

Math 4707 Spring 2018 (Darij Grinberg): homework set 2 with solutions

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Recall that \mathbb{N} denotes the set $\{0, 1, 2, \dots\}$.

Recall that if $n \in \mathbb{N}$, then $[n]$ denotes the n -element set $\{1, 2, \dots, n\}$.

0.1. Instances of the “Laurent phenomenon”: Sequences that produce integers despite division in their definition

There is a whole genre of theorem where you define a sequence recursively and then it turns out that all entries of the sequence are integers, although this is not obvious from the definition.

Here are two results from this genre, illustrating strategic use of induction.

Exercise 1. Define a sequence (t_0, t_1, t_2, \dots) of positive rational numbers recursively by setting

$$t_0 = 1, \quad t_1 = 1, \quad t_2 = 1, \quad \text{and}$$

$$t_n = \frac{1 + t_{n-1}t_{n-2}}{t_{n-3}} \quad \text{for each } n \geq 3.$$

(For example, $t_3 = \frac{1 + t_2t_1}{t_0} = \frac{1 + 1 \cdot 1}{1} = 2$ and $t_4 = \frac{1 + t_3t_2}{t_1} = \frac{1 + 2 \cdot 1}{1} = 3$.)

(a) Prove that $t_{n+2} = 4t_n - t_{n-2}$ for each $n \geq 2$.

(b) Prove that $t_n \in \mathbb{N}$ for each $n \in \mathbb{N}$.

[Hint: First prove part **(a)** by induction on n . Then prove part **(b)** by induction on n , using part **(a)**.]

Remark 0.1. The sequence (t_0, t_1, t_2, \dots) defined in Exercise 1 is the sequence A005246 in the OEIS (Online Encyclopedia of Integer Sequences). Its first values are

$$\begin{aligned} t_0 &= 1, & t_1 &= 1, & t_2 &= 1, & t_3 &= 2, & t_4 &= 3, \\ t_5 &= 7, & t_6 &= 11, & t_7 &= 26, & t_8 &= 41, & t_9 &= 97. \end{aligned}$$

Exercise 1 **(b)** is an instance of the *Laurent phenomenon* (see, e.g., [FomZel02, Example 3.2]).

Exercise 1 appears in [Grinbe16, Proposition 2.64].

Solution to Exercise 1. First, we notice that the recursive definition of the sequence (t_0, t_1, t_2, \dots) yields

$$\begin{aligned} t_3 &= \frac{1 + t_{3-1}t_{3-2}}{t_{3-3}} = \frac{1 + t_2t_1}{t_0} = \frac{1 + 1 \cdot 1}{1} && \text{(since } t_0 = 1 \text{ and } t_1 = 1 \text{ and } t_2 = 1) \\ &= 2. \end{aligned}$$

Furthermore, the recursive definition of the sequence (t_0, t_1, t_2, \dots) yields

$$\begin{aligned} t_4 &= \frac{1 + t_{4-1}t_{4-2}}{t_{4-3}} = \frac{1 + t_3t_2}{t_1} = \frac{1 + 2 \cdot 1}{1} && \text{(since } t_1 = 1 \text{ and } t_2 = 1 \text{ and } t_3 = 2) \\ &= 3. \end{aligned}$$

Thus, $t_{2+2} = t_4 = 3$. Comparing this with $4 \underbrace{t_2}_{=1} - \underbrace{t_{2-2}}_{=t_0=1} = 4 \cdot 1 - 1 = 3$, we obtain

$$t_{2+2} = 4t_2 - t_{2-2}.$$

(a) We shall solve Exercise 1 **(a)** by induction on n :

Induction base: We have already shown that $t_{2+2} = 4t_2 - t_{2-2}$. In other words, Exercise 1 **(a)** holds for $n = 2$. This completes the induction base.

Induction step: Let $m \geq 2$ be an integer. Assume that Exercise 1 **(a)** holds for $n = m$. We must prove that Exercise 1 **(a)** holds for $n = m + 1$.

We have assumed that Exercise 1 **(a)** holds for $n = m$. In other words, we have $t_{m+2} = 4t_m - t_{m-2}$.

We have $m \geq 2$ and thus $m + 1 \geq 2 + 1 = 3$. Thus, the recursive definition of the sequence (t_0, t_1, t_2, \dots) yields

$$t_{m+1} = \frac{1 + t_{(m+1)-1}t_{(m+1)-2}}{t_{(m+1)-3}} = \frac{1 + t_m t_{m-1}}{t_{m-2}}.$$

Multiplying this equality by t_{m-2} , we obtain $t_{m-2}t_{m+1} = 1 + t_m t_{m-1}$. In other words,

$$t_{m-2}t_{m+1} - 1 = t_m t_{m-1}. \quad (1)$$

Also, $m + 3 \geq 3$. Thus, the recursive definition of the sequence (t_0, t_1, t_2, \dots) yields

$$\begin{aligned} t_{m+3} &= \frac{1 + t_{(m+3)-1}t_{(m+3)-2}}{t_{(m+3)-3}} = \frac{1 + t_{m+2}t_{m+1}}{t_m} = \frac{1}{t_m} \left(1 + \underbrace{t_{m+2} t_{m+1}}_{=4t_m - t_{m-2}} \right) \\ &= \frac{1}{t_m} \underbrace{(1 + (4t_m - t_{m-2})t_{m+1})}_{=4t_m t_{m+1} - (t_{m-2}t_{m+1} - 1)} = \frac{1}{t_m} (4t_m t_{m+1} - (t_{m-2}t_{m+1} - 1)) \\ &= 4t_{m+1} - \frac{1}{t_m} \underbrace{(t_{m-2}t_{m+1} - 1)}_{=t_m t_{m-1} \text{ (by (1))}} = 4t_{m+1} - \underbrace{\frac{1}{t_m} t_m t_{m-1}}_{=1} = 4t_{m+1} - t_{m-1}. \end{aligned}$$

In view of $m + 3 = (m + 1) + 2$ and $m - 1 = (m + 1) - 2$, this rewrites as $t_{(m+1)+2} = 4t_{m+1} - t_{(m+1)-2}$. In other words, Exercise 1 (a) holds for $n = m + 1$. This completes the induction step. Hence, Exercise 1 (a) is solved by induction.

(b) We shall solve Exercise 1 (b) by strong induction on n :

*Induction step:*¹ Let $m \in \mathbb{N}$. Assume that Exercise 1 (b) holds whenever $n < m$. We must now show that Exercise 1 (b) holds for $n = m$.

We have assumed that Exercise 1 (b) holds whenever $n < m$. In other words, we have

$$t_n \in \mathbb{N} \text{ for each } n \in \mathbb{N} \text{ satisfying } n < m. \quad (2)$$

We must now show that Exercise 1 (b) holds for $n = m$. In other words, we must show that $t_m \in \mathbb{N}$.

Recall that (t_0, t_1, t_2, \dots) is a sequence of positive rational numbers. Thus, t_m is a positive rational number.

We are in one of the following five cases:

Case 1: We have $m = 0$.

Case 2: We have $m = 1$.

Case 3: We have $m = 2$.

Case 4: We have $m = 3$.

Case 5: We have $m > 3$.

Let us first consider Case 1. In this case, we have $m = 0$. Thus, $t_m = t_0 = 1 \in \mathbb{N}$. Hence, $t_m \in \mathbb{N}$ is proven in Case 1.

Similarly, we can prove $t_m \in \mathbb{N}$ in Case 2 (using $t_1 = 1$) and in Case 3 (using $t_2 = 1$) and in Case 4 (using $t_3 = 2$). It thus remains to prove $t_m \in \mathbb{N}$ in Case 5.

So let us consider Case 5. In this case, we have $m > 3$. Thus, $m \geq 4$ (since m is an integer), so that $m - 2 \geq 4 - 2 = 2$. Hence, Exercise 1 (a) (applied to $n = m - 2$) yields $t_{(m-2)+2} = 4t_{m-2} - t_{(m-2)-2}$. In view of $(m - 2) + 2 = m$ and $(m - 2) - 2 = m - 4$, this rewrites as $t_m = 4t_{m-2} - t_{m-4}$.

¹A strong induction does not strictly require an induction base. (But we will need to separate five cases in the induction step, and the first four of these cases can be interpreted as "induction bases" if one so desires.)

But $m \geq 4$, so that $m - 4 \in \mathbb{N}$, and $m - 4 < m$. Hence, (2) (applied to $n = m - 4$) yields $t_{m-4} \in \mathbb{N} \subseteq \mathbb{Z}$. Similarly, $t_{m-2} \in \mathbb{Z}$.

So we know that t_{m-2} and t_{m-4} are both integers (since $t_{m-2} \in \mathbb{Z}$ and $t_{m-4} \in \mathbb{Z}$). Hence, $4t_{m-2} - t_{m-4}$ is an integer as well. In other words, t_m is an integer (because $t_m = 4t_{m-2} - t_{m-4}$). Since t_m is positive, we thus conclude that t_m is a positive integer. Hence, $t_m \in \mathbb{N}$. This shows that $t_m \in \mathbb{N}$ in Case 5.

We now have proven $t_m \in \mathbb{N}$ in each of the five Cases 1, 2, 3, 4 and 5. Thus, $t_m \in \mathbb{N}$ always holds. In other words, Exercise 1 (b) holds for $n = m$. This completes the induction step. Thus, Exercise 1 (b) is solved by strong induction. \square

Exercise 2. Fix a positive integer r . Define a sequence (b_0, b_1, b_2, \dots) of positive rational numbers recursively by setting

$$b_0 = 1, \quad b_1 = 1, \quad \text{and} \\ b_n = \frac{b_{n-1}^r + 1}{b_{n-2}} \quad \text{for each } n \geq 2.$$

(For example, $b_2 = \frac{b_1^r + 1}{b_0} = \frac{1^r + 1}{1} = 2$ and $b_3 = \frac{b_2^r + 1}{b_1} = \frac{2^r + 1}{1} = 2^r + 1$.)

(a) Prove that $b_n \in \mathbb{N}$ for each $n \in \mathbb{N}$.

(b) If $r \geq 2$, then prove that $b_n \mid b_{n-2} + b_{n+2}$ for each $n \geq 2$.

[Hint: For every nonzero $x \in \mathbb{Q}$, we set $H(x) = \frac{(x+1)^r - 1}{x}$. Show that $H(x) \in \mathbb{Z}$ whenever x is a nonzero integer. Next, show that $b_{n+2} = b_{n-2}b_{n+1}^r - b_n^{r-1}H(b_n)$ for each $n \geq 2$. Use this to prove (a).]

Remark 0.2. If $r = 1$, then the sequence (b_0, b_1, b_2, \dots) defined in Exercise 2 is

$$(1, 1, 2, 3, 2, 1, 1, 2, 3, 2, 1, 1, 2, 3, 2, \dots)$$

(this is a periodic sequence, which consists of the five terms 1, 1, 2, 3, 2 repeated over and over); this can easily be proven by induction. Despite its simplicity, this sequence is the sequence A076839 in the OEIS.

If $r = 2$, then the sequence (b_0, b_1, b_2, \dots) defined in Exercise 2 is

$$(1, f_1, f_3, f_5, f_7, \dots) = (1, 1, 2, 5, 13, 34, 89, 233, 610, 1597, \dots)$$

consisting of all Fibonacci numbers at odd positions (i.e., Fibonacci numbers of the form f_{2n-1} for $n \in \mathbb{N}$) with an extra 1 at the front. This, again, can be proven by induction. Also, this sequence satisfies the recurrence relation $b_n = 3b_{n-1} - b_{n-2}$ for all $n \geq 2$. This is the sequence A001519 in the OEIS.

If $r = 3$, then the sequence (b_0, b_1, b_2, \dots) defined in Exercise 2 is

$$(1, 1, 2, 9, 365, 5403014, 432130991537958813, \dots);$$

its entries grow so fast that I am only showing the first seven. This is the sequence A003818 in the OEIS. Unlike the cases of $r = 1$ and $r = 2$, not much can be said about this sequence, other than what has been said in Exercise 2.

Exercise 2 (a) is an instance of the *Laurent phenomenon for cluster algebras* (see, e.g., [FomZel01, Example 2.5]; also, see [Marsh13] and [FoWiZe16] for expositions). See also [MusPro07] for a study of the specific recurrence equation from Exercise 2 (a) (actually, a slightly more general equation).

Exercise 2 appears in [Grinbe16, Proposition 2.66].

As the hint to Exercise 2 suggests, we first show the following lemma before solving the exercise:

Lemma 0.3. Let $r \in \mathbb{N}$. For every nonzero $x \in \mathbb{Q}$, we set $H(x) = \frac{(x+1)^r - 1}{x}$.

Then, $H(x) \in \mathbb{Z}$ whenever x is a nonzero integer.

First proof of Lemma 0.3. Here is a quick proof using modular arithmetic:

Let x be a nonzero integer. Then, $x + 1 \equiv 1 \pmod{x}$.

But it is well-known that if a, b and m are three integers satisfying $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$ for each $k \in \mathbb{N}$. Applying this to $a = x + 1$, $b = 1$ and $m = x$, we conclude that $(x + 1)^k \equiv 1^k \pmod{x}$ for each $k \in \mathbb{N}$. Applying this to $k = r$, we obtain $(x + 1)^r \equiv 1^r = 1 \pmod{x}$. In other words, $(x + 1)^r - 1$ is divisible by x .

In other words, $\frac{(x + 1)^r - 1}{x} \in \mathbb{Z}$. Thus, $H(x) = \frac{(x + 1)^r - 1}{x} \in \mathbb{Z}$. This proves Lemma 0.3. \square

Second proof of Lemma 0.3. Let x be a nonzero integer. The binomial formula yields

$$\begin{aligned} (x + 1)^r &= \sum_{k=0}^r \binom{r}{k} x^k \underbrace{1^{r-k}}_{=1} = \sum_{k=0}^r \binom{r}{k} x^k \\ &= \underbrace{\binom{r}{0}}_{=1} \underbrace{x^0}_{=1} + \sum_{k=1}^r \binom{r}{k} \underbrace{x^k}_{=xx^{k-1} \text{ (since } k \geq 1)} = 1 + \underbrace{\sum_{k=1}^r \binom{r}{k} xx^{k-1}}_{=x \sum_{k=1}^r \binom{r}{k} x^{k-1}} \\ &= 1 + x \sum_{k=1}^r \binom{r}{k} x^{k-1}. \end{aligned}$$

Subtracting 1 from this equality, we obtain

$$(x + 1)^r - 1 = x \sum_{k=1}^r \binom{r}{k} x^{k-1}.$$

Dividing this equality by x , we find

$$\frac{(x + 1)^r - 1}{x} = \sum_{k=1}^r \binom{r}{k} x^{k-1} \in \mathbb{Z}$$

(since $x \in \mathbb{Z}$ and since the binomial coefficients $\binom{r}{k}$ are integers as well). Thus,
 $H(x) = \frac{(x+1)^r - 1}{x} \in \mathbb{Z}$. This proves Lemma 0.3. \square

Solution to Exercise 2. First, we notice that the recursive definition of the sequence (b_0, b_1, b_2, \dots) yields

$$\begin{aligned} b_2 &= \frac{b_{2-1}^r + 1}{b_{2-2}} = \frac{b_1^r + 1}{b_0} = \frac{1^r + 1}{1} && (\text{since } b_0 = 1 \text{ and } b_1 = 1) \\ &= \frac{1 + 1}{1} && (\text{since } 1^r = 1) \\ &= 2. \end{aligned}$$

Furthermore, the recursive definition of the sequence (b_0, b_1, b_2, \dots) yields

$$\begin{aligned} b_3 &= \frac{b_{3-1}^r + 1}{b_{3-2}} = \frac{b_2^r + 1}{b_1} = \frac{2^r + 1}{1} && (\text{since } b_1 = 1 \text{ and } b_2 = 2) \\ &= 2^r + 1. \end{aligned}$$

For every nonzero $x \in \mathbb{Q}$, we set $H(x) = \frac{(x+1)^r - 1}{x}$.

For every integer $m \geq 1$, we have

$$b_m^r + 1 = b_{m+1}b_{m-1}. \quad (3)$$

[*Proof of (3):* Let $m \geq 1$ be an integer. From $m \geq 1$, we obtain $m+1 \geq 1+1=2$. Hence, the recursive definition of the sequence (b_0, b_1, b_2, \dots) yields

$$b_{m+1} = \frac{b_{(m+1)-1}^r + 1}{b_{(m+1)-2}} = \frac{b_m^r + 1}{b_{m-1}}.$$

Multiplying both sides of this equality by b_{m-1} , we obtain $b_{m+1}b_{m-1} = b_m^r + 1$. This proves (3).]

Let us first prove the following observation:

Observation 1: Each integer $n \geq 2$ satisfies $b_{n+2} = b_{n-2}b_{n+1}^r - b_n^{r-1}H(b_n^r)$.

[*Proof of Observation 1:* Let $n \geq 2$ be an integer. Thus, $n \geq 2 \geq 1$. Thus, (3) (applied to $m = n$) yields

$$b_n^r + 1 = b_{n+1}b_{n-1}. \quad (4)$$

On the other hand, $n+1 \geq n \geq 2 \geq 1$. Hence, (3) (applied to $m = n+1$) yields

$$b_{n+1}^r + 1 = b_{(n+1)+1}b_{(n+1)-1} = b_{n+2}b_n.$$

Hence,

$$b_{n+1}^r = b_{n+2}b_n - 1. \quad (5)$$

Also, $n - 1 \geq 1$ (since $n \geq 2 = 1 + 1$). Hence, (3) (applied to $m = n - 1$) yields

$$b_{n-1}^r + 1 = b_{(n-1)+1} b_{(n-1)-1} = b_n b_{n-2}.$$

Hence,

$$b_{n-1}^r = b_n b_{n-2} - 1. \tag{6}$$

But b_n is a positive rational number (since (b_0, b_1, b_2, \dots) is a sequence of positive rational numbers). Thus, b_n^r is also a positive rational number. Hence, $b_n^r \in \mathbb{Q}$ is nonzero. The definition of $H(b_n^r)$ yields $H(b_n^r) = \frac{(b_n^r + 1)^r - 1}{b_n^r}$; therefore,

$$\begin{aligned} b_n^{r-1} \frac{H(b_n^r)}{b_n^r} &= b_n^{r-1} \cdot \frac{(b_n^r + 1)^r - 1}{b_n^r} = \frac{b_n^{r-1}}{b_n^r} \cdot \left(\left(\underbrace{b_n^r + 1}_{=b_{n+1}b_{n-1} \text{ (by (4))}} \right)^r - 1 \right) \\ &= \frac{(b_n^r + 1)^r - 1}{b_n^r} \\ &= \frac{1}{b_n} \cdot \left(\underbrace{(b_{n+1}b_{n-1})^r - 1}_{=b_{n+1}^r b_{n-1}^r} \right) = \frac{1}{b_n} \cdot \left(\underbrace{b_{n+1}^r}_{=b_{n+2}b_{n-1}} \underbrace{b_{n-1}^r}_{=b_n b_{n-2} - 1 \text{ (by (6))}} - 1 \right) \\ &= \frac{1}{b_n} \cdot \underbrace{((b_{n+2}b_n - 1)(b_n b_{n-2} - 1) - 1)}_{=b_n(b_n b_{n+2} b_{n-2} - b_{n+2} - b_{n-2})} \\ &= \frac{1}{b_n} \cdot b_n (b_n b_{n+2} b_{n-2} - b_{n+2} - b_{n-2}) \\ &= b_n b_{n+2} b_{n-2} - b_{n+2} - b_{n-2} = b_{n-2} \underbrace{(b_{n+2} b_n - 1)}_{=b_{n+1}^r \text{ (by (5))}} - b_{n+2} \\ &= b_{n-2} b_{n+1}^r - b_{n+2}. \end{aligned}$$

Solving this equation for b_{n+2} , we obtain $b_{n+2} = b_{n-2} b_{n+1}^r - b_n^{r-1} H(b_n^r)$. This proves Observation 1.]

(a) We shall solve Exercise 2 (a) by strong induction on n :

Induction step: Let $m \in \mathbb{N}$. Assume that Exercise 2 (a) holds whenever $n < m$. We must now prove that Exercise 2 (a) holds for $n = m$.

We have assumed that Exercise 2 (a) holds whenever $n < m$. In other words, we have

$$b_n \in \mathbb{N} \text{ for each } n \in \mathbb{N} \text{ satisfying } n < m. \tag{7}$$

We must now show that Exercise 2 (a) holds for $n = m$. In other words, we must show that $b_m \in \mathbb{N}$.

Recall that (b_0, b_1, b_2, \dots) is a sequence of positive rational numbers. Thus, b_m is a positive rational number.

We are in one of the following five cases:

Case 1: We have $m = 0$.

Case 2: We have $m = 1$.

Case 3: We have $m = 2$.

Case 4: We have $m = 3$.

Case 5: We have $m > 3$.

Let us first consider Case 1. In this case, we have $m = 0$. Thus, $b_m = b_0 = 1 \in \mathbb{N}$. Hence, $b_m \in \mathbb{N}$ is proven in Case 1.

Similarly, we can prove $b_m \in \mathbb{N}$ in Case 2 (using $b_1 = 1$) and in Case 3 (using $b_2 = 2$) and in Case 4 (using $b_3 = 2^r + 1$). It thus remains to prove $b_m \in \mathbb{N}$ in Case 5.

So let us consider Case 5. In this case, we have $m > 3$. Thus, $m \geq 4$ (since m is an integer), so that $m - 2 \geq 4 - 2 = 2$. Hence, Observation 1 (applied to $n = m - 2$) yields $b_{(m-2)+2} = b_{(m-2)-2} b_{(m-2)+1}^r - b_{m-2}^{r-1} H(b_{m-2}^r)$. In view of $(m - 2) + 2 = m$ and $(m - 2) - 2 = m - 4$ and $(m - 2) + 1 = m - 1$, this rewrites as

$$b_m = b_{m-4} b_{m-1}^r - b_{m-2}^{r-1} H(b_{m-2}^r). \quad (8)$$

But $m - 2 \in \mathbb{N}$ (since $m \geq 4 \geq 2$) and $m - 2 < m$. Hence, (7) (applied to $n = m - 2$) yields $b_{m-2} \in \mathbb{N} \subseteq \mathbb{Z}$. Also, b_{m-2} is a positive rational number (since (b_0, b_1, b_2, \dots) is a sequence of positive rational numbers) and thus a positive integer (since $b_{m-2} \in \mathbb{N}$), hence a nonzero integer. Thus, b_{m-2}^r is a nonzero integer as well. Therefore, Lemma 0.3 (applied to $x = b_{m-2}^r$) shows that $H(b_{m-2}^r) \in \mathbb{Z}$. Also, $r - 1 \geq 0$ (since $r \geq 1$), and thus $r - 1 \in \mathbb{N}$. Hence, b_{m-2}^{r-1} is an integer (since b_{m-2} is an integer).

Also, $m - 4 \in \mathbb{N}$ (since $m \geq 4$) and $m - 4 < m$. Hence, (7) (applied to $n = m - 4$) yields $b_{m-4} \in \mathbb{N} \subseteq \mathbb{Z}$.

Similarly, $b_{m-1} \in \mathbb{Z}$. Thus, $b_{m-1}^r \in \mathbb{Z}$.

We now know that the four numbers b_{m-4} , b_{m-1}^r , b_{m-2}^{r-1} and $H(b_{m-2}^r)$ all are integers (since we have shown that $b_{m-4} \in \mathbb{Z}$, $b_{m-1}^r \in \mathbb{Z}$, $b_{m-2}^{r-1} \in \mathbb{Z}$ and $H(b_{m-2}^r) \in \mathbb{Z}$). Thus, the number $b_{m-4} b_{m-1}^r - b_{m-2}^{r-1} H(b_{m-2}^r)$ also is an integer (since it is obtained from these four numbers by multiplication and subtraction). In view of (8), this rewrites as follows: The number b_m is an integer. Since b_m is positive, we thus conclude that b_m is a positive integer. Hence, $b_m \in \mathbb{N}$. This shows that $b_m \in \mathbb{N}$ in Case 5.

We now have proven $b_m \in \mathbb{N}$ in each of the five Cases 1, 2, 3, 4 and 5. Thus, $b_m \in \mathbb{N}$ always holds. In other words, Exercise 2 (a) holds for $n = m$. This completes the induction step. Thus, Exercise 2 (a) is solved by strong induction.

(b) Assume that $r \geq 2$. We must prove that $b_n \mid b_{n-2} + b_{n+2}$ for each $n \geq 2$.

So let $n \geq 2$ be an integer. We must show that $b_n \mid b_{n-2} + b_{n+2}$.

Exercise 2 (a) (applied to $n - 2$ instead of n) yields $b_{n-2} \in \mathbb{N}$. Exercise 2 (a) yields $b_n \in \mathbb{N}$. Similarly, $b_{n+1} \in \mathbb{N}$ and $b_{n+2} \in \mathbb{N}$. Thus, all of b_{n-2}, b_{n+1}, b_n are b_{n+2} are integers.

We have $n + 2 \geq n \geq 2$. Hence, the recursive definition of the sequence (b_0, b_1, b_2, \dots) yields $b_{n+2} = \frac{b_{(n+2)-1}^r + 1}{b_{(n+2)-2}} = \frac{b_{n+1}^r + 1}{b_n}$. Multiplying this equality by b_n , we obtain

$$b_n b_{n+2} = b_{n+1}^r + 1. \quad (9)$$

We have $r \geq 2$, so that $r - 2 \geq 0$. Therefore, b_n^{r-2} is an integer (since b_n is an integer).

Observation 1 yields $b_{n+2} = b_{n-2} b_{n+1}^r - b_n^{r-1} H(b_n^r)$. Thus,

$$\begin{aligned} & \underbrace{b_{n+2}} + b_{n-2} \\ &= b_{n-2} b_{n+1}^r - b_n^{r-1} H(b_n^r) \\ &= b_{n-2} b_{n+1}^r - b_n^{r-1} H(b_n^r) + b_{n-2} = \underbrace{b_{n-2} b_{n+1}^r + b_{n-2}}_{=b_{n-2}(b_{n+1}^r+1)} - b_n^{r-1} H(b_n^r) \\ &= b_{n-2} \underbrace{(b_{n+1}^r + 1)}_{=b_n b_{n+2} \text{ (by (9))}} - \underbrace{b_n^{r-1}}_{=b_n b_n^{r-2}} H(b_n^r) = b_{n-2} b_n b_{n+2} - b_n b_n^{r-2} H(b_n^r) \\ &= b_n \left(b_{n-2} b_{n+2} - b_n^{r-2} H(b_n^r) \right). \end{aligned} \quad (10)$$

But $b_{n-2} b_{n+2} - b_n^{r-2} H(b_n^r)$ is an integer (because $b_{n-2}, b_{n+2}, b_n^{r-2}$ and $H(b_n^r)$ are integers). Denote this integer by z . Thus, $z = b_{n-2} b_{n+2} - b_n^{r-2} H(b_n^r)$. Since b_n and z are integers, we have

$$\begin{aligned} b_n \mid b_n \quad & \underbrace{z} \\ &= b_{n-2} b_{n+2} - b_n^{r-2} H(b_n^r) \\ &= b_n \left(b_{n-2} b_{n+2} - b_n^{r-2} H(b_n^r) \right) = b_{n+2} + b_{n-2} \quad (\text{by (10)}) \\ &= b_{n-2} + b_{n+2}. \end{aligned}$$

This solves Exercise 2 (b). □

For a (slightly) different solution to Exercise 2, see <http://artofproblemsolving.com/community/c6h428645p3705719>.

0.2. Lacunar subsets with a given number of even and a given number of odd elements

Recall the following definition: A set S of integers is said to be *lacunar* if no two consecutive integers occur in S (that is, there exists no $i \in \mathbb{Z}$ such that both i and

$i + 1$ belong to S). For example, $\{1, 3, 6\}$ is lacunar, but $\{2, 4, 5\}$ is not. (The empty set and any 1-element set are lacunar, of course.)

Also recall the *Iverson bracket notation*: If \mathcal{A} is any logical statement, then the *truth value* of \mathcal{A} is defined to be the integer

$$\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases} \in \{0, 1\}.$$

This truth value is denoted by $[\mathcal{A}]$. For example, $[1 + 1 = 2] = 1$ (since $1 + 1 = 2$ is true), whereas $[1 + 1 = 1] = 0$ (since $1 + 1 = 1$ is false).

Exercise 3. For any $n \in \mathbb{N}$, $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, we let $N(n, a, b)$ denote the number of all lacunar subsets of $[n]$ that contain exactly a even and exactly b odd elements.

(a) Prove that $N(2m, a, b) = [a \leq m][b \leq m] \binom{m-a}{b} \binom{m-b}{a}$ for all $m \in \mathbb{N}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

(b) Prove that $N(2m+1, a, b) = [a \leq m][b \leq m+1] \binom{m+1-a}{b} \binom{m-b}{a}$ for all $m \in \mathbb{N}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

[Hint: One way is to prove parts (a) and (b) simultaneously by induction (that is, let $\mathcal{A}(m)$ be the statement " $N(2m, a, b) = [a \leq m][b \leq m] \binom{m-a}{b} \binom{m-b}{a}$ and $N(2m+1, a, b) = [a \leq m][b \leq m+1] \binom{m+1-a}{b} \binom{m-b}{a}$ for all $a \in \mathbb{N}$ and $b \in \mathbb{N}$ ", and prove this by induction on m). One part of the induction step is an expression for $N(2m+2, a, b)$ through $N(2m+1, a, b)$ and $N(2m, a-1, b)$. Another similar expression will be needed for $N(2m+3, a, b)$. Make sure to treat the base case properly, as well as justifying the switch between the truth values necessary at one point in the induction step. There is also a bijective proof.]

Exercise 3 originates in [MusPro07, Theorem 3] (although the authors of this paper forget the $[a \leq m][b \leq m]$ and $[a \leq m][b \leq m+1]$ factors), and also appears in [Lampe, Proposition 3.2.5] (where it is proven according to the Hint given above). The probably simplest solution is the proof given in [MusPro07]; however, this proof uses the concept of a multiset, which I (unfortunately) have not introduced in class yet. Thus, here are two other solutions (one inductive following the hint, and one bijective):

First solution to Exercise 3 (rough outline). We first claim the following:

Observation 1: Let $n \geq 2$ be an integer.

(a) If n is odd, then $N(n, a, b) = N(n-1, a, b) + N(n-2, a, b-1)$ for all $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.

(b) If n is even, then $N(n, a, b) = N(n-1, a, b) + N(n-2, a-1, b)$ for all $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.

[*Proof of Observation 1*: This is similar to [Fall2017-HW1s, solution to Exercise 4 (b)]. In fact, back in the exercise, we proved that $g(n) = g(n-1) + g(n-2)$, where $g(k)$ (for an integer k) denotes the number of all lacunar subsets of $[k]$. The gist of the argument was that

$$\begin{aligned}
 g(n) &= (\text{the number of lacunar subsets of } [n]) \\
 &= \underbrace{(\text{the number of lacunar subsets of } [n] \text{ that don't contain } n)}_{\substack{=(\text{the number of lacunar subsets of } [n-1]) \\ (\text{because a subset of } [n] \text{ that doesn't contain } n \text{ is exactly the same as a subset of } [n-1])}} \\
 &\quad + \underbrace{(\text{the number of lacunar subsets of } [n] \text{ that contain } n)}_{\substack{=(\text{the number of lacunar subsets of } [n-2]) \\ (\text{because there is a bijection from } \{\text{lacunar subsets of } [n] \text{ that contain } n\} \\ \text{to } \{\text{lacunar subsets of } [n-2]\}; \text{ namely, this bijection sends each } T \text{ to } T \setminus \{n\})}} \\
 &= \underbrace{(\text{the number of lacunar subsets of } [n-1])}_{\substack{=g(n-1) \\ (\text{by the definition of } g(n-1))}} \\
 &\quad + \underbrace{(\text{the number of lacunar subsets of } [n-2])}_{\substack{=g(n-2) \\ (\text{by the definition of } g(n-2))}} \\
 &= g(n-1) + g(n-2).
 \end{aligned}$$

We now need to tweak this argument so that it counts not all lacunar subsets of $[n]$, but only those that contain exactly a even and exactly b odd elements (for given a and b).

For brevity, we introduce a notation: Given two integers a and b and a set S of integers, we say that S is (a, b) -good if and only if S contains exactly a even and exactly b odd elements². Thus, if $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ and $k \in \mathbb{Z}$, then

$$N(k, a, b) = (\text{the number of } (a, b)\text{-good lacunar subsets of } [k]). \quad (11)$$

(This is simply the definition of $N(k, a, b)$, rewritten using the word “ (a, b) -good”.)

(a) Assume that n is odd. Aping our above argument for $g(n) = g(n-1) +$

²Mathematicians use the word “good” when they need an adjective and really have no inspiration. Any better suggestion? The word should be short.

$g(n-2)$, we compute

$$\begin{aligned}
N(n, a, b) &= (\text{the number of } (a, b)\text{-good lacunar subsets of } [n]) \\
&= \underbrace{(\text{the number of } (a, b)\text{-good lacunar subsets of } [n] \text{ that don't contain } n)}_{\substack{=(\text{the number of } (a, b)\text{-good lacunar subsets of } [n-1]) \\ (\text{because a subset of } [n] \text{ that doesn't contain } n \text{ is exactly the same as a subset of } [n-1])}} \\
&\quad + \underbrace{(\text{the number of } (a, b)\text{-good lacunar subsets of } [n] \text{ that contain } n)}_{\substack{=(\text{the number of } (a, b-1)\text{-good lacunar subsets of } [n-2]) \\ (\text{because there is a bijection from } \{\text{lacunar } (a, b)\text{-good subsets of } [n] \text{ that contain } n\} \\ \text{to } \{\text{lacunar } (a, b-1)\text{-good subsets of } [n-2]\}; \text{ namely, this bijection sends each } T \text{ to } T \setminus \{n\} \\ (\text{see Claim 1.1 below for details})}} \\
&= \underbrace{(\text{the number of } (a, b)\text{-good lacunar subsets of } [n-1])}_{\substack{=N(n-1, a, b) \\ (\text{by (11)}}}} \\
&\quad + \underbrace{(\text{the number of } (a, b-1)\text{-good lacunar subsets of } [n-2])}_{\substack{=N(n-2, a, b-1) \\ (\text{by (11)}}}} \\
&= N(n-1, a, b) + N(n-2, a, b-1). \tag{12}
\end{aligned}$$

Here, we have used the following claim:

Claim 1.1: There is a bijection from
 $\{\text{lacunar } (a, b)\text{-good subsets of } [n] \text{ that contain } n\}$ to
 $\{\text{lacunar } (a, b-1)\text{-good subsets of } [n-2]\}$; namely, this bijection sends
each T to $T \setminus \{n\}$.

Why is this claim true? Essentially, it can be proven in the same way as we proved that there is a bijection from $\{\text{lacunar subsets of } [n] \text{ that contain } n\}$ to $\{\text{lacunar subsets of } [n-2]\}$ sending each T to $T \setminus \{n\}$ (see [Fall2017-HW1s, solution to Exercise 4 (b)] for a detailed writeup of this argument); we only need to ensure that if T is an (a, b) -good subset of $[n]$ that contains n , then $T \setminus \{n\}$ is $(a, b-1)$ -good (and vice versa). But this is clear: Removing n from T results in the loss of one of the b odd elements of T (because n is odd); thus, the resulting set $T \setminus \{n\}$ has exactly a even elements and exactly $b-1$ odd elements. In other words, $T \setminus \{n\}$ is $(a, b-1)$ -good. The opposite direction (i.e., adding n to an $(a, b-1)$ -good subset of $[n-2]$ results in an (a, b) -good subset of $[n]$) should be equally clear.

So we have proven (12). This proves Observation 1 (a).

(b) The proof of Observation 1 (b) is analogous to the proof of (a) just given (but, of course, n is now even, so removing n from T leads to the loss of one of the a even elements rather than one of the b odd elements.)

Now, let's solve the actual exercise.

For each $m \in \mathbb{N}$, let $\mathcal{A}(m)$ be the statement

$$\begin{aligned} & "N(2m, a, b) = [a \leq m] [b \leq m] \binom{m-a}{b} \binom{m-b}{a} \text{ and } N(2m+1, a, b) = \\ & [a \leq m] [b \leq m+1] \binom{m+1-a}{b} \binom{m-b}{a} \text{ for all } a \in \mathbb{N} \text{ and } b \in \mathbb{N}." \end{aligned}$$

We want to show that $\mathcal{A}(m)$ holds for all $m \in \mathbb{N}$ (because this will clearly yield both parts **(a)** and **(b)** of Exercise 3).

We shall prove this by induction on m .

The *induction base* is the case $m = 0$, and requires proving the equalities

$$N(0, a, b) = [a \leq 0] [b \leq 0] \binom{0-a}{b} \binom{0-b}{a} \quad \text{and} \quad (13)$$

$$N(1, a, b) = [a \leq 0] [b \leq 1] \binom{1-a}{b} \binom{0-b}{a} \quad (14)$$

for all $a \in \mathbb{N}$ and $b \in \mathbb{N}$. This is straightforward: The number $N(1, a, b)$ (for example) counts all lacunar subsets of $[1]$ that contain exactly a even and exactly b odd elements. But there are only two subsets of $[1]$, both of them lacunar, and so the number $N(1, a, b)$ is nonzero only in the case when $a = 0$ and $b \in \{0, 1\}$, in which case this number is 1. The right hand side of (14) behaves exactly the same: It is nonzero only in the case when $a = 0$ and $b \in \{0, 1\}$ (because in all other cases, the factor $[a \leq 0] [b \leq 1]$ vanishes, causing the whole right hand side to vanish), and in this case it equals 1 (which is easily checked by hand). Thus, the equality (14) holds. The equality (13) is similar but even simpler, and so we leave it to the reader.

Induction step: Let m be a positive integer. Assume that $\mathcal{A}(m-1)$ holds. We must prove that $\mathcal{A}(m)$ holds.

We have assumed that $\mathcal{A}(m-1)$ holds. In other words, we have

$$N(2(m-1), a, b) = [a \leq m-1] [b \leq m-1] \binom{m-1-a}{b} \binom{m-1-b}{a} \quad \text{and}$$

$$N(2(m-1)+1, a, b) = [a \leq m-1] [b \leq (m-1)+1] \binom{(m-1)+1-a}{b} \binom{m-1-b}{a}$$

for all $a \in \mathbb{N}$ and $b \in \mathbb{N}$. These two equalities simplify to

$$N(2m-2, a, b) = [a \leq m-1] [b \leq m-1] \binom{m-1-a}{b} \binom{m-1-b}{a} \quad \text{and} \quad (15)$$

$$N(2m-1, a, b) = [a \leq m-1] [b \leq m] \binom{m-a}{b} \binom{m-1-b}{a} \quad (16)$$

for all $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

We must prove that $\mathcal{A}(m)$ holds. In other words, we must prove that

$$N(2m, a, b) = [a \leq m] [b \leq m] \binom{m-a}{b} \binom{m-b}{a} \quad \text{and} \quad (17)$$

$$N(2m+1, a, b) = [a \leq m] [b \leq m+1] \binom{m+1-a}{b} \binom{m-b}{a} \quad (18)$$

for all $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

Let us start by proving (17). Fix $a \in \mathbb{N}$ and $b \in \mathbb{N}$. Observation 1 **(b)** (applied to $n = 2m$) yields

$$N(2m, a, b) = N(2m-1, a, b) + N(2m-2, a-1, b). \quad (19)$$

We now want to apply (15) to $a-1$ instead of a , in order to expand the $N(2m-2, a-1, b)$ on the right hand side. Unfortunately, $a-1$ might not be an element of \mathbb{N} , which would preclude such an application; thus, we need to deal with the case $a=0$ separately (since this is precisely the case when $a-1$ is not an element of \mathbb{N}). Fortunately, this case is easy (left to the reader). Thus, we WLOG assume that we don't have $a=0$. Hence, (19) becomes

$$\begin{aligned} & N(2m, a, b) \\ &= \underbrace{N(2m-1, a, b)}_{\substack{=[a \leq m-1][b \leq m] \\ \text{(by (16))}}} \binom{m-a}{b} \binom{m-1-b}{a} \\ &+ \underbrace{N(2m-2, a-1, b)}_{\substack{=[a-1 \leq m-1][b \leq m-1] \\ \text{(by (15), applied to } a-1 \text{ instead of } a)}} \binom{m-1-(a-1)}{b} \binom{m-1-b}{a-1} \\ &= [a \leq m-1] [b \leq m] \binom{m-a}{b} \binom{m-1-b}{a} \\ &+ \left[\underbrace{a-1 \leq m-1}_{\iff (a \leq m)} \right] [b \leq m-1] \underbrace{\binom{m-1-(a-1)}{b}}_{=\binom{m-a}{b}} \binom{m-1-b}{a-1} \\ &= [a \leq m-1] [b \leq m] \binom{m-a}{b} \binom{m-1-b}{a} + [a \leq m] [b \leq m-1] \binom{m-a}{b} \binom{m-1-b}{a-1}. \end{aligned}$$

The right hand side of this computation begins to resemble the right hand side of (17) (which we are trying to prove), but we're not quite there. For one, the truth value $[a \leq m-1]$ and $[b \leq m-1]$ are not quite the same as $[a \leq m]$ and $[b \leq m]$, respectively. However, they are "almost" the same: $[a \leq m-1]$ differs from $[a \leq m]$

only when $a = m$, and $[b \leq m - 1]$ differs from $[b \leq m]$ only if $b = m$. So let us treat the cases $a = m$ and $b = m$ separately.

Treating the case $a = m$ means studying (m, b) -good subsets of $[2m]$.

An (m, b) -good subset of $[2m]$ must contain exactly $a = m$ even elements; thus, it must contain **all** the even numbers $2, 4, \dots, 2m$ (because $[2m]$ has only these m even elements). Furthermore, such a subset is lacunar only if it is precisely $\{2, 4, \dots, 2m\}$ (because if it contains any further element, then this further element is adjacent to one of $2, 4, \dots, 2m$, and thus the subset cannot be lacunar). Thus, an (m, b) -good lacunar subset of $[2m]$ exists only if $b = 0$, and in this case there is only 1 such subset. Thus, for the number of such subsets, we get the formula $N(2m, m, b) = [b = 0]$. This formula leads to an easy and straightforward proof of (17) in the case when $a = m$. We leave this proof to the reader, and from now on WLOG assume that $a \neq m$.

Similarly, the case $b = m$ can be eliminated, and so we WLOG assume that $b \neq m$.

Since $a \neq m$, we have $[a \leq m - 1] = [a \leq m]$. Similarly, $[b \leq m - 1] = [b \leq m]$.

Thus, we can continue our computation as follows:

$$\begin{aligned}
 N(2m, a, b) &= \underbrace{[a \leq m - 1]}_{=[a \leq m]} [b \leq m] \binom{m - a}{b} \binom{m - 1 - b}{a} + [a \leq m] \underbrace{[b \leq m - 1]}_{=[b \leq m]} \binom{m - a}{b} \binom{m - 1 - b}{a - 1} \\
 &= [a \leq m] [b \leq m] \binom{m - a}{b} \binom{m - 1 - b}{a} + [a \leq m] [b \leq m] \binom{m - a}{b} \binom{m - 1 - b}{a - 1} \\
 &= [a \leq m] [b \leq m] \binom{m - a}{b} \underbrace{\left(\binom{m - 1 - b}{a} + \binom{m - 1 - b}{a - 1} \right)}_{=\binom{m - b}{a}} \\
 &\hspace{15em} \text{(by the recurrence relation of the binomial coefficients)} \\
 &= [a \leq m] [b \leq m] \binom{m - a}{b} \binom{m - b}{a}.
 \end{aligned}$$

This proves (17).

The proof of (18) is similar, except that we now have to use Observation 1 **(a)** instead of Observation 1 **(b)**, and we have to use (17) and (16) instead of (16) and (15) in the computation that ensues. The reader can check the details. (The cases that need to be handled separately this time are the cases $b = 0$, $a = m$ and $b = m + 1$.)

With both (17) and (18) proven, we conclude that $\mathcal{A}(m)$ holds. This completes the induction step. Thus, by induction, we have shown that $\mathcal{A}(m)$ holds for all $m \in \mathbb{N}$. This proves both parts of Exercise 3. □

Second solution to Exercise 3 (rough outline). Here is a (sketch of) a bijective proof:

(a) Let $m \in \mathbb{N}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$. We WLOG assume that $a \leq m$ and $b \leq m$ (because in the other cases, it is easy to see that $N(2m, a, b) = 0$).

Let A be the set of all $(m - a)$ -tuples which consist of $m - a - b$ times the entry 0 and b times the entry 1. Then, $|A| = \binom{m - a}{b}$ (because we can construct an $(m - a)$ -tuple in A by specifying which of its $m - a$ positions will contain the entry 1; these must be b positions).

Let B be the set of all $(m - b)$ -tuples which consist of $m - a - b$ times the entry 0 and a times the entry 2. Then, $|B| = \binom{m - b}{a}$ (by a similar argument).

Let C be the set of all lacunar subsets of $[2m]$ that contain exactly a even and exactly b odd elements. Clearly, $N(2m, a, b) = |C|$ (by the definition of $N(2m, a, b)$).

Now, we assign to every $S \in C$ an m -tuple $w_S = (w_{S,1}, w_{S,2}, \dots, w_{S,m})$, whose entries are defined by the equality

$$w_{S,i} = \begin{cases} 0, & \text{if } 2i - 1 \notin S \text{ and } 2i \notin S; \\ 1, & \text{if } 2i - 1 \in S \text{ (and thus } 2i \notin S); \\ 2, & \text{if } 2i \in S \text{ (and thus } 2i - 1 \notin S) \end{cases} \quad \text{for each } i \in [m]$$

³ This m -tuple w_S contains the entry 1 exactly b times (because each entry 1 corresponds to an odd element of S) and the entry 2 exactly a times (similarly), and it never has a 1 directly following a 2 (because if it had, then there would be an $i \in [m]$ such that $w_{S,i} = 2$ and $w_{S,i+1} = 1$; but this would mean that $2i \in S$ and $2i + 1 \in S$, which would contradict the fact that S is lacunar). Other than this, this m -tuple w_S can be arbitrary, and the map that assigns w_S to $S \in C$ is injective (i.e., if S_1 and S_2 are two different sets in C , then $w_{S_1} \neq w_{S_2}$).

[Example: If $m = 5$, $a = 3$, $b = 1$ and $S = \{2, 4, 7, 10\}$, then $w_S = (2, 2, 0, 1, 2)$.]

Now, we can define a map $\Phi : C \rightarrow A \times B$ by sending each $S \in C$ to the pair (w'_S, w''_S) , where the $(m - a)$ -tuple w'_S is obtained from w_S by removing all 2's, and where the $(m - b)$ -tuple w''_S is obtained from w_S by removing all 1's.

[Example: If $m = 5$, $a = 3$, $b = 1$ and $S = \{2, 4, 7, 10\}$, then $w'_S = (0, 1)$ and $w''_S = (2, 2, 0, 2)$, so that $\Phi(S) = ((0, 1), (2, 2, 0, 2))$.]

We claim that this map Φ is bijective. This may be somewhat surprising, but it's true. You may want to try constructing its inverse yourself rather than reading on; you might be done faster this way.

[Proof of the fact that Φ is bijective: What is the inverse map Φ^{-1} ? Well, let $(\alpha, \beta) \in A \times B$ be a pair of an $(m - a)$ -tuple $\alpha \in A$ and an $(m - b)$ -tuple $\beta \in B$. Then, we want to define $\Phi^{-1}(\alpha, \beta)$ to be a set $S \in C$ such that $\Phi(S) = (\alpha, \beta)$. To find such an S , let us first find its corresponding m -tuple w_S .

We want S to satisfy $\Phi(S) = (\alpha, \beta)$. Thus, we want S to satisfy $(\alpha, \beta) = \Phi(S) = (w'_S, w''_S)$. In other words, we want to have $\alpha = w'_S$ and $\beta = w''_S$. Thus, w_S should be an m -tuple containing the entry 1 exactly b times, the entry 2 exactly a times, and never having a 1 directly following a 2, and it should have the property that removing all 2's results in α whereas removing all 1's results in β .

To construct such an m -tuple w_S , we observe that each of the tuples α and β contains the entry 0 exactly $m - a - b$ times. Thus, we can represent each of the tuples α and β as a sequence of $m - a - b$

³Why " $2i - 1 \in S$ (and thus $2i \notin S$)"? Well, S must be lacunar (since $S \in C$), and therefore S cannot contain the two consecutive integers $2i - 1$ and $2i$ at the same time. Hence, if $2i - 1 \in S$, then $2i \notin S$. Similarly, if $2i \in S$, then $2i - 1 \notin S$.

many 0's, interrupted by other entries (namely, by 1's in the case of α , and by 2's in the case of β). Specifically:

- Reading α from left to right, let's say we first encounter g_0 many 1's, then the first 0, then g_1 many 1's, then the second 0, then g_2 many 1's, then the third 0, and so on, finally ending with g_{m-a-b} many 1's. The numbers $g_0, g_1, \dots, g_{m-a-b}$ here are nonnegative integers; they may be 0 (for example, if $g_3 = 0$, then there are no 1's between the third and the fourth 0). Thus, the $(m-a)$ -tuple α looks as follows:

$$\alpha = \left(\underbrace{1, 1, \dots, 1}_{g_0 \text{ times}}, 0, \underbrace{1, 1, \dots, 1}_{g_1 \text{ times}}, 0, \dots, \underbrace{1, 1, \dots, 1}_{g_{m-a-b} \text{ times}} \right).$$

- Reading β from left to right, let's say we first encounter h_0 many 2's, then the first 0, then h_1 many 2's, then the second 0, then h_2 many 2's, then the third 0, and so on, finally ending with h_{m-a-b} many 2's. The numbers $h_0, h_1, \dots, h_{m-a-b}$ here are nonnegative integers; they may be 0. Thus, the $(m-b)$ -tuple β looks as follows:

$$\beta = \left(\underbrace{2, 2, \dots, 2}_{h_0 \text{ times}}, 0, \underbrace{2, 2, \dots, 2}_{h_1 \text{ times}}, 0, \dots, \underbrace{2, 2, \dots, 2}_{h_{m-a-b} \text{ times}} \right).$$

Now, let w_S be the m -tuple that (when read from left to right) looks as follows:

- first, g_0 many 1's, then h_0 many 2's,
- then the first 0,
- then g_1 many 1's, then h_1 many 2's,
- then the second 0,
- then g_2 many 1's, then h_2 many 2's,
- then the third 0,
- and so on, finally ending with g_{m-a-b} many 1's and h_{m-a-b} many 2's.

In other words, w_S is the m -tuple

$$\left(\underbrace{1, 1, \dots, 1}_{g_0 \text{ times}}, \underbrace{2, 2, \dots, 2}_{h_0 \text{ times}}, 0, \underbrace{1, 1, \dots, 1}_{g_1 \text{ times}}, \underbrace{2, 2, \dots, 2}_{h_1 \text{ times}}, 0, \dots, \underbrace{1, 1, \dots, 1}_{g_{m-a-b} \text{ times}}, \underbrace{2, 2, \dots, 2}_{h_{m-a-b} \text{ times}} \right).$$

We still need to construct the set $S \in C$ that leads to this m -tuple w_S ; but this is easy: just let

$$S = \{2i - 1 \mid i \in [m], \text{ and the } i\text{-th entry of } w_S \text{ is } 1\} \\ \cup \{2i \mid i \in [m], \text{ and the } i\text{-th entry of } w_S \text{ is } 2\}.$$

(This is just inverting the construction of w_S .)

It is not hard to see that this correctly defines a map $A \times B \rightarrow C$ which is inverse to the map Φ .

Thus, the map Φ is bijective.]

Hence, there is a bijection from C to $A \times B$ (namely, Φ). Consequently,

$$|C| = |A \times B| = \underbrace{|A|}_{\binom{m-a}{b}} \cdot \underbrace{|B|}_{\binom{m-b}{a}} = \binom{m-a}{b} \binom{m-b}{a} \\ = \binom{m-a}{b} = \binom{m-b}{a}$$

Now, as we have seen,

$$N(2m, a, b) = |C| = \binom{m-a}{b} \binom{m-b}{a} = [a \leq m] [b \leq m] \binom{m-a}{b} \binom{m-b}{a}$$

(here, we have introduced the $[a \leq m]$ and $[b \leq m]$ factors, which are both 1 and therefore do not change the product). This solves Exercise 3 (a).

(b) The proof is similar to the above proof of (a), with the difference that w_s now has to be an $(m + 1)$ -tuple and is not allowed to end with a 2. □

0.3. Delannoy numbers

Fix two positive integers r and s .

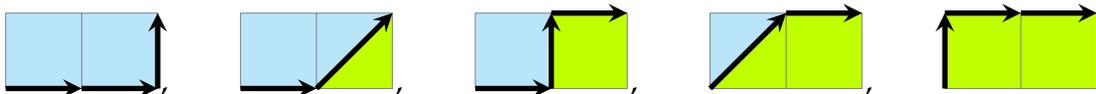
If $(a, b) \in \mathbb{Z}^2$ and $(c, d) \in \mathbb{Z}^2$ are two points on the integer lattice, then a (r, s) -Delannoy path from (a, b) to (c, d) is a path from (a, b) to (c, d) that uses only three kinds of steps:

- up-steps (U), which have the form $(x, y) \mapsto (x, y + 1)$;
- right-steps (R), which have the form $(x, y) \mapsto (x + 1, y)$;
- diagonal steps (D), which have the form $(x, y) \mapsto (x + r, y + s)$.

Thus, strictly speaking, a (r, s) -Delannoy path from (a, b) to (c, d) is a sequence (v_0, v_1, \dots, v_n) of points $v_i \in \mathbb{Z}^2$ such that for each $i \in [n]$, the difference vector $v_i - v_{i-1}$ is either $(0, 1)$ or $(1, 0)$ or (r, s) .

For two integers n and m , we let $d_{n,m}$ be the number of (r, s) -Delannoy paths from $(0, 0)$ to (n, m) . (This depends on r and s , too, but we regard r and s as fixed.) Note that $d_{n,m} = 0$ if (at least) one of n and m is negative (because neither the x -coordinate nor the y -coordinate can ever decrease along an (r, s) -Delannoy path).

For example, if $r = 1$ and $s = 1$, then $d_{2,1} = 5$, the five $(1, 1)$ -Delannoy paths being RRU, RD, RUR, DR and URR . Here are these five paths drawn in the plane:



Exercise 4. (a) Show that $d_{n,m} = d_{n-1,m} + d_{n,m-1} + d_{n-r,m-s}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, unless $(n, m) = (0, 0)$.

(b) Show that

$$d_{n,m} = \sum_{k=0}^n [n+m \geq (r+s-1)k] \binom{n-(r-1)k}{k} \binom{n+m-(r+s-1)k}{n-(r-1)k}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

(c) Assume that $r = s$. Show that $d_{n,m} = d_{m,n}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

[Hint: The case $r = 1$ and $s = 1$ is studied in [Galvin17, §28].]

We will outline a solution of this exercise; but first, we need to recall the symmetry of Pascal's triangle:

Proposition 0.4. Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then,

$$\binom{n}{k} = \binom{n}{n-k}.$$

We have proven Proposition 0.4 in class (see the proof of Theorem 1.13 in the 24 January 2018 lecture). If we don't have the luxury of knowing that $n \in \mathbb{N}$, we have at least the following:

Proposition 0.5. Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then,

$$[n \geq 0] \binom{n}{k} = [n \geq 0] \binom{n}{n-k}.$$

Proof of Proposition 0.5. If $n \in \mathbb{N}$, then Proposition 0.5 follows from Proposition 0.4. If not, then we have $n < 0$ (since $n \in \mathbb{Z}$), and thus both sides in Proposition 0.5 are 0 (because the $[n \geq 0]$ factors vanish). Thus, in either case, Proposition 0.5 holds. \square

Solution to Exercise 4 (sketched). In the following, the word "point" will always mean a pair $(x, y) \in \mathbb{Z}^2$ (and will be regarded as a point in the Euclidean plane \mathbb{R}^2). The word "path" will always mean an (r, s) -Delannoy path. Moreover, if n and m are two integers, then "path to (n, m) " shall always mean "path from $(0, 0)$ to (n, m) ". (So, paths start at $(0, 0)$ by default.)

Thus,

$$d_{n,m} = (\text{the number of all paths to } (n, m)) \tag{20}$$

for any integers n and m .

A *step* in a path (v_0, v_1, \dots, v_n) means a pair of the form (v_{i-1}, v_i) for some $i \in [n]$. More precisely, this pair (v_{i-1}, v_i) will be called the *i -th step* of the path.

We say that a path (v_0, v_1, \dots, v_n) *passes through* a point w if $w \in \{v_0, v_1, \dots, v_n\}$.

(a) Fix $n \in \mathbb{N}$ and $m \in \mathbb{N}$ that don't satisfy $(n, m) = (0, 0)$. Then, any (r, s) -Delannoy path from $(0, 0)$ to (n, m) contains at least one step, and thus has a **last** step. This last step must be either an up-step, or a right-step, or a diagonal step. Hence,

$$\begin{aligned} & (\text{the number of all paths to } (n, m)) \\ &= (\text{the number of all paths to } (n, m) \text{ whose last step is an up-step}) \\ & \quad + (\text{the number of all paths to } (n, m) \text{ whose last step is a right-step}) \\ & \quad + (\text{the number of all paths to } (n, m) \text{ whose last step is a diagonal step}). \end{aligned} \tag{21}$$

In order to profit from this observation, we need to actually compute the three numbers on the right hand side.

Any path \mathbf{p} to (n, m) whose last step is an up-step must pass through $(n, m - 1)$ (because this is the point from which an up-step leads to (n, m)). Thus, this path \mathbf{p} consists of two parts: the first part is a path to $(n, m - 1)$; the second part is a single up-step from $(n, m - 1)$ to (n, m) . Let us denote the first part by $L(\mathbf{p})$. Hence, we have defined a map

$$L : \{\text{paths to } (n, m) \text{ whose last step is an up-step}\} \rightarrow \{\text{paths to } (n, m - 1)\}$$

(which simply removes the last step from a path). This map L is a bijection (indeed, the inverse map simply adds an up-step at the end of a path). Thus,

$$\begin{aligned} & |\{\text{paths to } (n, m - 1)\}| \\ &= |\{\text{paths to } (n, m) \text{ whose last step is an up-step}\}| \\ &= (\text{the number of all paths to } (n, m) \text{ whose last step is an up-step}). \end{aligned}$$

Comparing this with

$$\begin{aligned} |\{\text{paths to } (n, m - 1)\}| &= (\text{the number of all paths to } (n, m - 1)) \\ &= d_{n, m-1} \quad (\text{by (20)}), \end{aligned}$$

we obtain

$$(\text{the number of all paths to } (n, m) \text{ whose last step is an up-step}) = d_{n, m-1}.$$

Similarly,

$$(\text{the number of all paths to } (n, m) \text{ whose last step is a right-step}) = d_{n-1, m}$$

and

$$(\text{the number of all paths to } (n, m) \text{ whose last step is a diagonal step}) = d_{n-r, m-s}.$$

Hence, (21) becomes

$$\begin{aligned} & (\text{the number of all paths to } (n, m)) \\ &= \underbrace{(\text{the number of all paths to } (n, m) \text{ whose last step is an up-step})}_{=d_{n, m-1}} \\ & \quad + \underbrace{(\text{the number of all paths to } (n, m) \text{ whose last step is a right-step})}_{=d_{n-1, m}} \\ & \quad + \underbrace{(\text{the number of all paths to } (n, m) \text{ whose last step is a diagonal step})}_{=d_{n-r, m-s}} \\ &= d_{n, m-1} + d_{n-1, m} + d_{n-r, m-s} = d_{n-1, m} + d_{n, m-1} + d_{n-r, m-s}. \end{aligned}$$

Hence, (20) yields

$$d_{n,m} = (\text{the number of all paths to } (n, m)) = d_{n-1,m} + d_{n,m-1} + d_{n-r,m-s}.$$

This solves Exercise 4 (a).

(b) The argument we shall give is a straightforward generalization of an argument made (for the particular case when $r = 1$ and $s = 1$) in [Galvin17, §28 (after the sentence “Could we have seen this formula combinatorially?”)].

Fix $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Consider the following way to construct a path to (n, m) :

- First, decide **how many diagonal steps** this path will have. This number must be an integer in $\{0, 1, \dots, n\}$ ⁴. Denote it by k . So we have decided that our path should have k diagonal steps. Thus, it must have exactly $n - rk$ right-steps⁵ and exactly $m - sk$ up-steps⁶. Hence, it will have a total of

$$k + (n - rk) + (m - sk) = n + m - (r + s - 1)k$$

steps. If $n + m - (r + s - 1)k < 0$, then this is plainly impossible⁷, and thus there are no paths to (n, m) in this case.

- We now decide which $m - sk$ among the $n + m - (r + s - 1)k$ steps of our path will be **up-steps**⁸. This can be done in

$[n + m - (r + s - 1)k \geq 0] \binom{n + m - (r + s - 1)k}{m - sk}$ many ways⁹. The remaining

$$(n + m - (r + s - 1)k) - (m - sk) = n - (r - 1)k$$

steps must therefore be right-steps or diagonal steps.

- Finally, we decide which k among these $n - (r - 1)k$ remaining steps will be **diagonal steps**. This can be done in $\binom{n - (r - 1)k}{k}$ many ways (since

⁴Indeed, it cannot be $> n$, because any diagonal step raises the x-coordinate of the point by at least $r \geq 1$, so if we take $> n$ diagonal steps, then we will overshoot our target (n, m) at least in its x-coordinate.

⁵Indeed, our path starts at $(0, 0)$ and ends at (n, m) , so its x-coordinate must change from 0 to n . The only steps that change the x-coordinate are right-steps (which change it by 1) and diagonal steps (which change it by r); in fact, up-steps don't change the x-coordinate. Thus, if our path has α right-steps, then the total change of the x-coordinate is $\alpha \cdot 1 + k \cdot r$ (since the path has k diagonal steps). Thus, we must have $\alpha \cdot 1 + k \cdot r = n$, therefore $n = \alpha \cdot 1 + k \cdot r = \alpha + rk$, hence $\alpha = n - rk$. In other words, our path must have exactly $n - rk$ right-steps.

⁶for similar reasons

⁷In fact, a path cannot have a negative number of steps.

⁸i.e., for which $i \in [n + m - (r + s - 1)k]$ will the i -th step of our path be an up-step?

⁹Why? Because if $n + m - (r + s - 1)k \geq 0$, then we are simply choosing an $(m - sk)$ -element subset of a given $(n + m - (r + s - 1)k)$ -element set, which can be done in $\binom{n + m - (r + s - 1)k}{m - sk}$ many ways; but if $n + m - (r + s - 1)k < 0$, then (as explained above) there are no paths to (n, m) at all.

we are just choosing a k -element subset of a given $(n - (r - 1)k)$ -element set¹⁰). Once this has been decided, we automatically know that the remaining $(n - (r - 1)k) - k = n - rk$ steps of our paths will be right-steps. Thus, the path is uniquely determined.

It is fairly clear (by retracing this argument backwards) that this construction yields every path to (n, m) exactly once. Thus, the number of paths to (n, m) equals the number of possible ways to perform this construction. But the former number is $d_{n,m}$ (by (20)), whereas the latter number is

$$\sum_{k \in \{0,1,\dots,n\}} [n + m - (r + s - 1)k \geq 0] \binom{n + m - (r + s - 1)k}{m - sk} \binom{n - (r - 1)k}{k}$$

(this follows by multiplying the numbers of choices we had at each step of our construction). Thus, we conclude that

$$\begin{aligned} d_{n,m} &= \sum_{k \in \{0,1,\dots,n\}} [n + m - (r + s - 1)k \geq 0] \binom{n + m - (r + s - 1)k}{m - sk} \binom{n - (r - 1)k}{k} \\ &= \sum_{k=0}^n \underbrace{[n + m - (r + s - 1)k \geq 0]}_{\substack{\text{(by Proposition 0.5, applied} \\ \text{to } n+m-(r+s-1)k \text{ and } m-sk \text{ instead of } n \text{ and } k)}} \binom{n + m - (r + s - 1)k}{(n + m - (r + s - 1)k) - (m - sk)} \binom{n - (r - 1)k}{k} \\ &= \sum_{k=0}^n \left[\underbrace{n + m - (r + s - 1)k \geq 0}_{\iff (n+m) \geq (r+s-1)k} \right] \binom{n + m - (r + s - 1)k}{(n + m - (r + s - 1)k) - (m - sk)} \binom{n - (r - 1)k}{k} \\ &= \sum_{k=0}^n \left[\underbrace{n + m - (r + s - 1)k \geq 0}_{\iff (n+m) \geq (r+s-1)k} \right] \binom{n + m - (r + s - 1)k}{n - (r - 1)k} \binom{n - (r - 1)k}{k} \\ &= \sum_{k=0}^n [n + m \geq (r + s - 1)k] \binom{n + m - (r + s - 1)k}{n - (r - 1)k} \binom{n - (r - 1)k}{k}. \end{aligned}$$

This solves Exercise 4 (b).

(c) We have $r = s$. Thus, diagonal steps have the form $(x, y) \mapsto (x + s, y + s)$. In geometric terms, this means that they are parallel to the $y = x$ diagonal (i.e., they are at an angle of 45° against the x -axis). In combinatorial terms, this means that they change the x -coordinate and the y -coordinate by the same amount.

¹⁰We don't need any $[n - (r - 1)k \geq 0]$ factor this time, because we already know that $n - (r - 1)k \geq 0$ at this point in our construction (in fact, we have just chosen $m - sk$ among $n + m - (r + s - 1)k$ steps; therefore, $m - sk \leq n + m - (r + s - 1)k$, so that $0 \leq (n + m - (r + s - 1)k) - (m - sk) = n - (r - 1)k$).

Now, let R be the map $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ that sends every point (x, y) to (y, x) . In geometric terms, R is the reflection across the $y = x$ diagonal. This map R is inverse to itself (i.e., it satisfies $R \circ R = \text{id}$), like any reflection.

If $\mathbf{p} = (v_0, v_1, \dots, v_k)$ is a path to (n, m) , then $(R(v_0), R(v_1), \dots, R(v_k))$ is a path to $R((n, m)) = (m, n)$ (indeed, every up-step (v_{i-1}, v_i) of \mathbf{p} yields a corresponding right-step $(R(v_{i-1}), R(v_i))$ in $(R(v_0), R(v_1), \dots, R(v_k))$; every right-step (v_{i-1}, v_i) of \mathbf{p} yields a corresponding up-step $(R(v_{i-1}), R(v_i))$ in $(R(v_0), R(v_1), \dots, R(v_k))$; and finally, every diagonal step (v_{i-1}, v_i) of \mathbf{p} yields a corresponding diagonal step $(R(v_{i-1}), R(v_i))$ in $(R(v_0), R(v_1), \dots, R(v_k))$ ¹¹). We denote this latter path by $\mathbf{R}(\mathbf{p})$. Thus, we have defined a map

$$\mathbf{R} : \{\text{paths to } (n, m)\} \rightarrow \{\text{paths to } (m, n)\}$$

(which sends every path $\mathbf{p} = (v_0, v_1, \dots, v_k)$ to $\mathbf{R}(\mathbf{p}) = (R(v_0), R(v_1), \dots, R(v_k))$). Similarly, we can define a map

$$\mathbf{R}' : \{\text{paths to } (m, n)\} \rightarrow \{\text{paths to } (n, m)\}$$

(which sends every path $\mathbf{p} = (v_0, v_1, \dots, v_k)$ to $\mathbf{R}'(\mathbf{p}) = (R(v_0), R(v_1), \dots, R(v_k))$). These maps \mathbf{R} and \mathbf{R}' are mutually inverse (since $R \circ R = \text{id}$), and thus are bijections. Hence, there is a bijection $\{\text{paths to } (n, m)\} \rightarrow \{\text{paths to } (m, n)\}$ (namely, \mathbf{R}). Thus,

$$\begin{aligned} |\{\text{paths to } (n, m)\}| &= |\{\text{paths to } (m, n)\}| \\ &= (\text{the number of all paths to } (m, n)) = d_{m,n} \end{aligned}$$

(by (20), applied to m and n instead of n and m). Hence,

$$d_{m,n} = |\{\text{paths to } (n, m)\}| = (\text{the number of all paths to } (n, m)) = d_{n,m}$$

(by (20) again). This solves Exercise 4 (c). \square

0.4. On inclusion/exclusion

0.4.1. The Principle of Inclusion and Exclusion

One version of the Principle of Inclusion and Exclusion is the following theorem (see, e.g., [Grinbe16, Theorem 3.42 and Theorem 3.43] or [Galvin17, Theorem 16.1 and (11)]):

Theorem 0.6. Let $n \in \mathbb{N}$. Let A_1, A_2, \dots, A_n be finite sets.

(a) We have

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

¹¹Here, we are using $r = s$.

(b) Let S be a finite set. Assume that each of A_1, A_2, \dots, A_n is a subset of S . Then,

$$\left| S \setminus \bigcup_{i=1}^n A_i \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|.$$

Here, the “empty” intersection $\bigcap_{i \in \emptyset} A_i$ is understood to mean the set S .

(See also [White10, Theorem 1] for a proof of Theorem 0.6 (b). It is easy to see that parts (a) and (b) of Theorem 0.6 are equivalent, because $\left| S \setminus \bigcup_{i=1}^n A_i \right| = |S| - \left| \bigcup_{i=1}^n A_i \right|$.)

Here is another way to write Theorem 0.6 (a):

Corollary 0.7. Let $n \in \mathbb{N}$. Let A_1, A_2, \dots, A_n be finite sets. Then,

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|.$$

Indeed, Corollary 0.7 is equivalent to Theorem 0.6 (a) because the k -tuples (i_1, i_2, \dots, i_k) of integers satisfying $k \in [n]$ and $1 \leq i_1 < i_2 < \dots < i_k \leq n$ are in bijection with the nonempty subsets I of $[n]$ (namely, the bijection sends the k -tuple (i_1, i_2, \dots, i_k) to the subset $\{i_1, i_2, \dots, i_k\}$), and under this bijection, the term $(-1)^{k-1} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$ corresponds to $(-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|$.

0.4.2. The exercise

Recall that if I is a set of real numbers, then $\min I$ stands for the minimum of I (that is, the smallest element of I). This is not always defined: Not every set of numbers has a minimum. But a **nonempty finite** set I of real numbers always has a minimum $\min I$. (Infinite sets might not have minima; e.g., the set $\left\{ \frac{1}{n} \mid n \text{ is a positive integer} \right\}$ does not, nor does \mathbb{Z} . Also, the empty set has no minimum.)

Exercise 5. Let n be a positive integer. Let a_1, a_2, \dots, a_n be n integers. Prove that

$$\begin{aligned} & \max \{a_1, a_2, \dots, a_n\} \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}. \end{aligned} \quad (22)$$

For example, if $n = 3$, then this says that

$$\begin{aligned} \max \{a_1, a_2, a_3\} &= \min \{a_1\} + \min \{a_2\} + \min \{a_3\} \\ &\quad - \min \{a_1, a_2\} - \min \{a_1, a_3\} - \min \{a_2, a_3\} \\ &\quad + \min \{a_1, a_2, a_3\}. \end{aligned}$$

[Hint: You can derive this from Corollary 0.7 by constructing n sets A_1, A_2, \dots, A_n such that $|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$, if the a_i are nonnegative. If some a_i are negative, a slight tweak is required. Alternatively, and perhaps more easily, there is a proof without using the Principle of Inclusion and Exclusion.

Note that (22) can be rewritten as

$$\max \{a_1, a_2, \dots, a_n\} = \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} \min \{a_i \mid i \in I\}. \quad (23)$$

It might be easier to prove this equivalent form.]

0.4.3. First solution

First solution to Exercise 5. Let us forget that we fixed a_1, a_2, \dots, a_n . We shall first prove the particular case of the exercise where all the a_i are assumed to be $\in \mathbb{N}$:

Observation 1: Let a_1, a_2, \dots, a_n be n elements of \mathbb{N} . Then,

$$\max \{a_1, a_2, \dots, a_n\} = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}.$$

[*Proof of Observation 1:* For each $i \in [n]$, we define a finite set A_i by $A_i = [a_i]$. Thus, for each $i \in [n]$, we have

$$A_i = [a_i] = \{g \in \{1, 2, 3, \dots\} \mid g \leq a_i\}.$$

Hence,

$$\begin{aligned} A_1 \cup A_2 \cup \dots \cup A_n &= \{g \in \{1, 2, 3, \dots\} \mid g \leq a_1\} \cup \{g \in \{1, 2, 3, \dots\} \mid g \leq a_2\} \\ &\quad \cup \dots \cup \{g \in \{1, 2, 3, \dots\} \mid g \leq a_n\} \\ &= \{g \in \{1, 2, 3, \dots\} \mid g \leq a_1 \text{ or } g \leq a_2 \text{ or } \dots \text{ or } g \leq a_n\} \\ &= \{g \in \{1, 2, 3, \dots\} \mid g \leq \max \{a_1, a_2, \dots, a_n\}\} \end{aligned}$$

(because for any $g \in \{1, 2, 3, \dots\}$, the statement “ $g \leq a_1$ or $g \leq a_2$ or \dots or $g \leq a_n$ ”

is equivalent to the statement “ $g \leq \max \{a_1, a_2, \dots, a_n\}$ ”). Thus,

$$\begin{aligned} & |A_1 \cup A_2 \cup \dots \cup A_n| \\ &= \left| \underbrace{\{g \in \{1, 2, 3, \dots\} \mid g \leq \max \{a_1, a_2, \dots, a_n\}\}}_{=\{1, 2, \dots, \max\{a_1, a_2, \dots, a_n\}\}} \right| \\ &= |\{1, 2, \dots, \max \{a_1, a_2, \dots, a_n\}\}| = \max \{a_1, a_2, \dots, a_n\}. \end{aligned} \quad (24)$$

A similar argument shows that every $k \in [n]$ and every k -tuple (i_1, i_2, \dots, i_k) of elements of $[n]$ satisfy

$$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \quad (25)$$

12.

Now, Corollary 0.7 yields

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|.$$

¹²Proof of (25): Let $k \in [n]$. Let (i_1, i_2, \dots, i_k) be a k -tuple of elements of $[n]$. Then, for each $p \in [k]$, we have

$$\begin{aligned} A_{i_p} &= [a_{i_p}] \quad (\text{by the definition of } A_{i_p}) \\ &= \{g \in \{1, 2, 3, \dots\} \mid g \leq a_{i_p}\}. \end{aligned}$$

Hence,

$$\begin{aligned} & A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k} \\ &= \{g \in \{1, 2, 3, \dots\} \mid g \leq a_{i_1}\} \cap \{g \in \{1, 2, 3, \dots\} \mid g \leq a_{i_2}\} \\ &\quad \cap \dots \cap \{g \in \{1, 2, 3, \dots\} \mid g \leq a_{i_k}\} \\ &= \{g \in \{1, 2, 3, \dots\} \mid g \leq a_{i_1} \text{ and } g \leq a_{i_2} \text{ and } \dots \text{ and } g \leq a_{i_k}\} \\ &= \{g \in \{1, 2, 3, \dots\} \mid g \leq \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}\} \end{aligned}$$

(because for any $g \in \{1, 2, 3, \dots\}$, the statement “ $g \leq a_{i_1}$ and $g \leq a_{i_2}$ and \dots and $g \leq a_{i_k}$ ” is equivalent to the statement “ $g \leq \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$ ”). Thus,

$$\begin{aligned} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| &= \left| \underbrace{\{g \in \{1, 2, 3, \dots\} \mid g \leq \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}\}}_{=\{1, 2, \dots, \min\{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}\}} \right| \\ &= |\{1, 2, \dots, \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}\}| = \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}. \end{aligned}$$

This proves (25).

Comparing this with (24), we find

$$\begin{aligned} \max \{a_1, a_2, \dots, a_n\} &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \underbrace{|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|}_{=\min\{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}} \\ & \quad \text{(by (25))} \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}. \end{aligned}$$

This proves Observation 1.]

From Observation 1, we can get the following corollary (whose usefulness we will soon see):

Observation 2: Let $b \in \mathbb{Z}$. Then,

$$\sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} b = b.$$

[*Proof of Observation 2:* Apply Observation 1 to $a_i = 1$. The result is

$$\begin{aligned} \max \{1, 1, \dots, 1\} &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \underbrace{\min \{1, 1, \dots, 1\}}_{=1} \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} 1. \end{aligned}$$

Compared with $\max \{1, 1, \dots, 1\} = 1$, this yields

$$1 = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} 1.$$

Multiplying both sides of this equality by b , we find

$$b = \left(\sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} 1 \right) b = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} b.$$

This proves Observation 2.]

Finally, we need the following elementary fact:

Observation 3: Let a_1, a_2, \dots, a_n be n integers. Then, there exists some $b \in \mathbb{Z}$ such that the n integers $a_1 + b, a_2 + b, \dots, a_n + b$ belong to \mathbb{N} .

[*Proof of Observation 3:* The set $\{a_1, a_2, \dots, a_n\}$ is a finite subset of \mathbb{Z} , and thus has a lower bound (since any finite subset of \mathbb{Z} has a lower bound). In other words,

there exists a $c \in \mathbb{Z}$ such that each element of $\{a_1, a_2, \dots, a_n\}$ is $\geq c$. Consider such a c .

Let $i \in [n]$. Then, each element of $\{a_1, a_2, \dots, a_n\}$ is $\geq c$; therefore, a_i is $\geq c$ (since a_i is an element of $\{a_1, a_2, \dots, a_n\}$). In other words, $a_i \geq c$. Hence, $a_i - c \geq 0$. In other words, $a_i + (-c) \geq 0$. Thus, $a_i + (-c) \in \mathbb{N}$ (since $a_i + (-c)$ is an integer).

Now, forget that we fixed i . We thus have shown that $a_i + (-c) \in \mathbb{N}$ for each $i \in [n]$. In other words, the n integers $a_1 + (-c), a_2 + (-c), \dots, a_n + (-c)$ belong to \mathbb{N} . Thus, there exists some $b \in \mathbb{Z}$ such that the n integers $a_1 + b, a_2 + b, \dots, a_n + b$ belong to \mathbb{N} (namely, $b = -c$). This proves Observation 3.]

We are now finally ready to solve the exercise. Let a_1, a_2, \dots, a_n be n integers. Observation 3 shows that there exists some $b \in \mathbb{Z}$ such that the n integers $a_1 + b, a_2 + b, \dots, a_n + b$ belong to \mathbb{N} . Consider such a b . Thus, Observation 1 (applied to $a_i + b$ instead of a_i) yields

$$\begin{aligned} & \max \{a_1 + b, a_2 + b, \dots, a_n + b\} \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \underbrace{\min \{a_{i_1} + b, a_{i_2} + b, \dots, a_{i_k} + b\}}_{=\min\{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} + b} \\ & \hspace{15em} \text{(because when several integers get shifted by } b, \\ & \hspace{15em} \text{their minimum also gets shifted by } b) \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (\min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} + b) \\ & \hspace{15em} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} + \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} b \\ &= \sum_{k=1}^n (-1)^{k-1} \left(\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} + \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} b \right) \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} + \underbrace{\sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} b}_{=b} \\ & \hspace{15em} \text{(by Observation 2)} \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} + b. \end{aligned}$$

Hence,

$$\begin{aligned} & \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \\ &= \underbrace{\max \{a_1 + b, a_2 + b, \dots, a_n + b\}}_{=\max\{a_1, a_2, \dots, a_n\} + b} - b \\ & \hspace{15em} \text{(because when several integers get shifted by } b, \\ & \hspace{15em} \text{their maximum also gets shifted by } b) \\ &= \max \{a_1, a_2, \dots, a_n\} + b - b = \max \{a_1, a_2, \dots, a_n\}. \end{aligned}$$

This solves Exercise 5. □

0.4.4. Second solution

Let me next show a different solution, which is somewhat similar to an argument I used in class (14 February 2018, 2nd proof of Corollary 3.3) – although it will be worded more economically (instead of splitting the sum into an “even” subsum and an “odd” subsum, I will keep it together and see how its terms cancel). Before the solution, let me state a lemma:

Lemma 0.8. Let S be a set. Let g be an element of S . Then, the map

$$\begin{aligned} \{J \subseteq S \mid g \notin J\} &\rightarrow \{J \subseteq S \mid g \in J\}, \\ K &\mapsto K \cup \{g\} \end{aligned}$$

is a bijection.

So Lemma 0.8 says that if g is an element of S , then those subsets of S that **don't** contain g are in 1-to-1 correspondence with those subsets of S that **do** contain g – and that this correspondence is given by adding g to the subset. This should be fairly clear. For the sake of completeness, here is a formalized version of this proof:

Proof of Lemma 0.8. Clearly, $\{g\}$ is a subset of S (since $g \in S$).

For each $K \in \{J \subseteq S \mid g \notin J\}$, we have $K \cup \{g\} \in \{J \subseteq S \mid g \in J\}$ ¹³. Thus, the map

$$\begin{aligned} \{J \subseteq S \mid g \notin J\} &\rightarrow \{J \subseteq S \mid g \in J\}, \\ K &\mapsto K \cup \{g\} \end{aligned}$$

is well-defined. Let us denote this map by α .

For each $L \in \{J \subseteq S \mid g \in J\}$, we have $L \setminus \{g\} \in \{J \subseteq S \mid g \notin J\}$ ¹⁴. Hence, the map

$$\begin{aligned} \{J \subseteq S \mid g \in J\} &\rightarrow \{J \subseteq S \mid g \notin J\}, \\ L &\mapsto L \setminus \{g\} \end{aligned}$$

is well-defined. Let us denote this map by β .

We have $\alpha \circ \beta = \text{id}$ ¹⁵ and $\beta \circ \alpha = \text{id}$ ¹⁶. Thus, the maps α and β are mutually inverse. Hence,

¹³*Proof.* Let $K \in \{J \subseteq S \mid g \notin J\}$. Thus, K is a subset of S satisfying $g \notin K$.

Now, both sets K and $\{g\}$ are subsets of S . Thus, their union $K \cup \{g\}$ is a subset of S as well.

Hence, $K \cup \{g\}$ is a subset of S satisfying $g \in K \cup \{g\}$ (since $g \in \{g\} \subseteq K \cup \{g\}$). In other words, $K \cup \{g\} \in \{J \subseteq S \mid g \in J\}$, qed.

¹⁴*Proof.* Let $L \in \{J \subseteq S \mid g \in J\}$. Thus, L is a subset of S satisfying $g \in L$.

Hence, $L \subseteq S$, so that $L \setminus \{g\} \subseteq L \subseteq S$. In other words, $L \setminus \{g\}$ is a subset of S . Furthermore, $g \notin L \setminus \{g\}$ (since $g \in \{g\}$). Hence, $L \setminus \{g\}$ is a subset of S satisfying $g \notin L \setminus \{g\}$. In other words, $L \setminus \{g\} \in \{J \subseteq S \mid g \notin J\}$, qed.

¹⁵*Proof.* Let $L \in \{J \subseteq S \mid g \in J\}$. Thus, L is a subset of S satisfying $g \in L$. Now, the definition of β shows that $\beta(L) = L \setminus \{g\}$. Furthermore, the definition of α shows that $\alpha(\beta(L)) = \underbrace{\beta(L)}_{=L \setminus \{g\}} \cup \{g\} = (L \setminus \{g\}) \cup \{g\} = L$ (since $g \in L$). Hence, $(\alpha \circ \beta)(L) = \alpha(\beta(L)) = L = \text{id}(L)$.

Forget that we fixed L . We thus have shown that $(\alpha \circ \beta)(L) = \text{id}(L)$ for each $L \in \{J \subseteq S \mid g \in J\}$. In other words, $\alpha \circ \beta = \text{id}$.

¹⁶*Proof.* Let $K \in \{J \subseteq S \mid g \notin J\}$. Thus, K is a subset of S satisfying $g \notin K$. Now, the definition

the map α is invertible, i.e., is a bijection. In other words, the map

$$\begin{aligned} \{J \subseteq S \mid g \notin J\} &\rightarrow \{J \subseteq S \mid g \in J\}, \\ K &\mapsto K \cup \{g\} \end{aligned}$$

is a bijection (since this map is precisely α). This proves Lemma 0.8. □

Lemma 0.8 helps us find lots of cancellations in certain kinds of sums:

Lemma 0.9. Let S be a finite set. Let g be an element of S . For any nonempty subset I of S , let b_I be a real number. Assume that for every subset K of S satisfying $g \notin K$ and $K \neq \emptyset$, we have

$$b_{K \cup \{g\}} = b_K. \tag{26}$$

Then,

$$\sum_{\substack{I \subseteq S; \\ I \neq \emptyset}} (-1)^{|I|-1} b_I = b_{\{g\}}. \tag{27}$$

We will later solve Exercise 5 (more precisely, prove (23)) by applying Lemma 0.8 to $S = [n]$ and $b_I = \min \{a_i \mid i \in I\}$ (and a carefully chosen $g \in S$).

Proof of Lemma 0.9. Each subset I of S satisfies either $g \in I$ or $g \notin I$ (but not both).

Hence, the sum $\sum_{\substack{I \subseteq S; \\ I \neq \emptyset}} (-1)^{|I|-1} b_I$ can be split as follows:

$$\sum_{\substack{I \subseteq S; \\ I \neq \emptyset}} (-1)^{|I|-1} b_I = \sum_{\substack{I \subseteq S; \\ I \neq \emptyset; \\ g \in I}} (-1)^{|I|-1} b_I + \sum_{\substack{I \subseteq S; \\ I \neq \emptyset; \\ g \notin I}} (-1)^{|I|-1} b_I. \tag{28}$$

For a subset I of S , the condition “ $I \neq \emptyset$ and $g \in I$ ” is equivalent to the condition “ $g \in I$ ”¹⁷. Hence, the summation sign $\sum_{\substack{I \subseteq S; \\ I \neq \emptyset; \\ g \in I}}$ can be replaced by the simpler

of α shows that $\alpha(K) = K \cup \{g\}$. Furthermore, the definition of β shows that $\beta(\alpha(K)) = \underbrace{\alpha(K) \setminus \{g\}}_{=K \cup \{g\}} = (K \cup \{g\}) \setminus \{g\} = K$ (since $g \notin K$). Hence, $(\beta \circ \alpha)(K) = \beta(\alpha(K)) = \text{id}(K)$.

Forget that we fixed K . We thus have shown that $(\beta \circ \alpha)(K) = \text{id}(K)$ for each $K \in \{J \subseteq S \mid g \notin J\}$. In other words, $\beta \circ \alpha = \text{id}$.

¹⁷*Proof.* Let I be a subset of S . If $g \in I$, then $(I \neq \emptyset \text{ and } g \in I)$ (because $g \in I$ clearly implies $I \neq \emptyset$). Thus, we have the logical implication $(g \in I) \implies (I \neq \emptyset \text{ and } g \in I)$. Combining this with the implication $(I \neq \emptyset \text{ and } g \in I) \implies (g \in I)$ (which is obvious), we obtain the equivalence $(I \neq \emptyset \text{ and } g \in I) \iff (g \in I)$. In other words, the condition “ $I \neq \emptyset$ and $g \in I$ ” is equivalent to the condition “ $g \in I$ ”, qed.

summation sign $\sum_{\substack{I \subseteq S; \\ g \in I}}$. Thus,

$$\begin{aligned} \sum_{\substack{I \subseteq S; \\ I \neq \emptyset; \\ g \in I}} (-1)^{|I|-1} b_I &= \sum_{\substack{I \subseteq S; \\ g \in I}} (-1)^{|I|-1} b_I \\ &= \underbrace{\sum_{I \in \{J \subseteq S \mid g \in J\}}}_{\substack{I \in \{J \subseteq S \mid g \in J\}}} (-1)^{|I|-1} b_I. \end{aligned} \quad (29)$$

But Lemma 0.8 shows that the map

$$\begin{aligned} \{J \subseteq S \mid g \notin J\} &\rightarrow \{J \subseteq S \mid g \in J\}, \\ K &\mapsto K \cup \{g\} \end{aligned}$$

is a bijection. Hence, we can substitute $K \cup S$ for I in the sum $\sum_{I \in \{J \subseteq S \mid g \in J\}} (-1)^{|I|-1} b_I$.

We thus obtain

$$\begin{aligned}
 & \sum_{I \in \{J \subseteq S \mid g \in J\}} (-1)^{|I|-1} b_I \\
 &= \sum_{\underbrace{K \in \{J \subseteq S \mid g \notin J\}}_{= \sum_{\substack{K \subseteq S; \\ g \notin K}}} } (-1)^{|K \cup \{g\}|-1} b_{K \cup \{g\}} \\
 &= \sum_{\substack{K \subseteq S; \\ g \notin K}} (-1)^{|K \cup \{g\}|-1} b_{K \cup \{g\}} = \sum_{\substack{K \subseteq S; \\ g \notin K}} \underbrace{(-1)^{|K \cup \{g\}|-1}}_{= (-1)^{(|K|+1)-1}} b_{K \cup \{g\}} \\
 & \hspace{10em} \text{(since } |K \cup \{g\}| = |K|+1 \text{ because } g \notin K) \\
 &= \sum_{\substack{K \subseteq S; \\ g \notin K}} \underbrace{(-1)^{(|K|+1)-1}}_{= (-1)^{|K|}} b_{K \cup \{g\}} = \sum_{\substack{K \subseteq S; \\ g \notin K}} (-1)^{|K|} b_{K \cup \{g\}} \\
 &= \underbrace{(-1)^{|\emptyset|}}_{=1} \underbrace{b_{\emptyset \cup \{g\}}}_{=b_{\{g\}}} + \sum_{\substack{K \subseteq S; \\ g \notin K; \\ K \neq \emptyset}} (-1)^{|K|} b_{K \cup \{g\}} \\
 & \hspace{10em} \text{(since } |\emptyset|=0 \text{) (since } \emptyset \cup \{g\} = \{g\} \text{)} \\
 & \hspace{10em} \left(\begin{array}{l} \text{here, we have split off the addend for } K = \emptyset \\ \text{from the sum, since } \emptyset \text{ is a subset of } S \text{ satisfying } g \notin \emptyset \end{array} \right) \\
 &= b_{\{g\}} + \sum_{\substack{K \subseteq S; \\ g \notin K; \\ K \neq \emptyset}} (-1)^{|K|} \underbrace{b_{K \cup \{g\}}}_{=b_K} = b_{\{g\}} + \sum_{\substack{K \subseteq S; \\ g \notin K; \\ K \neq \emptyset}} (-1)^{|K|} b_K \\
 &= b_{\{g\}} + \sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} (-1)^{|I|} b_I \tag{30}
 \end{aligned}$$

(here, we have renamed the summation index K as I). Now, (28) becomes

$$\begin{aligned}
 \sum_{\substack{I \subseteq S; \\ I \neq \emptyset}} (-1)^{|I|-1} b_I &= \underbrace{\sum_{\substack{I \subseteq S; \\ I \neq \emptyset; \\ g \in I}} (-1)^{|I|-1} b_I}_{\substack{= \sum_{I \in \{J \subseteq S \mid g \in J\}} \\ \text{(by (29))}}} (-1)^{|I|-1} b_I + \underbrace{\sum_{\substack{I \subseteq S; \\ I \neq \emptyset; \\ g \notin I}} (-1)^{|I|-1} b_I}_{= -(-1)^{|I|}} \\
 &= \sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} (-1)^{|I|-1} b_I + \sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} \left(-(-1)^{|I|} \right) b_I \\
 &= \underbrace{\sum_{I \in \{J \subseteq S \mid g \in J\}} (-1)^{|I|-1} b_I}_{= b_{\{g\}} + \sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} (-1)^{|I|} b_I} + \underbrace{\sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} \left(-(-1)^{|I|} \right) b_I}_{= - \sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} (-1)^{|I|} b_I} \\
 &= b_{\{g\}} + \sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} (-1)^{|I|} b_I + \left(- \sum_{\substack{I \subseteq S; \\ g \notin I; \\ I \neq \emptyset}} (-1)^{|I|} b_I \right) = b_{\{g\}}.
 \end{aligned}$$

This proves Lemma 0.9. □

Second solution to Exercise 5. We shall prove (23).

The set $\{a_1, a_2, \dots, a_n\}$ is nonempty (since n is positive) and finite. Thus, the set $\{a_1, a_2, \dots, a_n\}$ is a nonempty finite set of integers, and therefore has a maximum. In other words, there exists some $g \in [n]$ such that $a_g = \max \{a_1, a_2, \dots, a_n\}$. Consider such a g . (There may be several choices for g , but we choose one.)

For every subset K of $[n]$ satisfying $g \notin K$ and $K \neq \emptyset$, we have

$$\min \{a_i \mid i \in K \cup \{g\}\} = \min \{a_i \mid i \in K\}. \tag{31}$$

[*Proof of (31):* Roughly speaking, all that (31) is claiming is that if you add a_g to the set $\{a_i \mid i \in K\}$, then the minimum of this set does not change. This should be fairly clear, since a_g (being the maximum of $\{a_1, a_2, \dots, a_n\}$) is \geq to all elements of the set $\{a_i \mid i \in K\}$, and thus cannot pull the minimum of this set down. For the sake of completeness, let me give a fully formalized version of this argument:

Let K be a subset of $[n]$ satisfying $g \notin K$ and $K \neq \emptyset$. The set K is nonempty (since $K \neq \emptyset$) and finite; thus, $\min \{a_i \mid i \in K\}$ is well-defined.

Clearly, $\min \{a_i \mid i \in K\}$ is an element of the set $\{a_i \mid i \in K\}$. In other words, $\min \{a_i \mid i \in K\} = a_j$ for some $j \in K$. Consider this j .

The number a_j is the minimum of the set $\{a_i \mid i \in K\}$ (since $a_j = \min \{a_i \mid i \in K\}$), and thus is \leq to each element of this set. In other words,

$$a_j \leq a_i \quad \text{for each } i \in K. \tag{32}$$

But $a_g = \max \{a_1, a_2, \dots, a_n\} \geq a_p$ for each $p \in [n]$. Applying this to $p = j$, we obtain $a_g \geq a_j$ (since $j \in K \subseteq [n]$). Thus, $a_j \leq a_i$ for each $i \in K \cup \{g\}$ ¹⁸.

Also, $j \in K \subseteq K \cup \{g\}$; hence, a_j is an element of the set $\{a_i \mid i \in K \cup \{g\}\}$. Moreover, this element a_j is \leq to each element of this set (because $a_j \leq a_i$ for each $i \in K \cup \{g\}$). Hence, this element a_j is the minimum of this set $\{a_i \mid i \in K \cup \{g\}\}$. In other words, $a_j = \min \{a_i \mid i \in K \cup \{g\}\}$. Hence,

$$\min \{a_i \mid i \in K \cup \{g\}\} = a_j = \min \{a_i \mid i \in K\}.$$

This proves (31).]

Thus, Lemma 0.9 (applied to $S = [n]$ and $b_I = \min \{a_i \mid i \in I\}$) yields that

$$\begin{aligned} \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} \min \{a_i \mid i \in I\} &= \min \underbrace{\{a_i \mid i \in \{g\}\}}_{=\{a_g\}} = \min \{a_g\} \\ &= a_g = \max \{a_1, a_2, \dots, a_n\}. \end{aligned}$$

This proves (23). Thus, Exercise 5 is solved once again. \square

Remark 0.10. The above Second solution to Exercise 5 shows that Exercise 5 holds more generally if a_1, a_2, \dots, a_n are real numbers rather than integers. This is not immediately obvious from the First solution.

0.4.5. Third solution

Another solution of Exercise 5 relies on a few lemmas. First, two simple identities for binomial coefficients:

Proposition 0.11. We have

$$\binom{m}{n} = 0$$

for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m < n$.

This identity we have already seen many times.

Lemma 0.12. Let $n \in \mathbb{N}$. Then,

$$\sum_{h=0}^n (-1)^h \binom{n}{h} = [n = 0].$$

We have proven Lemma 0.12 in class (Corollary 3.3 in 14 February 2018). In a nutshell, it can be proven combinatorially, but it can also be obtained as a particular case of the binomial formula for $(x + y)^n$ when x is set to -1 and y is set to 1 .

¹⁸*Proof.* Let $i \in K \cup \{g\}$. We must prove that $a_j \leq a_i$.

If $i \in K$, then this follows from (32). Thus, we WLOG assume that we don't have $i \in K$. Hence, we have $i \notin K$. Combining this with $i \in K \cup \{g\}$, we obtain $i \in (K \cup \{g\}) \setminus K \subseteq \{g\}$. In other words, $i = g$. Hence, $a_i = a_g \geq a_j$, so that $a_j \leq a_i$. This completes our proof.

Lemma 0.13. Let $n \in \mathbb{N}$. Let $k \in [n]$. Then,

$$\sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1} = [k = n].$$

Example 0.14. For $n = 4$ and $k = 2$, Lemma 0.13 says that

$$(-1)^{|\{2\}|-1} + (-1)^{|\{2,3\}|-1} + (-1)^{|\{2,4\}|-1} + (-1)^{|\{2,3,4\}|-1} = [2 = 4]$$

(because the subsets I of $[4]$ satisfying $I \neq \emptyset$ and $\min I = 2$ are precisely the four subsets $\{2\}, \{2, 3\}, \{2, 4\}, \{2, 3, 4\}$). And this is easily verified to be true (since both sides are 0).

Proof of Lemma 0.13 (sketched). We first prove the following auxiliary observation:

Observation 1: Let $g \in \mathbb{N}$. Then,

$$|\{I \subseteq [n] \mid I \neq \emptyset \text{ and } \min I = k \text{ and } |I| = g\}| = \binom{n-k}{g-1}.$$

[*Proof of Observation 1:* Fix $g \in \mathbb{N}$. How do we construct a g -element subset I of $[n]$ satisfying $I \neq \emptyset$ and $\min I = k$ (that is, k is the smallest element of I) ?

One simple way is the following: Since k has to be an element of I , we only need to choose the remaining $g - 1$ elements of I . These $g - 1$ elements must be greater than k (since we want $\min I$ to be k); in other words, they must belong to $\{k + 1, k + 2, \dots, n\}$. So we just need to choose $g - 1$ elements from the $(n - k)$ -element set $\{k + 1, k + 2, \dots, n\}$. This can be done in $\binom{n-k}{g-1}$ ways¹⁹.

Clearly, this construction yields every g -element subset I of $[n]$ satisfying $I \neq \emptyset$ and $\min I = k$ in exactly one way. Thus, the number of g -element subsets I of $[n]$ satisfying $I \neq \emptyset$ and $\min I = k$ is precisely $\binom{n-k}{g-1}$. In other words, the number of all $I \subseteq [n]$ satisfying $I \neq \emptyset$ and $\min I = k$ and $|I| = g$ is precisely $\binom{n-k}{g-1}$. In other words,

$$|\{I \subseteq [n] \mid I \neq \emptyset \text{ and } \min I = k \text{ and } |I| = g\}| = \binom{n-k}{g-1}.$$

Observation 1 is thus proven.]

¹⁹Check that this is true even if $g = 0$.

Now, let us split the sum $\sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1}$ according to the size $|I|$ of I (which must be a positive integer, because the sum includes only those subsets I that satisfy $I \neq \emptyset$). We thus obtain

$$\begin{aligned}
 & \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1} \\
 &= \sum_{g \geq 1} \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k; \\ |I| = g}} \underbrace{(-1)^{|I|-1}}_{=(-1)^{g-1}} \\
 &= \sum_{g \geq 1} \underbrace{\sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k; \\ |I| = g}} (-1)^{g-1}}_{=|\{I \subseteq [n] \mid I \neq \emptyset \text{ and } \min I = k \text{ and } |I| = g\}| \cdot (-1)^{g-1}} \\
 &= \sum_{g \geq 1} \underbrace{|\{I \subseteq [n] \mid I \neq \emptyset \text{ and } \min I = k \text{ and } |I| = g\}|}_{= \binom{n-k}{g-1} \text{ (by Observation 1)}} \cdot (-1)^{g-1} \\
 &= \sum_{g \geq 1} \binom{n-k}{g-1} \cdot (-1)^{g-1} = \sum_{h \geq 0} \binom{n-k}{h} \cdot (-1)^h \\
 & \quad \text{(here, we have substituted } h \text{ for } g-1 \text{ in the sum)} \\
 &= \sum_{h \geq 0} (-1)^h \binom{n-k}{h} \\
 &= \underbrace{\sum_{h=0}^{n-k} (-1)^h \binom{n-k}{h}}_{= [n-k=0] \text{ (by Lemma 0.12, applied to } n-k \text{ instead of } n)} + \sum_{h=n-k+1}^{\infty} (-1)^h \underbrace{\binom{n-k}{h}}_{=0} \\
 & \quad \text{(by Proposition 0.11, applied to } n-k \text{ and } h \text{ instead of } m \text{ and } n \text{ (since } n-k < h))} \\
 &= \left[\underbrace{n-k=0}_{\iff (n=k)} \right] + \underbrace{\sum_{h=n-k+1}^{\infty} (-1)^h 0}_{=0} = [n = k].
 \end{aligned}$$

This proves Lemma 0.13. □

Third solution to Exercise 5 (sketched). We want to prove the identity (23). This identity clearly does not change when the numbers a_1, a_2, \dots, a_n are permuted (because

when this happens, the addends $(-1)^{|I|-1} \min \{a_i \mid i \in I\}$ on the right hand side get permuted as well, while the left hand side $\max \{a_1, a_2, \dots, a_n\}$ is preserved). Hence, we WLOG assume that $a_1 \leq a_2 \leq \dots \leq a_n$ (because we can always achieve this by permuting the numbers a_1, a_2, \dots, a_n : this is called sorting). Hence, each nonempty subset I of $[n]$ satisfies

$$\min \{a_i \mid i \in I\} = a_{\min I}. \tag{33}$$

(For example, $\min \{a_3, a_5, a_6\} = a_3 = a_{\min\{3,5,6\}}$.)

Now,

$$\begin{aligned} & \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} \underbrace{\min \{a_i \mid i \in I\}}_{\substack{= a_{\min I} \\ \text{(by (33))}}} \\ &= \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} a_{\min I} = \sum_{k \in [n]} \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1} \underbrace{a_{\min I}}_{\substack{= a_k \\ \text{(since } \min I = k)}} \\ & \quad \text{(here, we have split the sum according to the value of } \min I) \\ &= \sum_{k=1}^n \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1} a_k = \sum_{k=1}^n a_k \underbrace{\sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1}}_{\substack{= [k=n] \\ \text{(by Lemma 0.13)}}} \\ &= \sum_{k=1}^n a_k [k=n] = \sum_{k=1}^{n-1} a_k \underbrace{[k=n]}_{=0} + a_n \underbrace{[n=n]}_{=1} \\ & \quad \text{(since we don't have } k=n \text{ (because } k \leq n-1 < n)) \\ &= \underbrace{\sum_{k=1}^{n-1} a_k 0}_{=0} + a_n = a_n = \max \{a_1, a_2, \dots, a_n\} \quad \text{(since } a_1 \leq a_2 \leq \dots \leq a_n). \end{aligned}$$

This proves (23). Thus, Exercise 5 is solved. □

Remark 0.15. The above Third solution to Exercise 5 shows that Exercise 5 holds more generally if a_1, a_2, \dots, a_n are real numbers rather than integers. This is not immediately obvious from the First solution.

A fourth solution of Exercise 5 can be done by induction on n .

0.5. Not-quite-derangements

Exercise 6. Let n be a positive integer. An *nqd* (“not-quite-derangement”) of $[n]$ shall denote a permutation σ of $[n]$ such that every $i \in [n-1]$ satisfies $\sigma(i) \neq i+1$. Prove that the number of nqds of $[n]$ is

$$(n-1)! \sum_{k=0}^{n-1} (-1)^k \cdot \frac{n-k}{k!}.$$

This is similar to the formula, proven in [Galvin17, §16], which says that the number of derangements of $[n]$ is $n! \sum_{k=0}^n (-1)^k \cdot \frac{1}{k!}$. Unsurprisingly, the solution to Exercise 6 is also similar to the proof of the latter formula for the number of derangements of $[n]$.

In preparation for solving Exercise 6, let us restate Theorem 0.6 (b) as follows:

Lemma 0.16. Let $k \in \mathbb{N}$. Let S be a finite set. Let A_1, A_2, \dots, A_k be k subsets of S . Then,

$$\left| S \setminus \bigcup_{i=1}^k A_i \right| = \sum_{I \subseteq [k]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|.$$

Here, the “empty” intersection $\bigcap_{i \in \emptyset} A_i$ is understood to mean the set S .

Proof of Lemma 0.16. Theorem 0.6 (b) (applied to $n = k$) yields

$$\left| S \setminus \bigcup_{i=1}^k A_i \right| = \sum_{I \subseteq [k]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|.$$

This proves Lemma 0.16. □

We furthermore recall the following fact:

Proposition 0.17. We have

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

for any $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m \geq n$.

Next, let us show a simple identity for binomial coefficients:

Lemma 0.18. Let n be a positive integer. Let $k \in \{0, 1, \dots, n-1\}$. Then,

$$\binom{n-1}{k} (n-k)! = (n-1)! \cdot \frac{n-k}{k!}.$$

Proof of Lemma 0.18. We have $k \in \{0, 1, \dots, n-1\} \subseteq \mathbb{N}$, and $k \leq n-1$ (since $k \in \{0, 1, \dots, n-1\}$). Thus, $n-1 \geq k \geq 0$ (since $k \in \mathbb{N}$), so that $n-1 \in \mathbb{N}$. Now, Proposition 0.17 (applied to $n-1$ and k instead of m and n) yields

$$\binom{n-1}{k} = \frac{(n-1)!}{k!((n-1)-k)!} = \frac{(n-1)!}{k!((n-k)-1)!}$$

(since $(n-1) - k = (n-k) - 1$).

But $n-k \geq 1$ (since $n-1 \geq k$) and thus $(n-k)! = (n-k) \cdot ((n-k)-1)!$. Hence,

$$\begin{aligned} & \frac{\binom{n-1}{k}}{(n-1)!} = \frac{\underbrace{(n-k)!}_{=(n-k) \cdot ((n-k)-1)!}}{k!((n-k)-1)!} \\ & = \frac{(n-1)!}{k!((n-k)-1)!} (n-k) \cdot ((n-k)-1)! = (n-1)! \cdot \frac{n-k}{k!}. \end{aligned}$$

This proves Lemma 0.18. □

Next, we shall count permutations σ of $[n]$ that **do** satisfy $\sigma(i) = i+1$ for certain values of $i \in [n]$ (so, in a sense, the opposite of nqds). Specifically, we shall need the following lemma:

Lemma 0.19. Let $n \in \mathbb{N}$. Let I be a subset of $[n-1]$. Then,

$$|\{\sigma \in S_n \mid \sigma(i) = i+1 \text{ for all } i \in I\}| = (n - |I|)!.$$

In order to prove this lemma, let us generalize it a bit:

Lemma 0.20. Let $n \in \mathbb{N}$. Let I be a subset of $[n]$. Let h_i be an element of $[n]$ for each $i \in I$. Assume that the h_i for different $i \in I$ are distinct. Then,

$$|\{\sigma \in S_n \mid \sigma(i) = h_i \text{ for all } i \in I\}| = (n - |I|)!.$$

Proof of Lemma 0.20 (sketched). We say that a permutation $\sigma \in S_n$ is *fine* if it satisfies $(\sigma(i) = h_i \text{ for all } i \in I)$. (Keep in mind that I is fixed here, so we don't need to mention it every time.)

Let $(g_1, g_2, \dots, g_{n-|I|})$ be the list of all the $n - |I|$ elements of the set $[n] \setminus I$ in some arbitrarily chosen order (with no repetitions).

How can we construct a fine permutation $\sigma \in S_n$? One way to construct an arbitrary permutation in S_n is to choose its values one by one, each time choosing a value that has not already been chosen (to ensure that the values are distinct). We

are going to choose the values of $\sigma \in S_n$ in a specific order: namely, first we shall choose the values $\sigma(i)$ at elements i of I , and then we will choose the remaining values $\sigma(g_1), \sigma(g_2), \dots, \sigma(g_{n-|I|})$.

Here is the algorithm we will use to construct a fine permutation $\sigma \in S_n$:

- For each $i \in I$, we choose the value $\sigma(i)$ to be h_i (since we want σ to be fine). This is allowed, because the h_i for different $i \in I$ are distinct (so we are always choosing a value that has not already been chosen).

Notice that we have only 1 option at this step (because $\sigma(i)$ must be h_i for each $i \in I$).

- It remains to choose the values $\sigma(j)$ of σ on all the elements $j \in [n] \setminus I$. We do this step by step: First, we choose $\sigma(g_1)$; then, we choose $\sigma(g_2)$; then, we choose $\sigma(g_3)$, and so on. Here, we have $n - |I|$ options when choosing $\sigma(g_1)$ (because $\sigma(g_1)$ can be any element of $[n]$ except for the $|I|$ already chosen values of σ); then, we have $n - |I| - 1$ options when choosing $\sigma(g_2)$ (because $\sigma(g_2)$ can be any element of $[n]$ except for the $|I| + 1$ already chosen values of σ); then, we have $n - |I| - 2$ options when choosing $\sigma(g_3)$ at the next step, and so on. Thus, we have

$$\begin{aligned} & (n - |I|)(n - |I| - 1) \cdots (n - |I| - (n - |I| - 1)) \\ &= (n - |I|)(n - |I| - 1) \cdots 1 = (n - |I|)! \end{aligned}$$

options altogether.

Hence, this algorithm allows for $1 \cdot (n - |I|)! = (n - |I|)!$ many choices. Since this algorithm constructs each fine permutation $\sigma \in S_n$ exactly once, we thus conclude that the number of fine permutations $\sigma \in S_n$ is $(n - |I|)!$. Hence,

$$\begin{aligned} (n - |I|)! &= (\text{the number of fine permutations } \sigma \in S_n) \\ &= (\text{the number of } \sigma \in S_n \text{ such that } (\sigma(i) = h_i \text{ for all } i \in I)) \\ &\quad (\text{by the definition of "fine permutations"}) \\ &= |\{\sigma \in S_n \mid \sigma(i) = h_i \text{ for all } i \in I\}|. \end{aligned}$$

This proves Lemma 0.20. □

Proof of Lemma 0.19 (sketched). We have $I \subseteq [n - 1] \subseteq [n]$. For each $i \in I$, we have $i + 1 \in [n]$ ²⁰. Moreover, the $i + 1$ for different $i \in I$ are distinct. Hence, Lemma 0.20 (applied to $h_i = i + 1$) shows that $|\{\sigma \in S_n \mid \sigma(i) = i + 1 \text{ for all } i \in I\}| = (n - |I|)!$. This proves Lemma 0.19. □

Solution to Exercise 6 (sketched). For each $i \in [n - 1]$, we define a subset A_i of S_n by

$$A_i = \{\sigma \in S_n \mid \sigma(i) = i + 1\}. \quad (34)$$

²⁰*Proof.* Let $i \in I$. Thus, $i \in I \subseteq [n - 1] = \{1, 2, \dots, n - 1\}$, so that $i + 1 \in \{2, 3, \dots, n\} \subseteq [n]$, qed.

Then, for each subset I of $[n-1]$, we have

$$\begin{aligned} \bigcap_{i \in I} A_i &= \bigcap_{i \in I} \{\sigma \in S_n \mid \sigma(i) = i+1\} && \text{(by (34))} \\ &= \{\sigma \in S_n \mid \sigma(i) = i+1 \text{ for all } i \in I\} \end{aligned}$$

and therefore

$$\left| \bigcap_{i \in I} A_i \right| = |\{\sigma \in S_n \mid \sigma(i) = i+1 \text{ for all } i \in I\}| = (n - |I|)! \quad (35)$$

(by Lemma 0.19).

On the other hand, recall that an nqd of $[n]$ means a permutation σ of $[n]$ such that every $i \in [n-1]$ satisfies $\sigma(i) \neq i+1$. Thus,

$$\begin{aligned} &\{\text{nqds of } [n]\} \\ &= \{\text{permutations } \sigma \text{ of } [n] \text{ such that every } i \in [n-1] \text{ satisfies } \sigma(i) \neq i+1\} \\ &= \left\{ \sigma \in S_n \mid \underbrace{\sigma(i) \neq i+1 \text{ for all } i \in [n-1]}_{\Leftrightarrow (\text{not } (\sigma(i)=i+1 \text{ for some } i \in [n-1]))} \right\} \\ &= \{\sigma \in S_n \mid \text{not } (\sigma(i) = i+1 \text{ for some } i \in [n-1])\} \\ &= S_n \setminus \underbrace{\{\sigma \in S_n \mid \sigma(i) = i+1 \text{ for some } i \in [n-1]\}}_{= \bigcup_{i \in [n-1]} \{\sigma \in S_n \mid \sigma(i) = i+1\}} \\ &= S_n \setminus \underbrace{\bigcup_{i \in [n-1]} \{\sigma \in S_n \mid \sigma(i) = i+1\}}_{= A_i \text{ (by (34))}} = S_n \setminus \bigcup_{i=1}^{n-1} A_i. \end{aligned}$$

Hence,

$$\begin{aligned}
 & |\{\text{nqds of } [n]\}| \\
 &= \left| S_n \setminus \bigcup_{i=1}^{n-1} A_i \right| = \sum_{I \subseteq [n-1]} (-1)^{|I|} \underbrace{\left| \bigcap_{i \in I} A_i \right|}_{=(n-|I|)! \text{ (by (35))}} \\
 &\quad \text{(by Lemma 0.16, applied to } S = S_n \text{ and } k = n - 1) \\
 &= \sum_{I \subseteq [n-1]} (-1)^{|I|} (n - |I|)! = \sum_{k=0}^{n-1} \sum_{\substack{I \subseteq [n-1]; \\ |I|=k}} \underbrace{(-1)^{|I|}}_{=(-1)^k \text{ (since } |I|=k)} \underbrace{(n - |I|)!}_{=(n-k)! \text{ (since } |I|=k)} \\
 &\quad = \sum_{k=0}^{n-1} \sum_{\substack{I \subseteq [n-1]; \\ |I|=k}} (-1)^k (n - k)! \\
 &\quad \quad \quad \underbrace{= |\{I \subseteq [n-1] \mid |I|=k\}| \cdot (-1)^k (n-k)!} \\
 &= \sum_{k=0}^{n-1} \underbrace{|\{I \subseteq [n-1] \mid |I|=k\}|}_{=(\text{the number of all } k\text{-element subsets } I \text{ of } [n-1])} \cdot (-1)^k (n - k)! \\
 &\quad \quad \quad = \binom{n-1}{k} \quad \text{(since } n-1 \in \mathbb{N}) \\
 &= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot (-1)^k (n - k)! = \sum_{k=0}^{n-1} (-1)^k \underbrace{\binom{n-1}{k} (n - k)!}_{=(n-1)! \cdot \frac{n-k}{k!} \text{ (by Lemma 0.18)}} \\
 &= \sum_{k=0}^{n-1} (-1)^k (n - 1)! \cdot \frac{n - k}{k!} = (n - 1)! \sum_{k=0}^{n-1} (-1)^k \cdot \frac{n - k}{k!}.
 \end{aligned}$$

In other words, the number of nqds of $[n]$ is $(n - 1)! \sum_{k=0}^{n-1} (-1)^k \cdot \frac{n - k}{k!}$. This solves Exercise 6. □

0.6. Socks

Exercise 7. Let n and s be two even positive integers. Let q and r be the quotient and the remainder of division of n by s . (Thus, $q \in \mathbb{Z}$, $r \in \{0, 1, \dots, s-1\}$ and $n = qs + r$.)

Assume that n socks are hanging on a clothesline, with $n/2$ of these socks being black and the remaining $n/2$ white.

A *balanced window* will mean a choice of s consecutive socks on the clothesline such that $s/2$ of these socks are black and the remaining $s/2$ are white.

(a) If $r < 2q$, then show that there is a balanced window.

(b) If $s \leq 2q + r$, then show that there is a balanced window.

[**Hint:** Number the socks by $1, 2, \dots, n$ in the order in which they appear on the clothesline. For each $i \in [n - s + 1]$, define the integer

$$b_i = (\text{the number of black socks among socks } i, i+1, \dots, i+s-1) - s/2.$$

Proceed as in class, and take a look at the last r socks on the clothesline (i.e., those not counted in $b_1, b_{s+1}, b_{2s+1}, \dots, b_{(q-1)s+1}$). For part (b), take a closer look at the last s socks on the clothesline.]

(In class (Example 2.13 in classwork from 7 February 2018), we mostly considered the case when $n = 30$ and $s = 10$; this falls under the situation of part (a). For an example of part (b), try $n = 26$ and $s = 10$.)

Remark 0.21. A converse can also be shown: If neither $r < 2q$ nor $s \leq 2q + r$ holds, then one can place n socks ($n/2$ black, $n/2$ white) on a clothesline in such a way that no balanced window exists. (This observation, and part (b) of the exercise, are due to Daniel Harrer.)

Our solution to Exercise 7 relies on the *discrete continuity principle* (Lemma 2.14 in classwork from 7 February 2018):

Lemma 0.22. Let w be a positive integer. Let b_1, b_2, \dots, b_w be w nonzero integers. Assume that $b_1 \geq 0$. Assume furthermore that

$$|b_{i+1} - b_i| \leq 1 \quad \text{for all } i \in [w-1]. \quad (36)$$

Then, $b_i > 0$ for all $i \in [w]$.

Proof of Lemma 0.22. We claim that

$$b_i > 0 \quad \text{for all } i \in [w]. \quad (37)$$

We shall prove (37) by induction on i :

Induction base: We have $b_1 \geq 0$ (by assumption). But b_1 is a nonzero integer (since b_1, b_2, \dots, b_w are w nonzero integers). Thus, $b_1 \neq 0$. Combining this with $b_1 \geq 0$, we obtain $b_1 > 0$. In other words, (37) holds for $i = 1$. This completes the induction base.

Induction step: Let $j \in [w-1]$. Assume that (37) holds for $i = j$. We must then prove that (37) holds for $i = j+1$.

We have assumed that (37) holds for $i = j$. In other words, $b_j > 0$. Hence, $b_j \geq 1$ (since b_j is an integer).

But (36) (applied to $i = j$) yields $|b_{j+1} - b_j| \leq 1$. But every $x \in \mathbb{Z}$ satisfies $|x| \geq -x$. Applying this to $x = b_{j+1} - b_j$, we obtain $|b_{j+1} - b_j| \geq -(b_{j+1} - b_j) = b_j - b_{j+1}$. Therefore, $b_j - b_{j+1} \leq |b_{j+1} - b_j| \leq 1$, so that $b_j \leq b_{j+1} + 1$. In other words, $b_{j+1} \geq b_j - 1 \geq 0$ (because $b_j \geq 1$).

However, recall that b_1, b_2, \dots, b_w are nonzero integers. Thus, b_{j+1} is a nonzero integer. Therefore, $b_{j+1} \neq 0$. Combining this with $b_{j+1} \geq 0$, we obtain $b_{j+1} > 0$. In other words, (37) holds for $i = j + 1$. This completes the induction step. Thus, we have proven (37) by induction.

But this clearly concludes the proof of Lemma 0.22. \square

Note that Lemma 0.22 is also a particular case of [Grinbe16, Proposition 2.75].

Solution to Exercise 7 (sketched). We shall prove the contrapositive: If there is no balanced window, then neither $r < 2q$ nor $s \leq 2q + r$ can hold.

So let us assume that there is no balanced window. We must then show that neither $r < 2q$ nor $s \leq 2q + r$ can hold. In other words, we must show that $r \geq 2q$ and $s > 2q + r$.

Notice that $s/2 \in \mathbb{Z}$ (since s is even). Also, q and r are the quotient and the remainder of division of n by s . Thus, $q \in \mathbb{Z}$, $r \in \{0, 1, \dots, s-1\}$ and $n = qs + r$. From $r \in \{0, 1, \dots, s-1\}$, we obtain $r < s$. Thus, $s > r$.

The integer q is nonnegative (since it is the quotient of dividing the positive integer n by the positive integer s).

We must prove that $r \geq 2q$ and $s > 2q + r$. If $q = 0$, then this is obvious (because if $q = 0$, then $r \geq 0 = 2 \cdot \underbrace{0}_{=q} = 2q$ and $s > r = 2 \cdot \underbrace{0}_{=q} + r = 2q + r$). Hence, we

WLOG assume that $q \neq 0$. Thus, $q \geq 1$ (since q is a nonnegative integer). Therefore, $n = \underbrace{q}_{\geq 1} s + \underbrace{r}_{\geq 0} \geq s$. Hence, $n - s + 1 \geq 1$.

Number the socks by $1, 2, \dots, n$ in the order in which they appear on the clothesline.

For each $i \in [n - s + 1]$, define the integer

$$b_i = (\text{the number of black socks among socks } i, i + 1, \dots, i + s - 1) - s/2. \quad (38)$$

(This is indeed an integer, because $s/2 \in \mathbb{Z}$.) Then, each $i \in [n - s + 1]$ satisfies

$$b_i = \frac{1}{2} (\text{the number of black socks among socks } i, i + 1, \dots, i + s - 1) - \frac{1}{2} (\text{the number of white socks among socks } i, i + 1, \dots, i + s - 1). \quad (39)$$

[Proof of (39): Let $i \in [n - s + 1]$. Then,

$$\begin{aligned} & (\text{the number of black socks among socks } i, i + 1, \dots, i + s - 1) \\ & \quad + (\text{the number of white socks among socks } i, i + 1, \dots, i + s - 1) \\ & = (\text{the total number of socks } i, i + 1, \dots, i + s - 1) = s. \end{aligned}$$

Hence,

$$\begin{aligned} & (\text{the number of white socks among socks } i, i+1, \dots, i+s-1) \\ &= s - (\text{the number of black socks among socks } i, i+1, \dots, i+s-1). \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{1}{2} (\text{the number of black socks among socks } i, i+1, \dots, i+s-1) \\ & \quad - \frac{1}{2} \underbrace{(\text{the number of white socks among socks } i, i+1, \dots, i+s-1)}_{=s - (\text{the number of black socks among socks } i, i+1, \dots, i+s-1)} \\ &= \frac{1}{2} (\text{the number of black socks among socks } i, i+1, \dots, i+s-1) \\ & \quad - \frac{1}{2} (s - (\text{the number of black socks among socks } i, i+1, \dots, i+s-1)) \\ &= (\text{the number of black socks among socks } i, i+1, \dots, i+s-1) - s/2 \\ &= b_i \quad (\text{by (38)}). \end{aligned}$$

This proves (39).]

The equality (39) shows that if we invert the colors of all socks (simultaneously), then all the numbers $b_1, b_2, \dots, b_{n-s+1}$ change signs. Hence, we can WLOG assume that $b_1 \geq 0$ (since otherwise, we can invert the colors of all socks, and then b_1 will change sign)²¹. Assume this.

Note that each $i \in [n-s+1]$ satisfies

$$(\text{the number of black socks among socks } i, i+1, \dots, i+s-1) = b_i + s/2 \quad (40)$$

(by (38)).

For each $i \in [n-s+1]$, we have $b_i \neq 0$ (because if we had $b_i = 0$, then the s consecutive socks $i, i+1, \dots, i+s-1$ would form a balanced window; but this would contradict our assumption that there is no balanced window). Thus, $b_1, b_2, \dots, b_{n-s+1}$ are nonzero integers. Furthermore,

$$|b_{i+1} - b_i| \leq 1 \quad \text{for all } i \in [(n-s+1) - 1]$$

²² Hence, Lemma 0.22 shows that $b_i > 0$ for all $i \in [n-s+1]$. Since the b_i are integers, this shows that

$$b_i \geq 1 \text{ for all } i \in [n-s+1]. \quad (41)$$

²¹Here we are tacitly using the fact that b_1 exists (since $n-s+1 \geq 1$).

²²*Proof.* Let $i \in [(n-s+1) - 1]$. Thus, $i \in [n-s]$. Let r be the number of black socks among the $s-1$ socks $i+1, i+2, \dots, i+s-1$. Then, the definition of b_i yields

$$\begin{aligned} b_i &= \underbrace{(\text{the number of black socks among socks } i, i+1, \dots, i+s-1)}_{=r+} - s/2 \\ &= r + \begin{cases} 1, & \text{if sock } i \text{ is black;} \\ 0, & \text{if sock } i \text{ is white.} \end{cases} - s/2. \end{aligned}$$

Now, let g be the number of black socks among the r socks $qs + 1, qs + 2, \dots, qs + r$. Thus, clearly, $0 \leq g \leq r$.

Meanwhile, the definition of b_{i+1} yields

$$\begin{aligned} b_{i+1} &= \underbrace{\left(\text{the number of black socks among socks } i+1, i+2, \dots, i+s \right)}_{=r+} - s/2 \\ &= r + \begin{cases} 1, & \text{if sock } i+s \text{ is black;} \\ 0, & \text{if sock } i+s \text{ is white.} \end{cases} - s/2. \end{aligned}$$

Subtracting the first of these two equalities from the second, we find

$$\begin{aligned} b_{i+1} - b_i &= \left(r + \begin{cases} 1, & \text{if sock } i \text{ is black;} \\ 0, & \text{if sock } i \text{ is white.} \end{cases} - s/2 \right) - \left(r + \begin{cases} 1, & \text{if sock } i+s \text{ is black;} \\ 0, & \text{if sock } i+s \text{ is white.} \end{cases} - s/2 \right) \\ &= \begin{cases} 1, & \text{if sock } i \text{ is black;} \\ 0, & \text{if sock } i \text{ is white.} \end{cases} - \begin{cases} 1, & \text{if sock } i+s \text{ is black;} \\ 0, & \text{if sock } i+s \text{ is white.} \end{cases} \\ &\in \{1-1, 1-0, 0-1, 0-0\} = \{1, -1, 0\}, \end{aligned}$$

and thus $|b_{i+1} - b_i| \leq 1$, qed.

Recall that the total number of black socks on the clothesline is $n/2$. Thus,

$$\begin{aligned}
 n/2 &= (\text{the total number of black socks}) \\
 &= (\text{the number of black socks among socks } 1, 2, \dots, n) \\
 &= (\text{the number of black socks among socks } 1, 2, \dots, qs + r) \\
 &\quad (\text{since } n = qs + r) \\
 &= (\text{the number of black socks among socks } 1, 2, \dots, s) \\
 &\quad + (\text{the number of black socks among socks } s + 1, s + 2, \dots, 2s) \\
 &\quad + (\text{the number of black socks among socks } 2s + 1, 2s + 2, \dots, 3s) \\
 &\quad + \dots \\
 &\quad + (\text{the number of black socks among socks } (q - 1)s + 1, (q - 1)s + 2, \dots, qs) \\
 &\quad + (\text{the number of black socks among socks } qs + 1, qs + 2, \dots, qs + r) \\
 &= \sum_{h=0}^{q-1} \underbrace{(\text{the number of black socks among socks } hs + 1, hs + 2, \dots, (h + 1)s)}_{=b_{hs+1} + s/2 \text{ (by (40))}} \\
 &\quad + \underbrace{(\text{the number of black socks among socks } qs + 1, qs + 2, \dots, qs + r)}_{=g} \\
 &= \sum_{h=0}^{q-1} \left(\underbrace{b_{hs+1}}_{\substack{\geq 1 \\ \text{(by (41))}}} + s/2 \right) + g \\
 &\geq \underbrace{\sum_{h=0}^{q-1} (1 + s/2)}_{\substack{=q(1+s/2) \\ =q+qs/2}} + g = q + qs/2 + g.
 \end{aligned}$$

Hence,

$$q + qs/2 + g \leq \underbrace{n}_{=qs+r} / 2 = (qs + r) / 2 = qs/2 + r/2.$$

Subtracting $qs/2$ from both sides of this inequality, we find

$$q + g \leq r/2. \tag{42}$$

Hence, $r/2 \geq q + \underbrace{g}_{\geq 0} \geq q$, so that $r \geq 2q$.

It remains to prove that $s > 2q + r$. Note that there is a sock $n - s + 1$ on our clothesline (since $n - s + 1 \geq 1$).

From $n = qs + r$, we obtain $n - r = qs$. Thus,

$$\begin{aligned} & (\text{the number of black socks among socks } n - r + 1, n - r + 2, \dots, n) \\ &= (\text{the number of black socks among socks } qs + 1, qs + 2, \dots, n) \\ &= (\text{the number of black socks among socks } qs + 1, qs + 2, \dots, qs + r) \\ &\quad (\text{since } n = qs + r) \\ &= g. \end{aligned}$$

Recall that there is a sock $n - s + 1$ on our clothesline. Let p be the number of black socks among the $s - r$ socks $n - s + 1, n - s + 2, \dots, n - r$. Thus, clearly, $p \leq s - r$.

From (40), we obtain

$$\begin{aligned} & (\text{the number of black socks among socks } n - s + 1, n - s + 2, \dots, n) \\ &= \underbrace{b_{n-s+1}}_{\substack{\geq 1 \\ \text{(by (41))}}} + s/2 \geq 1 + s/2. \end{aligned}$$

Hence,

$$\begin{aligned} & 1 + s/2 \\ & \leq (\text{the number of black socks among socks } n - s + 1, n - s + 2, \dots, n) \\ &= \underbrace{(\text{the number of black socks among socks } n - s + 1, n - s + 2, \dots, n - r)}_{=p} \\ &\quad + \underbrace{(\text{the number of black socks among socks } n - r + 1, n - r + 2, \dots, n)}_{=g} \\ &= \underbrace{p}_{\leq s-r} + \underbrace{g}_{\substack{\leq r/2-q \\ \text{(by (42))}}} \leq (s - r) + (r/2 - q) = s - r/2 - q. \end{aligned}$$

Subtracting $s/2$ from both sides of this inequality, we find $1 \leq s/2 - r/2 - q$, so that $s/2 - r/2 - q \geq 1 > 0$. Multiplying this inequality by 2, we obtain $s - r - 2q > 0$, so that $s > 2q + r$. This completes the solution to Exercise 7. \square

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