Math 4707 Spring 2018 (Darij Grinberg): homework set 2
due date: Wednesday 21 February 2018 at the beginning of class, or before that by email or moodle
Please solve **at most 3** of the 7 exercises!

Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

See [Fall2017-HW1s, solution to Exercise 8] for an example of how a counting proof can be written.

Recall that if $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$.

0.1. Instances of the "Laurent phenomenon": Sequences that produce integers despite division in their definition

There is a whole genre of theorem where you define a sequence recursively and then it turns out that all entries of the sequence are integers, although this is not obvious from the definition.

Here are two results from this genre, illustrating strategic use of induction.

Exercise 1. Define a sequence $(t_0, t_1, t_2, \ldots)$ of positive rational numbers recursively by setting

$$t_0 = 1, \qquad t_1 = 1, \qquad t_2 = 1, \qquad \text{and}$$
$$t_n = \frac{1 + t_{n-1}t_{n-2}}{t_{n-3}} \qquad \text{for each } n \geq 3.$$

(For example, $t_3 = \frac{1 + t_2 t_1}{t_0} = \frac{1 + 1 \cdot 1}{1} = 2$ and $t_4 = \frac{1 + t_3 t_2}{t_1} = \frac{1 + 2 \cdot 1}{1} = 3$.)

(a) Prove that $t_{n+2} = 4t_n - t_{n-2}$ for each $n \geq 2$.

(b) Prove that $t_n \in \mathbb{N}$ for each $n \in \mathbb{N}$.

[Hint: First prove part **(a)** by induction on n. Then prove part **(b)** by induction on n, using part **(a)**.]

Exercise 2. Fix a positive integer r. Define a sequence $(b_0, b_1, b_2, \ldots)$ of positive rational numbers recursively by setting

$$b_0 = 1, \qquad b_1 = 1, \qquad \text{and}$$
$$b_n = \frac{b_{n-1}^r + 1}{b_{n-2}} \qquad \text{for each } n \geq 2.$$

(For example, $b_2 = \frac{b_1^r + 1}{b_0} = \frac{1^r + 1}{1} = 2$ and $b_3 = \frac{b_2^r + 1}{b_1} = \frac{2^r + 1}{1} = 2^r + 1$.)

(a) Prove that $b_n \in \mathbb{N}$ for each $n \in \mathbb{N}$.

(b) If $r \geq 2$, then prove that $b_n \mid b_{n-2} + b_{n+2}$ for each $n \geq 2$.

[Hint: For every nonzero $x \in \mathbb{Q}$, we set $H(x) = \dfrac{(x+1)^r - 1}{x}$. Show that $H(x) \in \mathbb{Z}$ whenever x is a nonzero integer. Next, show that $b_{n+2} = b_{n-2} b_{n+1}^r - b_n^{r-1} H(b_n^r)$ for each $n \geq 2$. Use this to prove **(a)**.]

0.2. Lacunar subsets with a given number of even and a given number of odd elements

Recall the following definition: A set S of integers is said to be *lacunar* if no two consecutive integers occur in S (that is, there exists no $i \in \mathbb{Z}$ such that both i and $i+1$ belong to S). For example, $\{1,3,6\}$ is lacunar, but $\{2,4,5\}$ is not. (The empty set and any 1-element set are lacunar, of course.)

Also recall the *Iverson bracket notation:* If $\mathcal{A}$ is any logical statement, then the *truth value* of $\mathcal{A}$ is defined to be the integer

$$\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases} \in \{0,1\}.$$

This truth value is denoted by $[\mathcal{A}]$. For example, $[1+1=2] = 1$ (since $1+1=2$ is true), whereas $[1+1=1] = 0$ (since $1+1=1$ is false).

Exercise 3. For any $n \in \mathbb{N}$, $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, we let $N(n,a,b)$ denote the number of all lacunar subsets of $[n]$ that contain exactly a even and exactly b odd elements.

(a) Prove that $N(2m,a,b) = [a \leq m][b \leq m] \dbinom{m-a}{b} \dbinom{m-b}{a}$ for all $m \in \mathbb{N}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

(b) Prove that $N(2m+1,a,b) = [a \leq m][b \leq m+1] \dbinom{m+1-a}{b} \dbinom{m-b}{a}$ for all $m \in \mathbb{N}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

[Hint: One way is to prove parts **(a)** and **(b)** simultaneously by induction (that is, let $\mathcal{A}(m)$ be the statement "$N(2m,a,b) = [a \leq m][b \leq m] \dbinom{m-a}{b} \dbinom{m-b}{a}$ and $N(2m+1,a,b) = [a \leq m][b \leq m+1] \dbinom{m+1-a}{b} \dbinom{m-b}{a}$ for all $a \in \mathbb{N}$ and $b \in \mathbb{N}$", and prove this by induction on m). One part of the induction step is an expression for $N(2m+2,a,b)$ through $N(2m+1,a,b)$ and $N(2m,a-1,b)$. Another similar expression will be needed for $N(2m+3,a,b)$. Make sure to treat the base case properly, as well as justifying the switch between the truth values necessary at one point in the induction step. There is also a bijective proof.]

0.3. Delannoy numbers

Fix two positive integers r and s.

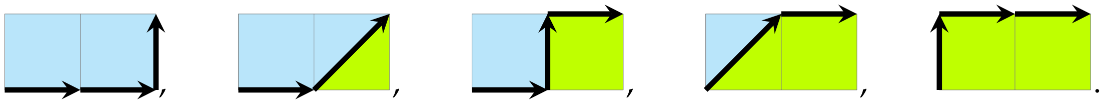
If $(a,b) \in \mathbb{Z}^2$ and $(c,d) \in \mathbb{Z}^2$ are two points on the integer lattice, then a (r,s)-*Delannoy path* from (a,b) to (c,d) is a path from (a,b) to (c,d) that uses only three kinds of steps:

- up-steps (U), which have the form $(x,y) \mapsto (x, y+1)$;
- right-steps (R), which have the form $(x,y) \mapsto (x+1, y)$;
- diagonal steps (D), which have the form $(x,y) \mapsto (x+r, y+s)$.

Thus, strictly speaking, a (r,s)-*Delannoy path* from (a,b) to (c,d) is a sequence $(v_0, v_1, \ldots, v_n)$ of points $v_i \in \mathbb{Z}^2$ such that for each $i \in [n]$, the difference vector $v_i - v_{i-1}$ is either $(0,1)$ or $(1,0)$ or (r,s).

For two integers n and m, we let $d_{n,m}$ be the number of (r,s)-Delannoy paths from $(0,0)$ to (n,m). (This depends on r and s, too, but we regard r and s as fixed.) Note that $d_{n,m} = 0$ if (at least) one of n and m is negative (because neither the x-coordinate nor the y-coordinate can ever decrease along an (r,s)-Delannoy path).

For example, if $r = 1$ and $s = 1$, then $d_{2,1} = 5$, the five $(1,1)$-Delannoy paths being RRU, RD, RUR, DR and URR. Here are these five paths drawn in the plane:



Exercise 4. **(a)** Show that $d_{n,m} = d_{n-1,m} + d_{n,m-1} + d_{n-r,m-s}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, unless $(n,m) = (0,0)$.

(b) Show that

$$d_{n,m} = \sum_{k=0}^{n} [n + m \geq (r+s-1)k] \binom{n-(r-1)k}{k} \binom{n+m-(r+s-1)k}{n-(r-1)k}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

(c) Assume that $r = s$. Show that $d_{n,m} = d_{m,n}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

[**Hint:** The case $r = 1$ and $s = 1$ is studied in [Galvin17, §28].]

0.4. On inclusion/exclusion

One version of the Principle of Inclusion and Exclusion is the following theorem (see, e.g., [Galvin17, Theorem 16.1 and (11)]):

Theorem 0.1. Let $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_n$ be finite sets.

(a) We have

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

(b) Let S be a finite set. Assume that each of $A_1, A_2, \ldots, A_n$ is a subset of S. Then,

$$\left| S \setminus \bigcup_{i=1}^{n} A_i \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| .$$

Here, the "empty" intersection $\bigcap_{i \in \varnothing} A_i$ is understood to mean the set S.

Here is another way to write Theorem 0.1 **(a)**:

Corollary 0.2. Let $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \right| .$$

Exercise 5. Let n be a positive integer. Let $a_1, a_2, \ldots, a_n$ be n integers. Prove that

$$\max \{a_1, a_2, \ldots, a_n\} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \min \left\{ a_{i_1}, a_{i_2}, \ldots, a_{i_k} \right\} . \qquad (1)$$

For example, if $n = 3$, then this says that

$$\begin{aligned} \max \{a_1, a_2, a_3\} = \min \{a_1\} + \min \{a_2\} + \min \{a_3\} \\ - \min \{a_1, a_2\} - \min \{a_1, a_3\} - \min \{a_2, a_3\} \\ + \min \{a_1, a_2, a_3\} . \end{aligned}$$

[Hint: You can derive this from Corollary 0.2 by constructing n sets $A_1, A_2, \ldots, A_n$ such that $\left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \right| = \min \left\{ a_{i_1}, a_{i_2}, \ldots, a_{i_k} \right\}$, if the a_i are nonnegative. If some a_i are negative, a slight tweak is required. Alternatively, and perhaps more easily, there is a proof without using the Principle of Inclusion and Exclusion.

Note that (1) can be rewritten as

$$\max \{a_1, a_2, \ldots, a_n\} = \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} (-1)^{|I|-1} \min \{a_i \mid i \in I\} . \qquad (2)$$

It might be easier to prove this equivalent form.]

0.5. Not-quite-derangements

Exercise 6. Let n be a positive integer. An *nqd* ("not-quite-derangement") of $[n]$ shall denote a permutation σ of $[n]$ such that every $i \in [n-1]$ satisfies $\sigma(i) \neq i+1$. Prove that the number of nqds of $[n]$ is

$$(n-1)! \sum_{k=0}^{n-1} (-1)^k \cdot \frac{n-k}{k!}.$$

(Compare with the formula, proven in [Galvin17, §16], which says that the number of derangements of $[n]$ is $n! \sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!}$.)

0.6. Socks

Exercise 7. Let n and s be two even positive integers. Let q and r be the quotient and the remainder of division of n by s. (Thus, $q \in \mathbb{Z}$, $r \in \{0, 1, \ldots, s-1\}$ and $n = qs + r$.)

Assume that n socks are hanging on a clothesline, with $n/2$ of these socks being black and the remaining $n/2$ white.

A *balanced window* will mean a choice of s consecutive socks on the clothesline such that $s/2$ of these socks are black and the remaining $s/2$ are white.

(a) If $r < 2q$, then show that there is a balanced window.

(b) If $s \leq 2q + r$, then show that there is a balanced window.

[**Hint:** Number the socks by $1, 2, \ldots, n$ in the order in which they appear on the clothesline. For each $i \in [n-s+1]$, define the integer

$$b_i = (\text{the number of black socks among socks } i, i+1, \ldots, i+s-1) - s/2.$$

Proceed as in class, and take a look at the last r socks on the clothesline (i.e., those not counted in $b_1, b_{s+1}, b_{2s+1}, \ldots, b_{(q-1)s+1}$). For part **(b)**, take a closer look at the last s socks on the clothesline.]

(In class, we mostly considered the case when $n = 30$ and $s = 10$; this falls under the situation of part **(a)**. For an example of part **(b)**, try $n = 26$ and $s = 10$.)

Remark 0.3. A converse can also be shown: If neither $r < 2q$ nor $s \leq 2q + r$ holds, then one can place n socks ($n/2$ black, $n/2$ white) on a clothesline in such a way that no balanced window exists.

References

[Galvin17] David Galvin, *Basic discrete mathematics*, 13 December 2017. `http://www.cip.ifi.lmu.de/~grinberg/t/17f/60610lectures2017-Galvin.pdf`

[Fall2017-HW1s] Darij Grinberg, *Math 4707 & Math 4990 Fall 2017 (Darij Grinberg): homework set 1 with solutions.*
`http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw1s.pdf`