Math 4707 Spring 2018 (Darij Grinberg): homework set 1
due date: Wednesday 31 January 2018
Please solve **at most 4** of the 7 exercises!

Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

See [Fall2017-HW1s, solution to Exercise 8] for an example of how a counting proof can be written.

0.1. Binomial coefficient basics

Definition 0.1. The notation $\mathbb{N}$ shall always stand for the set $\{0, 1, 2, \ldots\}$ of nonnegative integers.

Definition 0.2. If $n \in \mathbb{N}$, then $n!$ shall denote the product $1 \cdot 2 \cdot \cdots \cdot n$. For example, $3! = 1 \cdot 2 \cdot 3 = 6$ and $1! = 1$ and $0! = (\text{empty product}) = 1$. (An empty product is defined to be 1.)

Definition 0.3. **(a)** We define the *binomial coefficient* $\dbinom{n}{k}$ by

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

for every $n \in \mathbb{Q}$ and $k \in \mathbb{N}$.

For example, $\dbinom{-3}{4} = \dfrac{(-3)(-4)(-5)(-6)}{4!} = 15$ and $\dbinom{4}{1} = \dfrac{4}{1!} = 4$ and $\dbinom{4}{0} = \dfrac{(\text{empty product})}{0!} = \dfrac{1}{1} = 1$.

(b) If $n \in \mathbb{Q}$ and $k \in \mathbb{Q} \setminus \mathbb{N}$, then $\dbinom{n}{k}$ is defined to be 0.

See [Grinbe16, Chapter 3] for various properties of binomial coefficients. You are free to use those shown in [Grinbe16, §3.1] without proof. In particular, you are free to use the following fact:

Proposition 0.4 (recurrence of the binomial coefficients)**.** Let $n \in \mathbb{Q}$ and $k \in \mathbb{Z}$. Then,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(This is exactly [Grinbe16, Proposition 3.11] when k is a positive integer. The cases $k = 0$ and $k < 0$ are easy.)

Exercise 1. **(a)** Show that $\binom{-1}{k} = (-1)^k$ for each $k \in \mathbb{N}$.

(b) Show that $\binom{-2}{k} = (-1)^k (k+1)$ for each $k \in \mathbb{N}$.

(c) Show that $\dfrac{1! \cdot 2! \cdot \cdots \cdot (2n)!}{n!}$ is a perfect square (i.e., the square of an integer) whenever $n \in \mathbb{N}$ is even.

0.2. A fraction appears

Definition 0.5. Let x be a real number. Then, $\lfloor x \rfloor$ is defined to be the unique integer n satisfying $n \leq x < n+1$. This integer $\lfloor x \rfloor$ is called the *floor* of x, or the *integer part* of x. For example,

$$\lfloor n \rfloor = n \qquad \text{for every } n \in \mathbb{Z};$$
$$\lfloor 1.32 \rfloor = 1; \qquad \lfloor \pi \rfloor = 3; \qquad \lfloor 0.98 \rfloor = 0;$$
$$\lfloor -2.3 \rfloor = -3; \qquad \lfloor -0.4 \rfloor = -1.$$

Exercise 2. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} (-1)^k (k+1) = (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor .$$

0.3. Lemmas for the Sierpinski gasket appearing in Pascal's triangle

Exercise 3. Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

(a) Prove that $\binom{2a}{2b} \equiv \binom{a}{b} \operatorname{mod} 2$.

(b) Prove that $\binom{2a+1}{2b} \equiv \binom{a}{b} \operatorname{mod} 2$.

(c) Prove that $\binom{2a}{2b+1} \equiv 0 \operatorname{mod} 2$.

(d) Prove that $\binom{2a+1}{2b+1} \equiv \binom{a}{b} \operatorname{mod} 2$.

[**Hint:** Prove all four parts simultaneously by an induction on a, using Proposition 0.4.]

Exercise 4. Let $n \in \mathbb{N}$. Let a and b be two elements of $\{0, 1, \ldots, 2^n - 1\}$. Prove that

$$\binom{2^n + a}{b} \equiv \binom{a}{b} \mod 2 \qquad \text{and}$$
$$\binom{2^n + a}{2^n + b} \equiv \binom{a}{b} \mod 2.$$

[Here, "$\{0, 1, \ldots, 2^n - 1\}$" means the set of all integers k satisfying $0 \leq k \leq 2^n - 1$.]

[**Hint:** Induction on n. You can use Exercise 3 here even if you have not solved it.]

Back in class, Exercise 4 helped us prove that Pascal's triangle becomes a Sierpinski gasket (see, e.g., the Wikipedia) if its entries are replaced by their parities.

0.4. More on Fibonacci numbers

Definition 0.6. The *Fibonacci sequence* is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Its first terms are

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3, \quad f_5 = 5,$$
$$f_6 = 8, \quad f_7 = 13, \quad f_8 = 21, \quad f_9 = 34, \quad f_{10} = 55,$$
$$f_{11} = 89, \quad f_{12} = 144, \quad f_{13} = 233.$$

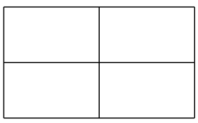
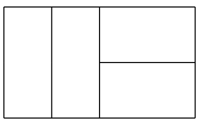
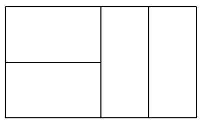
Exercise 5. Let $n \in \mathbb{N}$. Let $R_{n,2}$ denote the set $[n] \times [2]$, which we regard as a rectangle of width n and height 2 (by identifying the squares with pairs of coordinates). (Note: I might have been careless in class and confused width with height a few times. In case of doubt, follow the conventions just given.)

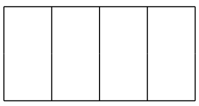
A *vertical domino* is a set of the form $\{(i, j), (i, j+1)\}$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

A *horizontal domino* is a set of the form $\{(i+1, j), (i, j)\}$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

A *domino tiling* of $R_{n,2}$ means a set of disjoint dominos (i.e., vertical dominos and horizontal dominos) whose union is $R_{n,2}$.

For example, there are 5 domino tilings of $R_{4,2}$, namely

 ,  ,  ,

 ,  .

Written as a set of dominos, the second of these tilings is

$$\{\{(1,1),(1,2)\},\{(2,1),(2,2)\},\{(3,1),(4,1)\},\{(3,2),(4,2)\}\}.$$

We have seen in class (January 17) that the number of domino tilings of $R_{n,2}$ is f_{n+1}.

A domino tiling S of $R_{n,2}$ is said to be *axisymmetric* if reflecting it across the vertical axis of the rectangle $R_{n,2}$ leaves it unchanged. (Formally, if S is regarded as a set, it means that for every domino $\{(i,j),(i',j')\} \in S$, its "mirror domino" $\{(n+1-i,j),(n+1-i',j')\}$ is also in S.) For example, among the 5 domino tilings of $R_{4,2}$ listed above, exactly 3 are axisymmetric (namely, the first, the fourth and the fifth).

Let s_n be the number of axisymmetric domino tilings of $R_{n,2}$.

(a) Prove that $s_n = f_{(n+1)/2}$ if n is odd.

(b) Prove that $s_n = f_{n/2+2}$ if n is even.

0.5. Some basic counting

Definition 0.7. Let $n \in \mathbb{Z}$. Then, $[n]$ shall denote the set $\{1,2,\ldots,n\}$. This is an empty set if $n \leq 0$.

Recall that if $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, and if S is an n-element set, then $\binom{n}{k}$ is the number of k-element subsets of S. (This can be used without proof; see, e.g., [Grinbe16, Proposition 3.12], or any text on combinatorics.)

Exercise 6. Let $n \in \mathbb{N}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$. Let N be the number of subsets of $[n]$ that contain exactly a even elements and exactly b odd elements.

(a) Prove that $N = \binom{n/2}{a}\binom{n/2}{b}$ if n is even.

(b) Compute N when n is odd.

Exercise 7. A set S of integers shall be called *self-starting* if its size $|S|$ is also its smallest element. (For example, $\{3,5,6\}$ is self-starting, while $\{2,3,4\}$ and $\{3\}$ are not.)

Let $n \in \mathbb{N}$.

(a) For any $k \in [n]$, find the number of self-starting subsets of $[n]$ having size k.

(b) Find the number of all self-starting subsets of $[n]$.

References

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.

http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10 .

[Fall2017-HW1s] Darij Grinberg, *Math 4707 & Math 4990 Fall 2017 (Darij Grinberg): homework set 1 with solutions.*
http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw1s.pdf