinct objects x_1, x_2, \dots, x_k form the same set as k objects y_1, y_2, \dots, y_k (which are not a-priori required to be distinct, but it follows easily that they are), then the objects y_1, y_2, \dots, y_k are just the objects x_1, x_2, \dots, x_k rearranged (the rearrangement is what the permutation σ is meant to take care of). Convince yourself that this is plausible before (or instead of) reading the following proof. Notice that x_1, x_2, \dots, x_k need to be distinct in Lemma 0.12; otherwise, the lemma would be easily disproven (e.g., we have $\{2, 2, 3\} = \{2, 3, 3\}$, but there is no way to get $2, 3, 3$ by rearranging $2, 2, 3$).

Proof of Lemma 0.12 (sketched). For each $i \in [k]$, there exists some $j \in [k]$ such that $y_i = x_j$ (since $y_i \in \{y_1, y_2, \dots, y_k\} = \{x_1, x_2, \dots, x_k\}$). Moreover, this j is unique (because if j_1 and j_2 were two such j ’s, then we would have $y_i = x_{j_1}$ and $y_i = x_{j_2}$, thus $x_{j_1} = y = x_{j_2}$, thus $j_1 = j_2$ because x_1, x_2, \dots, x_k are distinct). Let us denote this j by $\sigma(i)$. Thus, we have defined a $\sigma(i) \in [k]$ for each $i \in [k]$. In other words, we have defined a map $\sigma : [k] \rightarrow [k]$. Clearly, this map has the property that every $i \in [k]$ satisfies $y_i = x_{\sigma(i)}$ (because this is how $\sigma(i)$ was defined). Thus, in order to prove Lemma 0.12, it suffices to check that this map σ is a permutation of $[k]$.

Let $h \in [k]$. Then, $x_h \in \{x_1, x_2, \dots, x_k\} = \{y_1, y_2, \dots, y_k\}$. In other words, there exists some $i \in [k]$ such that $x_h = y_i$. Consider this i . Then, $\sigma(i)$ is the unique $j \in [k]$ such that $y_i = x_j$ (by the definition of $\sigma(i)$). But this unique j must be h (since $h \in [k]$ and $y_i = x_h$). Hence, $\sigma(i) = h$.

Now, forget that we fixed h . We thus have shown that for each $h \in [k]$, there exists some $i \in [k]$ such that $\sigma(i) = h$. In other words, the map σ is surjective. Since σ is a map from $[k]$ to $[k]$, this yields that σ is bijective (by the Pigeonhole Principle for surjections). In other words, σ is a permutation of $[k]$. As we said, this completes the proof of Lemma 0.12. \square

Proposition 0.13. Let X be a set, and let $k \in \mathbb{N}$. Let $\pi : \text{SC}_k(X) \rightarrow \text{SP}_k(X)$ be the map that sends each set composition (S_1, S_2, \dots, S_k) to the set partition $\{S_1, S_2, \dots, S_k\}$. (So what the map π does is forgetting the order of the parts. In the language of Example 0.3, this means forgetting the colors of the blobs.)

Then, for each set partition $\mathbf{P} \in \text{SP}_k(X)$, we have

$$(\text{the number of all } \mathbf{C} \in \text{SC}_k(X) \text{ satisfying } \pi(\mathbf{C}) = \mathbf{P}) = k!.$$

Example 0.14. For this example, pick $X = \{1, 2, 3\}$ and $k = 2$ and $\mathbf{P} = \{\{1, 3\}, \{2\}\}$. Then, Proposition 0.13 says that the number of all set compositions $\mathbf{C} \in \text{SC}_k(X)$ satisfying $\pi(\mathbf{C}) = \mathbf{P}$ is $2! = 2$. These two set compositions are $(\{1, 3\}, \{2\})$ and $(\{2\}, \{1, 3\})$.

Proof of Proposition 0.13 (sketched). Let $\mathbf{P} \in \text{SP}_k(X)$. Thus, \mathbf{P} is a set partition of X into k parts. We can thus write \mathbf{P} as $\mathbf{P} = \{T_1, T_2, \dots, T_k\}$, where T_1, T_2, \dots, T_k are some disjoint nonempty subsets of X satisfying $T_1 \cup T_2 \cup \dots \cup T_k = X$. Consider these T_1, T_2, \dots, T_k . These sets T_1, T_2, \dots, T_k are nonempty and disjoint, and therefore distinct (because if two of them were equal, then their disjointness would force them to be empty).

The rest is now easy: Informally speaking, the set compositions $\mathbf{C} \in \text{SC}_k(X)$ satisfying $\pi(\mathbf{C}) = \mathbf{P}$ are just all possible ways to rearrange the k -tuple (T_1, T_2, \dots, T_k) (this follows from Lemma 0.12); thus, there are $k!$ of them (in fact, the sets T_1, T_2, \dots, T_k are distinct, so any way of rearranging them results in a different k -tuple). This proves Proposition 0.13.

Here is a more formal way to make this argument (read this if you care; I don't require this level of rigor in solutions): The k -tuple (T_1, T_2, \dots, T_k) is a set composition of X into k parts. Thus, for each permutation σ of $[k]$, the k -tuple $(T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)})$ also is a set composition of X into k parts, hence belongs to $\text{SC}_k(X)$. Moreover, this set composition $(T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)})$ is a $\mathbf{C} \in \text{SC}_k(X)$ satisfying $\pi(\mathbf{C}) = \mathbf{P}$ (because $\pi((T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)})) = \{T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)}\} = \{T_1, T_2, \dots, T_k\} = \mathbf{P}$). Hence, we can define a map

$$\alpha : \{\text{permutations of } [k]\} \rightarrow \{\mathbf{C} \in \text{SC}_k(X) \mid \pi(\mathbf{C}) = \mathbf{P}\},$$

$$\sigma \mapsto (T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)}).$$

This map is injective⁴ and surjective⁵; thus, it is bijective. We have therefore found a bijective map

⁴*Proof.* Let σ and τ be two permutations of $[k]$ such that $\alpha(\sigma) = \alpha(\tau)$. We must show that $\sigma = \tau$.

Let $i \in [k]$. The definition of α yields $\alpha(\sigma) = (T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)})$ and $\alpha(\tau) = (T_{\tau(1)}, T_{\tau(2)}, \dots, T_{\tau(k)})$. Hence, the equality $\alpha(\sigma) = \alpha(\tau)$ rewrites as $(T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)}) = (T_{\tau(1)}, T_{\tau(2)}, \dots, T_{\tau(k)})$. Thus, $T_{\sigma(i)} = T_{\tau(i)}$. Since the sets T_1, T_2, \dots, T_k are distinct, this entails $\sigma(i) = \tau(i)$.

Now, forget that we fixed i . We thus have shown that $\sigma(i) = \tau(i)$ for each $i \in [k]$. Therefore, $\sigma = \tau$. This completes the proof of the injectivity of α .

⁵*Proof.* Let $\mathbf{D} \in \{\mathbf{C} \in \text{SC}_k(X) \mid \pi(\mathbf{C}) = \mathbf{P}\}$. We must prove that $\mathbf{D} = \alpha(\sigma)$ for some permutation

from $\{\text{permutations of } [k]\}$ to $\{\mathbf{C} \in \text{SC}_k(X) \mid \pi(\mathbf{C}) = \mathbf{P}\}$. Hence,

$$\begin{aligned} |\{\text{permutations of } [k]\}| &= |\{\mathbf{C} \in \text{SC}_k(X) \mid \pi(\mathbf{C}) = \mathbf{P}\}| \\ &= (\text{the number of all } \mathbf{C} \in \text{SC}_k(X) \text{ satisfying } \pi(\mathbf{C}) = \mathbf{P}), \end{aligned}$$

so that

$$(\text{the number of all } \mathbf{C} \in \text{SC}_k(X) \text{ satisfying } \pi(\mathbf{C}) = \mathbf{P}) = |\{\text{permutations of } [k]\}| = k!.$$

This proves Proposition 0.13. \square

Proof of Proposition 0.7. Proposition 0.6 yields $|\text{SC}_k(X)| = \text{sur}(|X|, k)$. Thus,

$$\begin{aligned} \text{sur}(|X|, k) &= |\text{SC}_k(X)| \\ &= (\text{the number of all } \mathbf{C} \in \text{SC}_k(X)) \\ &= \sum_{\mathbf{P} \in \text{SP}_k(X)} \underbrace{(\text{the number of all } \mathbf{C} \in \text{SC}_k(X) \text{ satisfying } \pi(\mathbf{C}) = \mathbf{P})}_{=k! \text{ (by Proposition 0.13)}} \\ &\quad \left(\begin{array}{c} \text{here, we have subdivided our count according} \\ \text{to the value of } \pi(\mathbf{C}) \end{array} \right) \\ &= \sum_{\mathbf{P} \in \text{SP}_k(X)} k! = |\text{SP}_k(X)| \cdot k!. \end{aligned}$$

Dividing this equality by $k!$, we find $\frac{\text{sur}(|X|, k)}{k!} = |\text{SP}_k(X)|$. This proves Proposition 0.7. \square

References

[Fall2017-HW6os] Darij Grinberg, *Math 4990 Fall 2017 (Darij Grinberg): homework set 6 with solutions*.

<http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw6os.pdf>

σ of $[k]$.

We have $\mathbf{D} \in \{\mathbf{C} \in \text{SC}_k(X) \mid \pi(\mathbf{C}) = \mathbf{P}\}$. In other words, $\mathbf{D} \in \text{SC}_k(X)$ and $\pi(\mathbf{D}) = \mathbf{P}$. From $\mathbf{D} \in \text{SC}_k(X)$, we conclude that \mathbf{D} is a set composition of X into k parts. We can thus write \mathbf{D} as $\mathbf{D} = (D_1, D_2, \dots, D_k)$, where D_1, D_2, \dots, D_k are some disjoint nonempty subsets of X satisfying $D_1 \cup D_2 \cup \dots \cup D_k = X$. Consider these D_1, D_2, \dots, D_k .

From $\mathbf{D} = (D_1, D_2, \dots, D_k)$, we conclude that $\pi(\mathbf{D}) = \pi((D_1, D_2, \dots, D_k)) = \{D_1, D_2, \dots, D_k\}$ (by the definition of π), so that $\{D_1, D_2, \dots, D_k\} = \pi(\mathbf{D}) = \mathbf{P} = \{T_1, T_2, \dots, T_k\}$. In other words, $\{T_1, T_2, \dots, T_k\} = \{D_1, D_2, \dots, D_k\}$. Since T_1, T_2, \dots, T_k are distinct, we can thus apply Lemma 0.12 to $x_i = T_i$ and $y_i = D_i$. We conclude that there exists a permutation σ of $[k]$ such that every $i \in [k]$ satisfies $D_i = T_{\sigma(i)}$. Consider this σ . We have $(D_1, D_2, \dots, D_k) = (T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)})$ (since every $i \in [k]$ satisfies $D_i = T_{\sigma(i)}$). The definition of α yields

$$\alpha(\sigma) = (T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(k)}) = (D_1, D_2, \dots, D_k) = \mathbf{D}.$$

Hence, we have found a permutation σ of $[k]$ such that $\mathbf{D} = \alpha(\sigma)$. This completes our proof of the surjectivity of α .
