Notes on network flows

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Contents

1. Introduction

1.1. What is this?

In these notes, we will explain the basics of the theory of network flows. We will barely scratch the surface; for a more comprehensive survey, see Schrijver's [Schrij17, Chapter 4]. Also, [Martin17, §8.2] gives a neat introduction, and [ForFul62] is the classical text on the subject. The thesis [Thalwi08] appears to be a thorough

treatment of the subject with lots of technical details. See also the "Lecture notes on maximum flows and minimum cut problems" in [Goeman17].

We shall mostly follow [Martin17, §8.2]. In particular, we will only use elementary methods. More advanced disciplines (such as linear optimization and polyhedral geometry) offer alternative points of view on the theory of network flows; we shall ignore these. We shall also ignore real-life applications, although there are many (see [Schrij17, Chapter 4] for a few, and see [Schrij12, §2] for the Cold War origins of the subject).

These notes have been written for undergraduate classes on graph theory and combinatorics. They are derivative of the notes [Grinbe17b], but are more self-contained than [Grinbe17b] and are concerned with a more general setting.

1.2. Notations

We let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$ of all nonnegative integers.

We let $\mathbb{Q}_+$ denote the set $\{x \in \mathbb{Q} \mid x \geq 0\}$ of all nonnegative rational numbers.

We let $\mathbb{R}_+$ denote the set $\{x \in \mathbb{R} \mid x \geq 0\}$ of all nonnegative real numbers.

1.3. Simple digraphs and multidigraphs

The theory of network flows can be built either on the notion of a *simple digraph*, or on the (somewhat more general) notion of a *multidigraph*. We shall take the latter choice (thus obtaining a slightly more general theory), but we will define both simple digraphs and multidigraphs.

First, let us define simple digraphs, since these are the simpler object:

Definition 1.1. A *simple digraph* is defined to be a pair (V, A) consisting of a finite set V and of a subset A of $V \times V$. The elements of V are called the *vertices* of this simple digraph; the elements of A are called its *arcs*. If $a = (u, v)$ is an arc of a simple digraph (V, A), then u is called the *source* of this arc a, and v is called the *target* of this arc a.

Example 1.2. (a) The pair

$$(\{1, 2, 3\}, \{(1, 2), (2, 3), (1, 3)\})$$

is a simple digraph. Its vertices are $1, 2, 3$. Its arcs are $(1, 2)$, $(2, 3)$ and $(1, 3)$. The arc $(2, 3)$ has source 2 and target 3.

(b) The pair

$$(\{1, 3, 5\}, \{(1, 5), (5, 5)\})$$

is a simple digraph. Its vertices are $1, 3, 5$. Its arcs are $(1, 5)$ and $(5, 5)$.

A simple digraph (V, A) can be visually represented as follows:

- For each vertex $v \in V$, choose a point in the plane and label it with a "v".

- For each arc $(u, v) \in A$, draw an arrow from the point labelled "u" to the point labelled "v". (The arrow can be straight or curved.)

(Of course, the drawing should be made with readability in mind: The points labelled by the vertices should be chosen sufficiently far apart that the labels don't overlap. The arrows are allowed to cross[1], but they should cross in such a way that it is clear which arrow goes where. In short, the representation should unambiguously determine the digraph.)
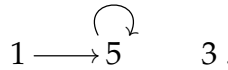
There are many ways to represent a given digraph.

Example 1.3. (a) The digraph $(\{1, 2, 3\}, \{(1, 2), (2, 3), (1, 3)\})$ from Example 1.2 **(a)** can be represented as follows:
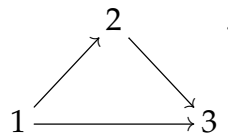
It can also be represented as follows:
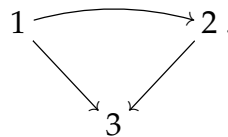
(b) The digraph $(\{1, 3, 5\}, \{(1, 5), (5, 5)\})$ from Example 1.2 **(b)** can be represented as follows:

Note that an arc of a simple digraph is uniquely determined by its source and its target: indeed, it is the pair consisting of its source and its target. Multidigraphs are similar to simple digraphs, except that this is no longer true: their arcs are not uniquely determined by their sources and their targets any more, but rather have "their own identities". Here is how multidigraphs are defined:

Definition 1.4. A *multidigraph* is a triple (V, A, ϕ), where V and A are finite sets and where ϕ is a map from A to $V \times V$. The elements of V are called the *vertices* of this multidigraph; the elements of A are called its *arcs*. If a is an arc of a multidigraph (V, A, ϕ), and if $(u, v) = \phi(a)$, then u is called the *source* of this arc a, and w is called the *target* of this arc a.

Example 1.5. (a) Let $\alpha, \beta, \gamma, \delta$ be any four distinct objects (it doesn't matter which objects we take; for example, $10, 11, 12, 13$ do the job). Let V be the set $\{1, 2, 3\}$,

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[1] In fact, this is unavoidable for certain digraphs.

and let A be the set $\{\alpha, \beta, \gamma, \delta\}$. Let $\phi : A \to V \times V$ be the map given by

$$\phi(\alpha) = (1,3), \qquad \phi(\beta) = (2,3),$$
$$\phi(\gamma) = (1,3), \qquad \phi(\delta) = (2,1).$$

Then, the triple (V, A, ϕ) is a multidigraph. Its vertices are $1, 2, 3$; its arcs are $\alpha, \beta, \gamma, \delta$. The arc α has source 1 and target 3; so does the arc γ.

(b) Let V and A be as in part **(a)**. But now, let $\phi : A \to V \times V$ be the map given by

$$\phi(\alpha) = (1,1), \qquad \phi(\beta) = (1,2),$$
$$\phi(\gamma) = (2,2), \qquad \phi(\delta) = (1,2).$$

Then, the triple (V, A, ϕ) is a multidigraph as well.

A multidigraph (V, A, ϕ) is visually represented in the same way as a simple digraph (V, A), with one difference: An arc $a \in A$ is now drawn as an arrow from the point labelled by its source to the point labelled by its target, and we furthermore label this arrow with an "a".

Example 1.6. (a) The multidigraph (V, A, ϕ) from Example 1.5 **(a)** can be represented as follows:
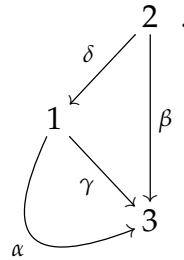
(b) The multidigraph (V, A, ϕ) from Example 1.5 **(b)** can be represented as follows:
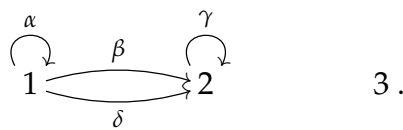
Let us summarize the difference between a simple digraph and a multidigraph: An arc of a simple digraph (V, A) is merely a pair consisting of its source and its target, whereas an arc of a multidigraph (V, A, ϕ) can be an arbitrary object (so it "has its own identity") whose source and target are assigned to it by the map ϕ. Thus, we can regard multidigraphs as a refined version of simple digraphs. Every simple digraph gives rise to a multidigraph as follows:

Definition 1.7. Let (V, A) be a simple digraph. Let $\iota : A \to V \times V$ be the inclusion map (i.e., the map that sends each $a \in A$ to a itself); this is well-defined because A is a subset of $V \times V$ (since (V, A) is a simple digraph). Then, (V, A, ι) is a multidigraph. This multidigraph (V, A, ι) is called the *multidigraph induced by* (V, A); we will often just identify it with the simple digraph (V, A) (so that each simple digraph becomes a multidigraph in this way).

Example 1.8. The simple digraph
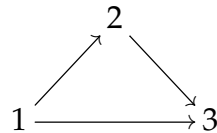
becomes identified with the multidigraph
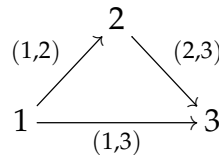
in this way.

Both simple digraphs and multidigraphs are subsumed under the concept of a *digraph*, which is an abbreviation for "directed graph".

1.4. Walks and paths

Two of the fundamental concepts regarding multidigraphs is that of a walk and that of a path. Let us define them:

Definition 1.9. Let $D = (V, A, \phi)$ be a multidigraph.

(a) A *walk* in D is defined to be a list of the form $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are vertices of D, and where a_i is an arc of D having source v_{i-1} and target v_i for each $i \in \{1, 2, \ldots, k\}$. (Note that $k = 0$ is allowed; in this case, the walk is a 1-tuple (v_0) consisting of a single vertex v_0.)

(b) Consider any walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ in D. The *vertices* of this walk are defined to be $v_0, v_1, \ldots, v_k$. Moreover, v_0 is called the *starting point* of the walk, and v_k is called the *ending point* of the walk. The *arcs* of this walk are defined to be $a_1, a_2, \ldots, a_k$. The *length* of this walk is defined to be k.

(c) A walk is said to be a *path* if its vertices are distinct. (In other words, a walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ is a path if and only if $v_0, v_1, \ldots, v_k$ are distinct.)

(d) Let s and t be two vertices of D. A *walk from s to t* in D means a walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ in D such that $v_0 = s$ and $v_k = t$. Likewise, a *path from*

s to t in D means a path $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ in D such that $v_0 = s$ and $v_k = t$.

Example 1.10. (a) Let D be the multidigraph (V, A, ϕ) from Example 1.5 **(a)**. Then,

$$(1), \qquad (1, \gamma, 3), \qquad (1, \delta, 3), \qquad (2) \qquad \text{and } (2, \delta, 1, \alpha, 3)$$

are five distinct walks in D. All of them are also paths in D. The lengths of these walks are $0, 1, 1, 0, 2$, respectively. The fifth of these walks is a walk from 2 to 3 (and also a path from 2 to 3). The vertices of the third walk are 1 and 3.

 (b) Now, let D be the multidigraph (V, A, ϕ) from Example 1.5 **(b)** instead. Then,

$$(1), \qquad (1, \alpha, 1), \qquad (1, \beta, 2), \qquad (2, \gamma, 2), \qquad \text{and } (3)$$

are five distinct walks in D. The first, third and fifth of these walks are paths in D; the other two are not. The lengths of these five walks are $0, 1, 1, 1, 0$, respectively. The fifth of these walks is a walk from 3 to 3 (and also a path from 3 to 3). The vertices of the second walk are 1 and 1.

 (c) Now, let D be the multidigraph
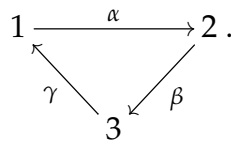
Then, $(1, \alpha, 2, \beta, 3, \gamma, 1)$ is a walk from 1 to 1, but not a path (since its vertices $1, 2, 3, 1$ are not distinct).

One of the most fundamental facts about walks in a multidigraph is the following:

Proposition 1.11. Let D be a multidigraph. Let s and t be two vertices of D. Assume that there exists a walk from s to t in D. Then, there exists a path from s to t in D.

Example 1.12. Let D be the multidigraph
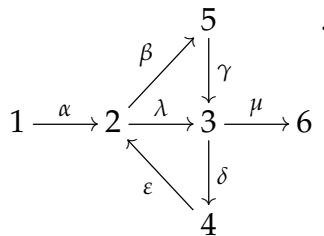
Then, $(1, \alpha, 2, \beta, 5, \gamma, 3, \delta, 4, \varepsilon, 2, \lambda, 3, \mu, 6)$ is a walk from 1 to 6 in D, but not a path.

Thus, Proposition 1.11 (applied to $s = 1$ and $t = 6$) yields that there exists a path from 1 to 6 in D. And indeed, $(1, \alpha, 2, \lambda, 3, \mu, 6)$ is such a path. (It is not the only path; another is $(1, \alpha, 2, \beta, 5, \gamma, 3, \mu, 6)$.)

Proof of Proposition 1.11 (sketched). We claim the following:

> *Claim 1:* Let $\mathbf{w}$ be a walk from s to t in D. Then, there exists a path from s to t in D.

[*Proof of Claim 1:* We shall prove Claim 1 by strong induction on the length of $\mathbf{w}$. Thus, we fix a $k \in \mathbb{N}$, and we assume that Claim 1 holds for all walks $\mathbf{w}$ having length $< k$. We must now prove Claim 1 for all walks $\mathbf{w}$ having length k.

We have assumed that Claim 1 holds for all walks $\mathbf{w}$ having length $< k$. In other words,

$$\left(\begin{array}{c} \text{if } \mathbf{v} \text{ is a walk from } s \text{ to } t \text{ in } D \text{ having length } < k, \\ \text{then there exists a path from } s \text{ to } t \text{ in } D \end{array} \right). \qquad (1)$$

Let $\mathbf{w}$ be a walk from s to t in D having length k. We want to prove that there exists a path from s to t in D.

Write the walk $\mathbf{w}$ in the form $\mathbf{w} = (v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$. (This can be done, since the walk $\mathbf{w}$ has length k.) Thus, $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k) = \mathbf{w}$ is a walk in D; hence, the definition of a walk shows that $v_0, v_1, \ldots, v_k$ are vertices of D, and that a_i is an arc of D having source v_{i-1} and target v_i for each $i \in \{1, 2, \ldots, k\}$. Furthermore, $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k) = \mathbf{w}$ is a walk from s to t; thus, $v_0 = s$ and $v_k = t$.

We are in one of the following two cases:

> *Case 1:* The vertices $v_0, v_1, \ldots, v_k$ are distinct.

> *Case 2:* The vertices $v_0, v_1, \ldots, v_k$ are not distinct.

We consider Case 1 first. In this case, the vertices $v_0, v_1, \ldots, v_k$ are distinct. Thus, the walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ is a path in D (by the definition of a "path"), and therefore is a path from s to t in D (because $v_0 = s$ and $v_k = t$). Hence, there exists a path from s to t in D. Thus, we have proven in Case 1 that there exists a path from s to t in D.

Let us now consider Case 2. In this case, the vertices $v_0, v_1, \ldots, v_k$ are not distinct. Hence, there exist two elements i and j of $\{0, 1, \ldots, k\}$ such that $i < j$ and $v_i = v_j$. Consider such i and j. Let v' be the vertex $v_i = v_j$ of D. Our walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ thus visits this vertex v' (at least) twice. We can therefore shorten this walk by removing the part between these two visits. The resulting shorter walk is

$$\left(\underbrace{v_0, a_1, v_1, a_2, v_2, \ldots, a_{i-1}, v_{i-1}, a_i}_{\substack{\text{This is the part of the} \\ \text{walk } \mathbf{w} \text{ until it reaches } v_i}}, v', \underbrace{a_{j+1}, v_{j+1}, a_{j+2}, v_{j+2}, \ldots, a_k, v_k}_{\substack{\text{This is the part of the} \\ \text{walk } \mathbf{w} \text{ after it leaves } v_j}} \right);$$

this is still a walk from s to t in D, but now has length $\underbrace{i}_{<j} + (k-j) < j + (k-j) =$
k. Denote this new walk by $\mathbf{v}$. Thus, $\mathbf{v}$ is a walk from s to t in D having length $< k$.
Hence, (1) yields that there exists a path from s to t in D. Thus, we have proven in
Case 2 that there exists a path from s to t in D.

We have now proven in both cases that there exists a path from s to t in D. Thus,
this always holds.

Now, forget that we fixed $\mathbf{w}$. Hence, we have shown that if $\mathbf{w}$ is a walk from s
to t in D having length k, then there exists a path from s to t in D. In other words,
Claim 1 holds for all walks $\mathbf{w}$ having length k. This completes the induction step;
thus, Claim 1 is proven.]

Proposition 1.11 immediately follows from Claim 1. □

We observe another simple fact:

> **Proposition 1.13.** Let D be a multidigraph. Let n be the number of vertices of D.
> Then:
>   **(a)** Any path in D has length $\leq n-1$.
>   **(b)** For any given $k \in \mathbb{N}$, there are only finitely many walks in D of length k.
>   **(c)** There are only finitely many paths in D.

Proof of Proposition 1.13 (sketched). Write the multidigraph D in the form $D = (V, A, \phi)$.
Then, V is the set of all vertices of D. Thus, $|V| = n$ (since n is the number of ver-
tices of D). Also, V and A are finite sets.

(a) Let $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ be a path in D. We must prove that this path
$(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ has length $\leq n-1$.

The vertices $v_0, v_1, \ldots, v_k$ are distinct (since $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ is a path).
Hence, $|\{v_0, v_1, \ldots, v_k\}| = k+1$. But $v_0, v_1, \ldots, v_k$ are vertices of D, thus elements
of V. Thus, $\{v_0, v_1, \ldots, v_k\} \subseteq V$, so that $|\{v_0, v_1, \ldots, v_k\}| \leq |V| = n$. Hence,
$k+1 = |\{v_0, v_1, \ldots, v_k\}| \leq n$, so that $k \leq n-1$.

Now, the path $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ has length $k \leq n-1$. This completes
the proof of Proposition 1.13 **(a)**.

(b) Let $k \in \mathbb{N}$. Each walk in D of length k has the form $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$
for some vertices $v_0, v_1, \ldots, v_k$ of D and some arcs $a_1, a_2, \ldots, a_k$ of D. Thus, each
walk in D of length k can be constructed by choosing $k+1$ vertices $v_0, v_1, \ldots, v_k$ of
D and k arcs $a_1, a_2, \ldots, a_k$ of D (although not every such choice will yield a walk in
D). There are only finitely many such choices (namely, $|V|^{k+1} \cdot |A|^k$ many). Thus,
there are only finitely many walks in D of length k. This proves Proposition 1.13
(b).

(c) Proposition 1.13 **(a)** yields that any path in D has length $\leq n-1$. In other
words, any path in D has length $\in \{0, 1, \ldots, n-1\}$. Hence, the set of all paths in
D can be decomposed as follows:

$$\{\text{paths in } D\} = \bigcup_{k \in \{0,1,\ldots,n-1\}} \{\text{paths in } D \text{ of length } k\}. \tag{2}$$

But for every $k \in \{0, 1, \ldots, n-1\}$, the set {paths in D of length k} is finite[2]. Hence, the equality (2) shows that {paths in D} is a union of finitely many finite sets. Thus, the set {paths in D} is itself finite. In other words, there are only finitely many paths in D. This proves Proposition 1.13 **(c)**. $\square$

2. Network flows

2.1. The concept of a network

> **Definition 2.1.** A *network* consists of:
>
> - a multidigraph (V, A, ϕ);
>
> - two distinct vertices $s \in V$ and $t \in V$, called the *source* and the *sink*, respectively;
>
> - a function $c : A \to \mathbb{Q}_+$, called the *capacity function*.

Note that the word "source" in Definition 2.1 is unrelated to the concept of the source of an arc. In particular, we are not requiring that the source s of a network is an actual source of an arc. (It is also allowed for s to be the target of some arc, or for t to be the source of some arc.)

We draw a network (with notations as in Definition 2.1) as follows: We first draw the multidigraph (V, A, ϕ) (with source s and sink t marked as such), and then we write "$a : c(a)$" atop each arc $a \in A$. For example, the picture

$$\tag{3}$$
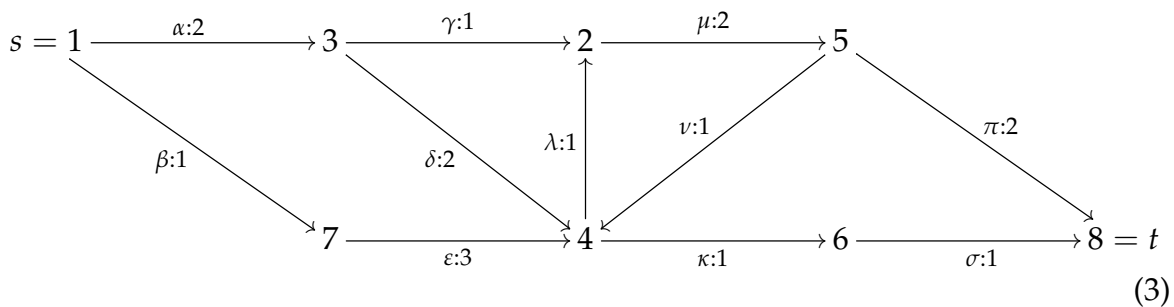
represents a network N whose underlying multidigraph (V, A, ϕ) has 8 vertices $1, 2, 3, 4, 5, 6, 7, 8$ and has 11 arcs $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \mu, \kappa, \nu, \pi, \sigma$ with

$$
\begin{array}{llll}
\phi(\alpha) = (1,3), & \phi(\beta) = (1,7), & \phi(\gamma) = (3,2), & \phi(\delta) = (3,4), \\
\phi(\varepsilon) = (7,4), & \phi(\lambda) = (4,2), & \phi(\mu) = (2,5), & \phi(\kappa) = (4,6), \\
\phi(\nu) = (5,4), & \phi(\pi) = (5,8), & \phi(\sigma) = (6,8), &
\end{array}
$$

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[2]*Proof.* Let $k \in \{0, 1, \ldots, n-1\}$. Then, Proposition 1.13 **(b)** yields that there are only finitely many walks in D of length k. Hence, there are only finitely many paths in D of length k (since every path in D is a walk in D). In other words, the set {paths in D of length k} is finite.

with 1 chosen as source of the network and 8 chosen as sink of the network, and with capacity function c given by

$$c(\alpha) = 2, \qquad c(\beta) = 1, \qquad c(\gamma) = 1, \qquad c(\delta) = 2,$$
$$c(\varepsilon) = 3, \qquad c(\lambda) = 1, \qquad c(\mu) = 2, \qquad c(\kappa) = 1,$$
$$c(\nu) = 1, \qquad c(\pi) = 2, \qquad c(\sigma) = 1.$$

Definition 2.2. Let N be a network consisting of a multidigraph (V, A, ϕ), a source $s \in V$ and a sink $t \in V$, and a capacity function $c : A \rightarrow \mathbb{Q}_+$. Then, we define the following notations:

- For any arc $a \in A$, we call the number $c(a) \in \mathbb{Q}_+$ the *capacity* of the arc a.

- For any subset S of V, we let $\overline{S}$ denote the subset $V \setminus S$ of V.

- If P and Q are two subsets of V, then $[P, Q]$ shall mean the set of all arcs $a \in A$ whose source belongs to P and whose target belongs to Q. (In other words, $[P, Q] = A \cap \phi^{-1}(P \times Q)$.)

- If P and Q are two subsets of V, and if $d : A \rightarrow \mathbb{Q}_+$ is any function, then the number $d(P, Q) \in \mathbb{Q}_+$ is defined by

$$d(P, Q) = \sum_{a \in [P,Q]} d(a).$$

2.2. The concept of a flow

Definition 2.3. Let N, V, A, ϕ, s, t and c be as in Definition 2.2.
   A *flow (on the network N)* means a function $f : A \rightarrow \mathbb{Q}_+$ with the following properties:

- We have $0 \leq f(a) \leq c(a)$ for all $a \in A$. This is called the *capacity constraints*.

- For any vertex $v \in V \setminus \{s, t\}$, we have
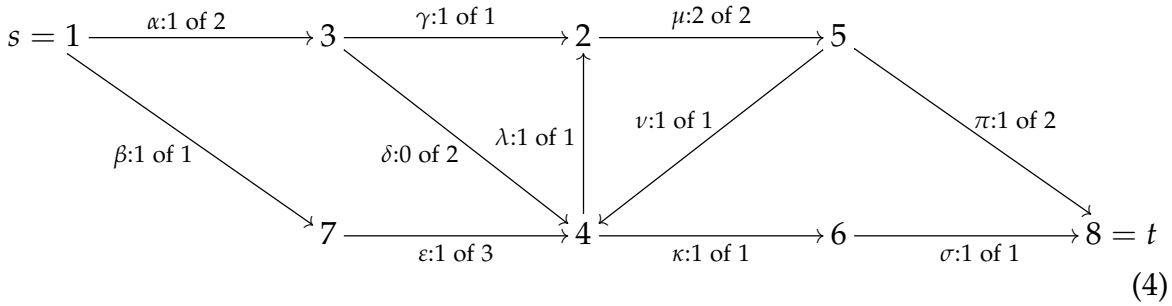
$$f^-(v) = f^+(v),$$

   where the rational numbers $f^-(v)$ and $f^+(v)$ are defined as follows:

$$f^-(v) = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} f(a) \qquad \text{and} \qquad f^+(v) = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } v}} f(a).$$

   This is called the *conservation constraints*.

We can visualize a network N as a collection of water pipes (the pipes are the arcs $a \in A$; the capacity $c(a)$ of a pipe a is how much water it can maximally transport in a second); then, a flow f on N can be visualized as water flowing through the pipes (namely, the amount of water traveling through a pipe a in a second is $f(a)$). The capacity constraints say that no pipe is over its capacity or carries a negative amount of water[3]. The conservation constraints say that at every vertex v other than s and t, the amount of water coming in (that is, $f^-(v)$) equals the amount of water moving out (that is, $f^+(v)$); that is, there are no leaks and no water being injected into the system other than at s and t. This is why s is called the "source" and t is the "sink"[4]. This visualization of flows suggests some real-life applications (although usually, the flow is not a flow of water, but, e.g., of traffic on a highway system[5]). Such applications can be found, for example, in [Schrij12, §2] and [Schrij17, Chapter 4].

To draw a flow f on a network N (with notations as in Definition 2.2), we proceed in the same way as when drawing the network N itself, but instead of writing "$a : c(a)$" atop each arc $a \in A$, we write "$a : f(a)$ of $c(a)$" atop each arc $a \in A$. For example, here is a flow f on the network N shown on (3):

$$(4)$$

(so, for example, $f(\alpha) = 1$, $f(\mu) = 2$ and $f(\delta) = 0$).

Let us make a definition (which we already made temporarily in Definition 2.3):

Definition 2.4. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Let $f : A \to \mathbb{Q}_+$ be any map. Let $v \in V$. Then, we define two rational numbers $f^-(v) \in \mathbb{Q}_+$ and $f^+(v) \in \mathbb{Q}_+$ as follows:

$$f^-(v) = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} f(a) \qquad \text{and} \qquad f^+(v) = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } v}} f(a).$$

---

[3]Notice that each pipe has a pre-determined direction; water can only flow in that direction!

[4]although these words are not to be taken fully at face value: it is possible that the source has more water coming in than moving out, and that the sink has more water moving out than coming in

[5]Somewhat unrealistically, the definition of a network requires that traffic can only enter and exit the network at s and t, and that each highway can only be used in one direction. But both of these issues are easy to fix: To model traffic that can enter at several points $s_1, s_2, \ldots, s_a$, we introduce a new "virtual" source s and arcs going from s to each of $s_1, s_2, \ldots, s_a$, so that any traffic that enters at s_i can be re-interpreted as traffic that enters at s instead (and then goes straight to s_i, before moving further through the network). Similarly, we can allow traffic to exit at several points $t_1, t_2, \ldots, t_b$. Finally, we model a bidirectional highway joining two vertices p and q by two arcs, one of which has source p and target q, while the other has source q and target p.

(We may call $f^-(v)$ the *inflow* of f into v, and we may call $f^+(v)$ the *outflow* of f from v.)

We can now define the *value* of a flow:

Definition 2.5. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Let $f : A \to \mathbb{Q}_+$ be a flow on N.

The *value* of the flow f is defined to be the number $f^+(s) - f^-(s)$. It is denoted by $|f|$.

For example, the value of the flow f shown in (4) is $|f| = f^+(s) - f^-(s) = 2 - 0 = 2$.

Proposition 2.6. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Let $f : A \to \mathbb{Q}_+$ be a flow on N. Then,

$$|f| = f^+(s) - f^-(s) \tag{5}$$
$$= f^-(t) - f^+(t). \tag{6}$$

Proof of Proposition 2.6 (sketched). The definition of $|f|$ yields $|f| = f^+(s) - f^-(s)$. It thus remains to show that $|f| = f^-(t) - f^+(t)$.

Each arc $a \in A$ has exactly one source. Thus,

$$\sum_{a \in A} f(a) = \sum_{v \in V} \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } v}} f(a)}_{\substack{= f^+(v) \\ \text{(by the definition of } f^+(v))}} = \sum_{v \in V} f^+(v). \tag{7}$$

Similarly,

$$\sum_{a \in A} f(a) = \sum_{v \in V} f^-(v). \tag{8}$$

Comparing this with (7), we obtain

$$\sum_{v \in V} f^+(v) = \sum_{v \in V} f^-(v).$$

Hence,

$$\sum_{v \in V} \left(f^+(v) - f^-(v) \right) = \underbrace{\sum_{v \in V} f^+(v)}_{= \sum_{v \in V} f^-(v)} - \sum_{v \in V} f^-(v) = 0. \tag{9}$$

But the conservation constraints (which hold, since f is a flow) say that for any vertex $v \in V \setminus \{s, t\}$, we have

$$f^-(v) = f^+(v). \tag{10}$$

Now, recall that s and t are distinct elements of V. Hence, we can split off the addends for $v = s$ and for $v = t$ from the sum $\sum\limits_{v \in V} (f^+ (v) - f^- (v))$. We thus obtain

$$\sum_{v \in V} (f^+ (v) - f^- (v))$$
$$= \underbrace{(f^+ (s) - f^- (s))}_{= |f|} + (f^+ (t) - f^- (t)) + \underbrace{\sum_{v \in V \setminus \{s,t\}} (f^+ (v) - f^- (v))}_{\substack{=0 \\ \text{(by (10))}}}$$
$$= |f| + (f^+ (t) - f^- (t)).$$

Comparing this with (9), we find $|f| + (f^+ (t) - f^- (t)) = 0$. Thus,

$$|f| = - (f^+ (t) - f^- (t)) = f^- (t) - f^+ (t).$$

This completes the proof of Proposition 2.6. $\qquad\qquad\qquad\qquad\qquad\square$

For the next proposition, let us recall the conventions we made in Definition 2.2. In particular, if S is any subset of V (where notations as in Definition 2.2), then $\overline{S}$ denotes the complement $V \setminus S$ of S. Also, for any two subsets P and Q of V and any map $d : A \to \mathbb{Q}_+$, we have $d (P, Q) = \sum\limits_{a \in [P,Q]} d (a)$.

Proposition 2.7. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Let $f : A \to \mathbb{Q}_+$ be a flow on N. Let S be a subset of V.
(a) We have
$$f (S, \overline{S}) - f (\overline{S}, S) = \sum_{v \in S} (f^+ (v) - f^- (v)).$$

(b) If $s \in S$ and $t \notin S$, then

$$|f| = f (S, \overline{S}) - f (\overline{S}, S).$$

(c) If $s \in S$ and $t \notin S$, then $|f| \le c (S, \overline{S})$.
(d) Assume that $s \in S$ and $t \notin S$. Then, $|f| = c (S, \overline{S})$ if and only if

$$(f (a) = 0 \text{ for all } a \in [\overline{S}, S]) \tag{11}$$

and

$$(f (a) = c (a) \text{ for all } a \in [S, \overline{S}]). \tag{12}$$

Proof of Proposition 2.7 (sketched). **(a)** We have

$$\sum_{v \in S} \left(f^+(v) - f^-(v) \right)$$

$$= \sum_{v \in S} \underbrace{f^+(v)}_{\substack{= \sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with source } v}} f(a) \\ \text{(by the definition of } f^+(v))}} - \sum_{v \in S} \underbrace{f^-(v)}_{\substack{= \sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} f(a) \\ \text{(by the definition of } f^-(v))}}$$

$$= \underbrace{\sum_{v \in S} \sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } v}} f(a)}_{\substack{= \sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with source in } S}}}} - \underbrace{\sum_{v \in S} \sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} f(a)}_{\substack{= \sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with target in } S}}}}$$

$$= \sum_{\substack{a \in A \text{ is an arc} \\ \text{with source in } S}} f(a) - \sum_{\substack{a \in A \text{ is an arc} \\ \text{with target in } S}} f(a). \tag{13}$$

But recall that $\overline{S}$ denotes the complement $V \setminus S$ of S. Thus, any vertex in V must lie either in S or in $\overline{S}$ (but not in both). Hence, any arc $a \in A$ must either have target in S or target in $\overline{S}$ (but not both). Hence, the sum $\sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with source in } S}} f(a)$ can be split as follows:

$$\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source in } S}} f(a)$$

$$= \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source in } S \\ \text{and target in } S}} f(a)}_{\substack{= \sum\limits_{a \in [S,S]} \\ \text{(by the definition of } [S,S])}} + \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source in } S \\ \text{and target in } \overline{S}}} f(a)}_{\substack{= \sum\limits_{a \in [S,\overline{S}]} \\ \text{(by the definition of } [S,\overline{S}])}}$$

$$= \sum_{a \in [S,S]} f(a) + \sum_{a \in [S,\overline{S}]} f(a). \tag{14}$$

Similarly, we can split $\sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with target in } S}} f(a)$ according to whether the source (not the target this time) of an arc lies in S or in $\overline{S}$. We thus obtain

$$\sum_{\substack{a \in A \text{ is an arc} \\ \text{with target in } S}} f(a) = \sum_{a \in [S,S]} f(a) + \sum_{a \in [\overline{S},S]} f(a). \tag{15}$$

Now, (13) becomes

$$\sum_{v \in S} \left(f^+ (v) - f^- (v) \right)$$

$$= \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source in } S}} f (a)}_{\substack{= \sum_{a \in [S,S]} f(a) + \sum_{a \in [S,\overline{S}]} f(a) \\ \text{(by (14))}}} - \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with target in } S}} f (a)}_{\substack{= \sum_{a \in [S,S]} f(a) + \sum_{a \in [\overline{S},S]} f(a) \\ \text{(by (15))}}}$$

$$= \left(\sum_{a \in [S,S]} f (a) + \sum_{a \in [S,\overline{S}]} f (a) \right) - \left(\sum_{a \in [S,S]} f (a) + \sum_{a \in [\overline{S},S]} f (a) \right)$$

$$= \sum_{a \in [S,\overline{S}]} f (a) - \sum_{a \in [\overline{S},S]} f (a) .$$

Comparing this with

$$\underbrace{f \left(S, \overline{S} \right)}_{\substack{= \sum_{a \in [S,\overline{S}]} f(a) \\ \text{(by the definition of } f(S,\overline{S}))}} - \underbrace{f \left(\overline{S}, S \right)}_{\substack{= \sum_{a \in [\overline{S},S]} f(a) \\ \text{(by the definition of } f(\overline{S},S))}} = \sum_{a \in [S,\overline{S}]} f (a) - \sum_{a \in [\overline{S},S]} f (a) ,$$

we obtain $f \left(S, \overline{S} \right) - f \left(\overline{S}, S \right) = \sum_{v \in S} \left(f^+ (v) - f^- (v) \right)$. This proves Proposition 2.7
(a).

(b) Assume that $s \in S$ and $t \notin S$. Thus, $S \setminus \{s, t\} = S \setminus \{s\}$ (since $t \notin S$).
We can split off the addend for $v = s$ from the sum $\sum_{v \in S} \left(f^+ (v) - f^- (v) \right)$ (since $s \in S$). We thus obtain

$$\sum_{v \in S} \left(f^+ (v) - f^- (v) \right) = \underbrace{\left(f^+ (s) - f^- (s) \right)}_{= |f|} + \sum_{v \in S \setminus \{s\}} \underbrace{\left(f^+ (v) - f^- (v) \right)}_{\substack{= 0 \\ \text{(by (10)} \\ \text{(since } v \in S \setminus \{s\} = S \setminus \{s,t\} \subseteq V \setminus \{s,t\}))}} = |f| .$$

Hence,

$$|f| = \sum_{v \in S} \left(f^+ (v) - f^- (v) \right) = f \left(S, \overline{S} \right) - f \left(\overline{S}, S \right)$$

(by Proposition 2.7 **(a)**). This proves Proposition 2.7 **(b)**.

(c) Assume that $s \in S$ and $t \notin S$. The definition of $f \left(S, \overline{S} \right)$ yields

$$f \left(S, \overline{S} \right) = \sum_{a \in [S,\overline{S}]} \underbrace{f (a)}_{\substack{\leq c(a) \\ \text{(by the capacity} \\ \text{constraints)}}} \leq \sum_{a \in [S,\overline{S}]} c (a) = c \left(S, \overline{S} \right) \qquad (16)$$

(since $c\left(S,\overline{S}\right)$ is defined to be $\sum\limits_{a\in\left[S,\overline{S}\right]}c\left(a\right)$). The definition of $f\left(\overline{S},S\right)$ yields

$$f\left(\overline{S},S\right)=\sum_{a\in\left[\overline{S},S\right]}\underbrace{f\left(a\right)}_{\substack{\geq 0 \\ \text{(by the capacity} \\ \text{constraints)}}}\geq\sum_{a\in\left[\overline{S},S\right]}0=0. \tag{17}$$

Proposition 2.7 **(b)** yields

$$|f|=\underbrace{f\left(S,\overline{S}\right)}_{\substack{\leq c\left(S,\overline{S}\right) \\ \text{(by (16))}}}-\underbrace{f\left(\overline{S},S\right)}_{\substack{\geq 0 \\ \text{(by (17))}}}\leq c\left(S,\overline{S}\right)-0=c\left(S,\overline{S}\right).$$

This proves Proposition 2.7 **(c)**.

    (d) Let us analyze our above proof of Proposition 2.7 **(c)**. We have obtained the inequality (16) by adding together the inequalities $f\left(a\right)\leq c\left(a\right)$ for all $a\in\left[S,\overline{S}\right]$. Thus, the inequality (16) becomes an equality if and only if all of the latter inequalities $f\left(a\right)\leq c\left(a\right)$ for all $a\in\left[S,\overline{S}\right]$ become equalities. Hence, we have the following chain of equivalences:

    (the inequality (16) becomes an equality)

    $\Longleftrightarrow$ (all of the inequalities $f\left(a\right)\leq c\left(a\right)$ for all $a\in\left[S,\overline{S}\right]$ become equalities)

    $\Longleftrightarrow$ $\left(f\left(a\right)=c\left(a\right)\text{ for all }a\in\left[S,\overline{S}\right]\right).$

    We have obtained the inequality (17) by adding together the inequalities $f\left(a\right)\geq 0$ for all $a\in\left[\overline{S},S\right]$. Thus, the inequality (17) becomes an equality if and only if all of the latter inequalities $f\left(a\right)\geq 0$ for all $a\in\left[\overline{S},S\right]$ become equalities. Hence, we have the following chain of equivalences:

    (the inequality (17) becomes an equality)

    $\Longleftrightarrow$ (all of the inequalities $f\left(a\right)\geq 0$ for all $a\in\left[\overline{S},S\right]$ become equalities)

    $\Longleftrightarrow$ $\left(f\left(a\right)=0\text{ for all }a\in\left[\overline{S},S\right]\right).$

    But we have proven the inequality $|f|\leq c\left(S,\overline{S}\right)$ by subtracting the inequality (17) from the inequality (16). Thus, the inequality $|f|\leq c\left(S,\overline{S}\right)$ becomes an equality if and only if both inequalities (17) and (16) become equalities. Hence, we have the

following chain of equivalences:

$$\left(\text{the inequality } |f| \le c\left(S,\overline{S}\right) \text{ becomes an equality}\right)$$
$$\Longleftrightarrow \left(\text{both inequalities (17) and (16) become equalities}\right)$$
$$\Longleftrightarrow \underbrace{\left(\text{the inequality (16) becomes an equality}\right)}$$
$$\Longleftrightarrow \left(f(a)=c(a) \text{ for all } a\in\left[S,\overline{S}\right]\right)$$
$$\wedge \underbrace{\left(\text{the inequality (17) becomes an equality}\right)}$$
$$\Longleftrightarrow \left(f(a)=0 \text{ for all } a\in\left[\overline{S},S\right]\right)$$
$$\Longleftrightarrow \underbrace{\left(f\left(a\right) = c\left(a\right) \text{ for all } a \in \left[S,\overline{S}\right]\right)}_{\Longleftrightarrow \text{ ((12) holds)}} \wedge \underbrace{\left(f\left(a\right) = 0 \text{ for all } a \in \left[\overline{S},S\right]\right)}_{\Longleftrightarrow \text{ ((11) holds)}}$$
$$\Longleftrightarrow \left(\text{(12) holds}\right) \wedge \left(\text{(11) holds}\right)$$
$$\Longleftrightarrow \left(\text{both (11) and (12) hold}\right).$$

In other words, the inequality $|f| \le c\left(S,\overline{S}\right)$ becomes an equality if and only if both (11) and (12) hold. In other words, $|f| = c\left(S,\overline{S}\right)$ if and only if both (11) and (12) hold. This proves Proposition 2.3 **(d)**. $\square$

Remark 2.8. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Let $f : A \to \mathbb{Q}_+$ be a flow on N. Assume that there is no arc $a \in A$ whose source and target are both equal to v. Then, it is easy to see that $f^+\left(v\right) = f\left(\{v\}, \overline{\{v\}}\right)$ and $f^-\left(v\right) = f\left(\overline{\{v\}}, \{v\}\right)$.

2.3. Cuts in networks

Definition 2.9. Let N, V, A, ϕ, s, t and c be as in Definition 2.2.
　(a) An *s-t-cutting subset of V* shall mean a subset S of V satisfying $s \in S$ and $t \notin S$.
　(b) A *cut of N* shall mean a subset of A having the form $\left[S,\overline{S}\right]$ for some *s-t-*cutting subset S of V.
　(c) The *capacity* of a cut $\left[S,\overline{S}\right]$ is defined to be $c\left(S,\overline{S}\right)$. (Note that this is indeed well-defined: In fact, $c\left(S,\overline{S}\right) = \sum\limits_{a\in\left[S,\overline{S}\right]} c\left(a\right)$ clearly depends only on the cut $\left[S,\overline{S}\right]$ rather than on the set S.)

For example, if N is the network shown in (3), then $\{1,3,4\}$ is an *s-t-*cutting subset; the cut $\left[\{1,3,4\}, \overline{\{1,3,4\}}\right]$ corresponding to this subset is $\{\beta, \gamma, \lambda, \kappa\}$ and has capacity

$$c\left(\{1,3,4\}, \overline{\{1,3,4\}}\right) = \sum_{a\in\{\beta,\gamma,\lambda,\kappa\}} c\left(a\right) = \underbrace{c\left(\beta\right)}_{=1} + \underbrace{c\left(\gamma\right)}_{=1} + \underbrace{c\left(\lambda\right)}_{=1} + \underbrace{c\left(\kappa\right)}_{=1} = 4.$$

We can illustrate this cut by drawing all arcs belonging to the cut as double arrows ($\Longrightarrow$):
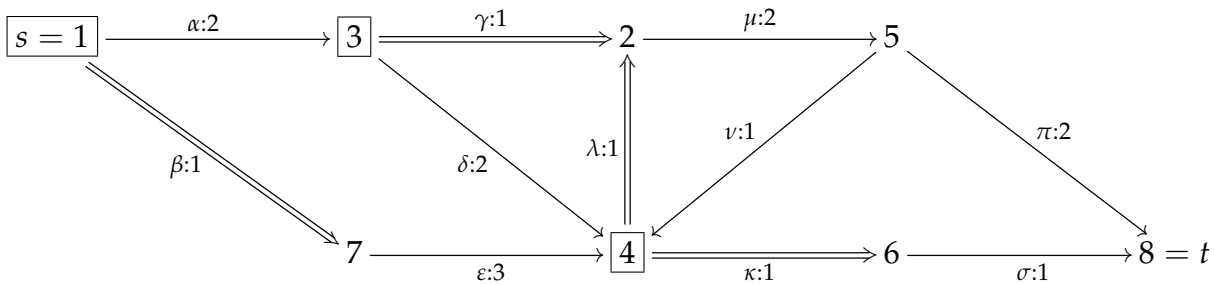
(where the vertices in $\{1, 3, 4\}$ have been marked by boxes).

Notice that every network N (with notations as in Definition 2.2) has two special cuts: the first is the cut

$$\left[\{s\}, \overline{\{s\}} \right] = \{a \in A \mid \text{the source of } a \text{ is } s, \text{ but the target is not} \}$$

corresponding to the s-t-cutting subset $\{s\}$; the second is the cut

$$\left[\overline{\{t\}}, \{t\} \right] = \{a \in A \mid \text{the target of } a \text{ is } t, \text{ but the source is not} \}$$

corresponding to the s-t-cutting subset $\overline{\{t\}}$.

2.4. The max-flow-min-cut theorems

Proposition 2.7 **(c)** thus says that the value of a flow is always $\leq$ to the capacity of a cut. But can we achieve equality?

One of the most important results in combinatorics – the *max-flow-min-cut theorem* – says that "yes": In each network, we can find a flow and a cut such that the value of the flow equals the capacity of the cut. More precisely, there are three "max-flow-min-cut theorems", corresponding to different kinds of flows. The first one is about the kind of flows we have defined above:

Theorem 2.10. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Then,

$$\max \{ |f| \mid f \text{ is a flow} \} = \min \left\{ c \left(S, \overline{S} \right) \mid S \subseteq V; \ s \in S; \ t \notin S \right\}. \qquad (18)$$

In particular, the left-hand side of this equation is well-defined (i.e., there exists a flow f for which $|f|$ is maximum). (Of course, the right-hand side of (18) is well-defined, because there are only finitely many subsets S of V satisfying $s \in S$ and $t \notin S$, and because there exists at least one such subset)

The equality (18) in Theorem 2.10 says that the maximum value of a flow equals the minimum value of $c \left(S, \overline{S} \right)$ where S ranges over the s-t-cutting subsets of V. In other words, the maximum value of a flow equals the minimum capacity of a cut.

Another variant of the max-flow-min-cut theorem makes the same claim about *integer flows* – i.e., flows $f : A \to \mathbb{Q}_+$ such that every arc $a \in A$ satisfies $f(a) \in \mathbb{N}$. Accordingly, it requires that the capacities $c(a)$ of arcs also are integers. Let us state it precisely:

Definition 2.11. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. An *integer flow* means a flow $f : A \to \mathbb{Q}_+$ satisfying ($f(a) \in \mathbb{N}$ for each $a \in A$).

Theorem 2.12. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Assume that $c(a) \in \mathbb{N}$ for each $a \in A$. Then,

$$\max \{|f| \mid f \text{ is an integer flow}\}$$
$$= \min \left\{ c(S, \overline{S}) \mid S \subseteq V; \, s \in S; \, t \notin S \right\}.$$

In particular, the left-hand side of this equation is well-defined (i.e., there exists a flow f for which $|f|$ is maximum).

Finally, a third variant (which we shall not prove) makes the same statement as Theorem 2.10 but with $\mathbb{Q}_+$ replaced by $\mathbb{R}_+$:

Theorem 2.13. Let N, V, A, ϕ, s, t and c be as in Definition 2.2, except that c is now a map $A \to \mathbb{R}_+$ instead of being a map $A \to \mathbb{Q}_+$. (That is, the capacities of arcs are now allowed to be irrational.) Also, let us temporarily modify the definition of a flow in such a way that a flow is a map $A \to \mathbb{R}_+$ instead of being a map $A \to \mathbb{Q}_+$. Then,

$$\max \{|f| \mid f \text{ is a flow}\} = \min \left\{ c(S, \overline{S}) \mid S \subseteq V; \, s \in S; \, t \notin S \right\}.$$

In particular, the left-hand side of this equation is well-defined (i.e., there exists a flow f for which $|f|$ is maximum).

There are various proofs of these three theorems. In particular, Theorem 2.10 and Theorem 2.13 can be viewed as an application of *linear programming duality*, a fundamental principle in linear optimization. We will instead proceed elementarily. Along the way, we will see how to **construct** a flow f maximizing $|f|$ and an *s-t*-cutting subset S minimizing $c(S, \overline{S})$ (two problems that occur in real life).

2.5. The residual digraph

First, let us define the so-called *residual digraph* of a flow:[6]

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[6]This definition is somewhat abstract. See the discussion below and Example 2.15 for what is really going on.

Definition 2.14. Let N, V, A, ϕ, s, t and c be as in Definition 2.2.

(a) If p is a pair $(u, v) \in V \times V$, then we let p^{-1} denote the pair $(v, u) \in V \times V$. (This is simply a suggestive notation; it has nothing to do with reciprocals of numbers.) We call p^{-1} the *reversal* of the pair p. Notice that $\left(p^{-1}\right)^{-1} = a$ for each $p \in V \times V$.

(b) Let $\overleftrightarrow{A}$ be the set $\{0, 1\} \times A$.

For each arc $a \in A$, we define $\overrightarrow{a} \in \overleftrightarrow{A}$ to be the pair $(0, a)$, and we define $\overleftarrow{a} \in \overleftrightarrow{A}$ to be the pair $(1, a)$.

We shall use (some of) these pairs $\overrightarrow{a}$ and $\overleftarrow{a}$ as arcs in a multidigraph we shall soon define. Notice that any element of $\overleftrightarrow{A}$ either has the form $\overrightarrow{a}$ for some uniquely determined $a \in A$, or has the form $\overleftarrow{a}$ for some uniquely determined $a \in A$, but not both at the same time.

(c) Let $f : A \to \mathbb{Q}_+$ be any flow on N. Define a subset A_f of $\overleftrightarrow{A}$ by

$$A_f = \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\} \cup \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}.$$

Define a map $\phi_f : A_f \to V \times V$ by

$$\left(\phi_f\left(\overrightarrow{a} \right) = \phi(a) \text{ for each } a \in A \text{ satisfying } f(a) < c(a) \right) \qquad \text{and}$$
$$\left(\phi_f\left(\overleftarrow{a} \right) = (\phi(a))^{-1} \text{ for each } a \in A \text{ satisfying } f(a) > 0 \right).$$

(d) We define the *residual digraph* D_f to be the multidigraph $\left(V, A_f, \phi_f \right)$.

Let us unpack this definition. The residual digraph D_f of a flow f has the same vertices as the multidigraph (V, A, ϕ) that underlies the network N; but its arcs are different. Namely, the arcs of D_f are described as follows:

- for any arc $a \in A$ satisfying $f(a) < c(a)$ (that is, for any arc $a \in A$ that is "used under capacity" by the flow f), the digraph D_f has an arc $\overrightarrow{a}$ whose source and target are the source and target of a;

- for any arc $a \in A$ satisfying $f(a) > 0$ (that is, for any arc $a \in A$ that sees some positive throughput by the flow f), the digraph D_f has an arc $\overleftarrow{a}$ whose source and target are the target and source of a.

Thus, informally speaking, the residual digraph D_f shows the "wiggle-room" for modifying f without breaking the capacity constraints (ignoring, for the time being, the conservation constraints). Indeed:

- an arc $a \in A$ can afford an increase in the flow going through it (without violating the capacity constraints) if and only if the corresponding $\overrightarrow{a}$ is an arc of D_f;

- an arc $a \in A$ can afford a reduction in the flow going through it (without violating the capacity constraints) if and only if the corresponding $\overleftarrow{a}$ is an arc of D_f.

Of course, if we modify the flow on a single arc, then we will most likely break the conservation constraints. The key to finding a maximum flow is thus to change a flow in such a way that both capacity and conservation constraints are preserved; the residual digraph D_f is merely the first step.

Notice that the arcs of D_f are not arcs of (V, A, ϕ).

Example 2.15. Let f be the following flow:
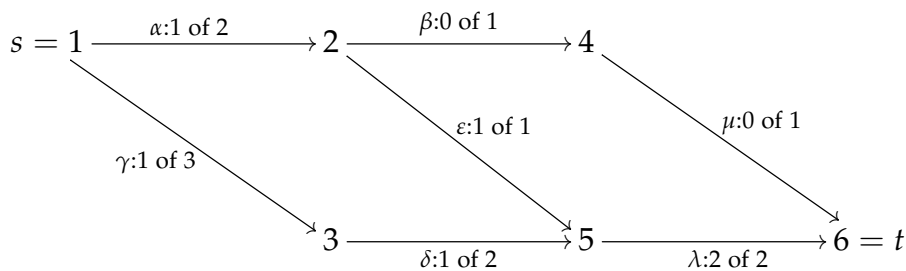
(where, as before, we write "$f(a)$ of $c(a)$" atop each arc a). Then, the residual digraph D_f is
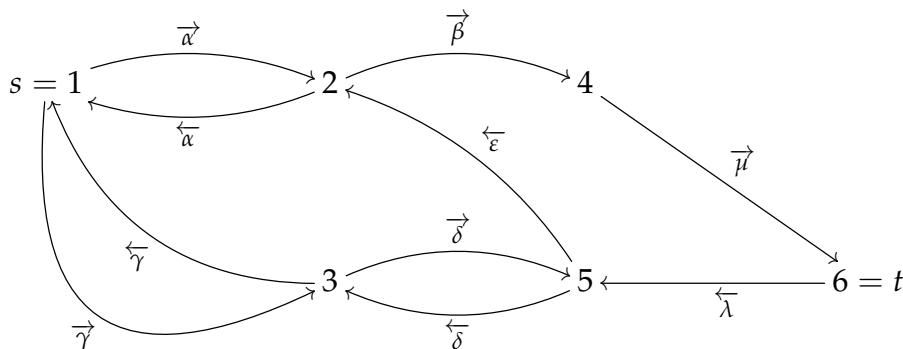
Notice that the multidigraph D_f has cycles even though (V, A, ϕ) has none!

2.6. The augmenting path lemma

The main workhorse of our proof will be the following lemma:

Lemma 2.16. Let N, V, A, ϕ, s, t and c be as in Definition 2.2.
Let $f : A \to \mathbb{Q}_+$ be a flow.
(a) If the multidigraph D_f has a path from s to t, then there is a flow f' with a larger value than f.
(b) If the multidigraph D_f has no path from s to t, then there exists a subset S of V satisfying $s \in V$ and $t \notin V$ and $c(S, \overline{S}) = |f|$.

Before we prove this, we need to lay some more groundwork.

Definition 2.17. Let N, V, A, ϕ, s, t and c be as in Definition 2.2.
   (a) We identify the maps $g : A \to \mathbb{Q}$ satisfying $(g(a) \geq 0$ for all $a \in A)$ (that is, the maps $g : A \to \mathbb{Q}$ satisfying $g(A) \subseteq \mathbb{Q}_+$) with the maps $A \to \mathbb{Q}_+$.
   (b) We extend the notations $f^-(v)$ and $f^+(v)$ from Definition 2.4 to arbitrary maps $f : A \to \mathbb{Q}$ (not just maps $f : A \to \mathbb{Q}_+$). Of course, $f^-(v)$ and $f^+(v)$ will then be elements of $\mathbb{Q}$, not necessarily of $\mathbb{Q}_+$.
   We also extend the notation $|f|$ from Definition 2.5 to arbitrary maps $f : A \to \mathbb{Q}$ (not just flows). Thus, $|f| = f^+(s) - f^-(s)$ for any map $f : A \to \mathbb{Q}$.
   (c) If $f : A \to \mathbb{Q}$ and $g : A \to \mathbb{Q}$ are two maps, then $f + g$ denotes a new map $A \to \mathbb{Q}$ that is defined by

$$(f + g)(a) = f(a) + g(a) \qquad \text{for all } a \in A.$$

   (d) We say that a map $f : A \to \mathbb{Q}$ *satisfies the conservation constraints* if and only if each $v \in V \setminus \{s, t\}$ satisfies $f^-(v) = f^+(v)$.

Notice that any flow $f : A \to \mathbb{Q}_+$ satisfies the conservation constraints.

Lemma 2.18. Let N, V, A, ϕ, s, t and c be as in Definition 2.2.
   (a) Any two maps $f : A \to \mathbb{Q}$ and $g : A \to \mathbb{Q}$ satisfy $|f + g| = |f| + |g|$.
   (b) Let $f : A \to \mathbb{Q}$ and $g : A \to \mathbb{Q}$ be two maps satisfying the conservation constraints. Then, the map $f + g : A \to \mathbb{Q}$ also satisfies the conservation constraints.

Proof of Lemma 2.18 (sketched). **(b)** This is exactly as straightforward as you would expect: Fix $v \in V \setminus \{s, t\}$. Then, the definition of $(f + g)^-(v)$ yields

$$(f + g)^-(v) = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} \underbrace{(f + g)(a)}_{\substack{=f(a)+g(a) \\ \text{(by the definition of } f+g)}} = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} (f(a) + g(a))$$

$$= \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} f(a)}_{\substack{=f^-(v) \\ \text{(by the definition of } f^-(v))}} + \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } v}} g(a)}_{\substack{=g^-(v) \\ \text{(by the definition of } g^-(v))}}$$

$$= f^-(v) + g^-(v).$$

Similarly, $(f + g)^+(v) = f^+(v) + g^+(v)$. But the map f satisfies the conservation constraints; hence, $f^-(v) = f^+(v)$. Similarly, $g^-(v) = g^+(v)$.
   Now, $(f + g)^-(v) = \underbrace{f^-(v)}_{=f^+(v)} + \underbrace{g^-(v)}_{=g^+(v)} = f^+(v) + g^+(v) = (f + g)^+(v)$.
   Now forget that we fixed v. We thus have proven that each $v \in V \setminus \{s, t\}$ satisfies $(f + g)^-(v) = (f + g)^+(v)$. In other words, the map $f + g$ satisfies the conservation constraints. This proves Lemma 2.18 **(b)**.

(a) The proof (which uses the same ideas as the proof of Lemma 2.18 **(b)**) is left to the reader. $\qquad\square$

Lemma 2.19. Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Let $f : A \to \mathbb{Q}_+$ be a flow.

Let **p** be a path from s to t in the multidigraph $D_f = (V, A_f, \phi_f)$.

Let P be the set of arcs of **p**. Thus, $P \subseteq A_f \subseteq \overleftrightarrow{A}$.

Let $\rho \in \mathbb{Q}$.

Define a map $g : A \to \mathbb{Q}$ by setting

$$g(a) = \begin{cases} \rho, & \text{if } \overrightarrow{a} \in P; \\ -\rho, & \text{if } \overleftarrow{a} \in P; \\ 0, & \text{otherwise} \end{cases} \qquad \text{for all } a \in A.$$

(This is well-defined, because the conditions $\overrightarrow{a} \in P$ and $\overleftarrow{a} \in P$ cannot hold at the same time[7].)

(a) This map g satisfies the conservation constraints (i.e., each $v \in V \setminus \{s, t\}$ satisfies $g^-(v) = g^+(v)$).

(b) We have $|g| = \rho$.

Proof of Lemma 2.19 (sketched). **(a)** Let $v \in V \setminus \{s, t\}$ be arbitrary. We must prove that $g^-(v) = g^+(v)$.

If v is not a vertex of the path **p**, then this is obvious (since in this case, each arc a having source v or target v satisfies $g(a) = 0$ (since neither $\overrightarrow{a} \in P$ nor $\overleftarrow{a} \in P$), and thus we have $g^-(v) = 0$ and $g^+(v) = 0$).

Thus, we WLOG assume that v is a vertex of the path **p**. Since v is neither the starting point nor the ending point of this path **p** (because $v \in V \setminus \{s, t\}$, but the starting and ending points of **p** are s and t), we thus conclude that there is exactly one arc $x \in P$ having source v, and exactly one arc $y \in P$ having target v (because **p** is a path, and P is the set of its arcs). Consider these arcs x and y. Notice that x and y are arcs of the multidigraph D_f, not arcs of (V, A, ϕ).

---

[7]*Proof.* Let $a \in A$. We must show that the conditions $\overrightarrow{a} \in P$ and $\overleftarrow{a} \in P$ cannot hold at the same time.

Assume the contrary. Thus, $\overrightarrow{a} \in P$ and $\overleftarrow{a} \in P$ both hold. The definition of ϕ_f shows that $\phi_f(\overrightarrow{a}) = \phi(a)$; thus, the target of $\overrightarrow{a}$ is the target of a. But the definition of ϕ_f also shows that $\phi_f(\overleftarrow{a}) = (\phi(a))^{-1}$; thus, the source of $\overleftarrow{a}$ is the target of a. Hence, the source of $\overleftarrow{a}$ and the target of $\overrightarrow{a}$ are identical (since they both are the target of a).

But both $\overleftarrow{a}$ and $\overrightarrow{a}$ are arcs of the path **p** (since $\overrightarrow{a} \in P$ and $\overleftarrow{a} \in P$). Since the vertices of a path are distinct, this shows that the source of $\overleftarrow{a}$ and the target of $\overrightarrow{a}$ are distinct unless $\overleftarrow{a}$ directly follows $\overrightarrow{a}$ on the path **p**. Therefore, $\overleftarrow{a}$ directly follows $\overrightarrow{a}$ on the path **p** (since the source of $\overleftarrow{a}$ and the target of $\overrightarrow{a}$ are equal). But an analogous argument shows that $\overrightarrow{a}$ directly follows $\overleftarrow{a}$ on the path **p**. The preceding two sentences contradict each other. This contradiction shows that our assumption was false. Qed.

The arcs x and y are distinct[8].
We have

$$y \in P \subseteq A_f = \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\} \cup \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}$$
$$= \left\{ \overrightarrow{b} \mid b \in A \text{ and } f(b) < c(b) \right\} \cup \left\{ \overleftarrow{b} \mid b \in A \text{ and } f(b) > 0 \right\}$$

(here, we renamed the indices a as b). Thus, either $y \in \left\{ \overrightarrow{b} \mid b \in A \text{ and } f(b) < c(b) \right\}$
or $y \in \left\{ \overleftarrow{b} \mid b \in A \text{ and } f(b) > 0 \right\}$.
We have

$$x \in P \subseteq A_f = \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\} \cup \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}.$$

Thus, either $x \in \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\}$ or $x \in \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}$.
Hence, we are in one of the following two cases:

Case 1: We have $x \in \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\}$.

Case 2: We have $x \in \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}$.

Let us consider Case 2. In this case, we have $x \in \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}$.
In other words, $x = \overleftarrow{a}$ for some $a \in A$ satisfying $f(a) > 0$. Consider this a. Since
$\overleftarrow{a} = x \in P$, we have $g(a) = -\rho$ (by the definition of g).
But recall that either $y \in \left\{ \overrightarrow{b} \mid b \in A \text{ and } f(b) < c(b) \right\}$ or
$y \in \left\{ \overleftarrow{b} \mid b \in A \text{ and } f(b) > 0 \right\}$. Hence, we are in one of the following two sub-cases:

Subcase 2.1: We have $y \in \left\{ \overrightarrow{b} \mid b \in A \text{ and } f(b) < c(b) \right\}$

Subcase 2.2: We have $y \in \left\{ \overleftarrow{b} \mid b \in A \text{ and } f(b) > 0 \right\}$.

Let us consider Subcase 2.2. In this subcase, we have $y \in \left\{ \overleftarrow{b} \mid b \in A \text{ and } f(b) > 0 \right\}$.
In other words, $y = \overleftarrow{b}$ for some $b \in A$ satisfying $f(b) > 0$. Consider this b. Since
$\overleftarrow{b} = y \in P$, we have $g(b) = -\rho$.

---

[8]*Proof.* Assume the contrary. Thus, $x = y$. Hence, the arc $x = y$ has source v (since x has source v)
and target v (since y has target v). Thus, the source and the target of this arc x are equal (because
they are both equal to v).
  But the arc x is an arc of the path $\mathbf{p}$ (since $x \in P$), and thus its source and its target are two
different vertices of $\mathbf{p}$. Therefore, its source and its target are distinct (since the vertices of $\mathbf{p}$ are
distinct). This contradicts the fact that its source and its target are equal. This is a contradiction,
qed.

Here is how the two arcs x and y look like in the multidigraph D_f:

$$\xrightarrow{\;\;y\;\;} v \xrightarrow{\;\;x\;\;} .$$

And here is how the corresponding arcs a and b look like in the underlying multidigraph (V, A, ϕ) of our network:

$$\xleftarrow{\;\;b\;\;} v \xleftarrow{\;\;a\;\;} \tag{19}$$

(because $\overleftarrow{a} = x$ and $\overleftarrow{b} = y$). Both arcs a and b are sent to $-\rho$ by g (since $g(a) = -\rho$ and $g(b) = -\rho$). All other arcs in A having source or target v are sent to 0 by g [9]. Hence, in particular, all arcs in A having target v are sent to 0 by g, except for the arc a. Thus, $g^-(v) = g(a) = -\rho$. A similar argument shows that $g^+(v) = -\rho$. Thus, $g^-(v) = -\rho = g^+(v)$. Thus, $g^-(v) = g^+(v)$ is proven in Subcase 2.2.

The proof in Subcase 2.1 is rather similar, with the little difference that now the picture (19) is replaced by

$$\xrightarrow{\;\;y\;\;} v \xleftarrow{\;\;x\;\;} ,$$

and so we have $g^-(v) = (-\rho) + \rho = 0$ and $g^+(v) = 0$. But again the result is the same.

Thus, Case 2 is settled. The proof in Case 1 is similar (again, we need to consider two subcases, depending on whether $y \in \left\{ \overrightarrow{b} \mid b \in A \text{ and } f(b) < c(b) \right\}$ or $y \in \left\{ \overleftarrow{b} \mid b \in A \text{ and } f(b) > 0 \right\}$).

Altogether, we have now proven $g^-(v) = g^+(v)$ in each possible case.

---

[9]*Proof.* Let d be any arc in A having source or target v. Assume that d is distinct from both a and b. We must prove that $g(d) = 0$.

From $d \neq a$, we obtain $\overleftarrow{d} \neq \overleftarrow{a} = x$. Also, clearly, $\overrightarrow{d} \neq \overleftarrow{a} = x$. From $d \neq b$, we obtain $\overleftarrow{d} \neq \overleftarrow{b} = y$. Also, clearly, $\overrightarrow{d} \neq \overleftarrow{b} = y$.

Let us first assume that d has source v. Thus, the arc $\overleftarrow{d}$ of D_f has target v (by the definition of D_f). But the only arc in P having target v is y (by the definition of y). Thus, if we had $\overleftarrow{d} \in P$, then we would have $\overleftarrow{d} = y$ (since $\overleftarrow{d}$ would be an arc in P having target v), which would contradict $\overleftarrow{d} \neq y$. Hence, we cannot have $\overleftarrow{d} \in P$.

Recall again that d has source v. Thus, the arc $\overrightarrow{d}$ of D_f has source v (by the definition of D_f). But the only arc in P having source v is x (by the definition of x). Thus, if we had $\overrightarrow{d} \in P$, then we would have $\overrightarrow{d} = x$ (since $\overrightarrow{d}$ would be an arc in P having source v), which would contradict $\overrightarrow{d} \neq x$. Hence, we cannot have $\overrightarrow{d} \in P$.

Hence, we have neither $\overleftarrow{d} \in P$ nor $\overrightarrow{d} \in P$. According to the definition of g, we thus conclude that $g(d) = 0$.

Now, forget our assumption that d has source v. We thus have shown that if d has source v, then $g(d) = 0$. A similar argument shows that if d has target v, then $g(d) = 0$. Combining these two statements, we conclude that we always have $g(d) = 0$ (because d has source or target v). This completes our proof of $g(d) = 0$.

Now, forget that we fixed v. We thus have shown that each $v \in V \setminus \{s, t\}$ satisfies $g^-(v) = g^+(v)$. In other words, g satisfies the conservation constraints. This proves Lemma 2.19 **(a)**.

(b) This is somewhat similar to our above proof of Lemma 2.19 **(a)**, with the vertex s playing the role of v.

The vertex s is the starting point of the path **p**, but not its ending point (since its ending point is $t \neq s$). Hence, there is exactly one arc $x \in P$ having source s, and there are no arcs $y \in P$ having target s.

We have

$$x \in P \subseteq A_f = \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\} \cup \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}.$$

Thus, either $x \in \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\}$ or $x \in \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}$. Hence, we are in one of the following two cases:

Case 1: We have $x \in \left\{ \overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a) \right\}$.

Case 2: We have $x \in \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}$.

Let us consider Case 2. In this case, we have $x \in \left\{ \overleftarrow{a} \mid a \in A \text{ and } f(a) > 0 \right\}$. In other words, $x = \overleftarrow{a}$ for some $a \in A$ satisfying $f(a) > 0$. Consider this a. Since $\overleftarrow{a} = x \in P$, we have $g(a) = -\rho$ (by the definition of g).

Thus, the arc a is sent to $-\rho$ by g. All other arcs in A having source or target s are sent to 0 by g [10]. Hence, in particular, all arcs in A having target s are sent to 0 by g, except for the arc a. Thus, $g^-(s) = g(a) = -\rho$. A similar argument shows that $g^+(s) = 0$. Now, the definition of $|g|$ yields $|g| = \underbrace{g^+(s)}_{=0} - \underbrace{g^-(s)}_{=-\rho} = 0 - (-\rho) = \rho$.

Thus, Lemma 2.19 **(b)** is proven in Case 2.

A similar argument works in Case 1. Thus, Lemma 2.19 **(b)** is always proven. $\square$

We are now ready to prove Lemma 2.16:

Proof of Lemma 2.16 (sketched). **(a)** Assume that the multidigraph D_f has a path from s to t. Fix such a path, and denote it by **p**.

Let P be the set of arcs of **p**. Thus, $P \subseteq A_f$ (since **p** is a path in $D_f = (V, A_f, \phi_f)$). The path **p** is a path from s to t, and thus has at least one arc (since $s \neq t$); hence, the set P is nonempty.

For each arc $b \in P$, we define a number $\rho_b \in \mathbb{Q}$ by

$$\rho_b = \begin{cases} c(a) - f(a), & \text{if } b = \overrightarrow{a} \text{ for some } a \in A; \\ f(a), & \text{if } b = \overleftarrow{a} \text{ for some } a \in A \end{cases}.$$

(This is well-defined, because each $b \in P$ either has the form $b = \overrightarrow{a}$ for some $a \in A$ or has the form $b = \overleftarrow{a}$ for some $a \in A$; this follows from $b \in P \subseteq A_f$.)

---

[10] This is easy to check – just as in the above proof of Lemma 2.19 **(a)**.

For each arc $b \in P$, the number ρ_b is positive[11]. We now define $\rho \in \mathbb{Q}$ by $\rho = \min \{\rho_b \mid b \in P\}$. (This is well-defined, since P is nonempty and finite.) Then, ρ is positive (since all ρ_b are positive), so that $\rho \geq 0$.

Now, define a map $g : A \to \mathbb{Q}$ as in Lemma 2.19. Then, Lemma 2.19 **(a)** shows that the map g satisfies the conservation constraints. Also, the map f satisfies the conservation constraints (since f is a flow). Hence, Lemma 2.18 **(b)** shows that the map $f + g$ satisfies the conservation constraints. Also, Lemma 2.18 **(a)** shows that
$$|f + g| = |f| + \underbrace{|g|}_{\substack{=\rho \\ \text{(by Lemma 2.19 \textbf{(b)})}}} = |f| + \rho.$$

Next, we claim that the map $f + g$ also satisfies the capacity constraints – i.e., that we have
$$0 \leq (f + g)(a) \leq c(a) \qquad \text{for each } a \in A. \tag{20}$$

[*Proof of (20):* Fix $a \in A$. Recall that f is a flow; thus, f satisfies the capacity constraints. Hence, $0 \leq f(a) \leq c(a)$.

Now, we are in one of the following three cases:

Case 1: We have $\overrightarrow{a} \in P$.

Case 2: We have $\overleftarrow{a} \in P$.

Case 3: We have neither $\overrightarrow{a} \in P$ nor $\overleftarrow{a} \in P$.

Let us first consider Case 3. In this case, the definition of g yields $g(a) = 0$. Thus, $(f + g)(a) = f(a) + \underbrace{g(a)}_{=0} = f(a)$. Hence, $0 \leq f(a) \leq c(a)$ rewrites as $0 \leq (f + g)(a) \leq c(a)$. Thus, (20) is proven in Case 3.

Let us now consider Case 2. In this case, we have $\overleftarrow{a} \in P$. Hence, the definition of g yields $g(a) = -\rho$. But the definition of ρ yields $\rho = \min \{\rho_b \mid b \in P\} \leq \rho_{\overleftarrow{a}}$ (since $\overleftarrow{a} \in P$ and thus $\rho_{\overleftarrow{a}} \in \{\rho_b \mid b \in P\}$). Also, the definition of $\rho_{\overleftarrow{a}}$ yields $\rho_{\overleftarrow{a}} = f(a)$.

---

[11]*Proof.* Let $b \in P$ be any arc. We must prove that ρ_b is positive.

We have $b \in P \subseteq A_f = \{\overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a)\} \cup \{\overleftarrow{a} \mid a \in A \text{ and } f(a) > 0\}$. Thus, either $b \in \{\overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a)\}$ or $b \in \{\overleftarrow{a} \mid a \in A \text{ and } f(a) > 0\}$. So we are in one of the following two cases:

Case 1: We have $b \in \{\overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a)\}$.

Case 2: We have $b \in \{\overleftarrow{a} \mid a \in A \text{ and } f(a) > 0\}$.

Let us first consider Case 1. In this case, we have $b \in \{\overrightarrow{a} \mid a \in A \text{ and } f(a) < c(a)\}$. Thus, $b = \overrightarrow{a}$ for some $a \in A$ satisfying $f(a) < c(a)$. Consider this a. The definition of ρ_b yields $\rho_b = c(a) - f(a)$ (since $b = \overrightarrow{a}$), and thus $\rho_b = c(a) - f(a) > 0$ (since $f(a) < c(a)$). Thus, we have shown that ρ_b is positive in Case 1.

Let us now consider Case 2. In this case, we have $b \in \{\overleftarrow{a} \mid a \in A \text{ and } f(a) > 0\}$. Thus, $b = \overleftarrow{a}$ for some $a \in A$ satisfying $f(a) > 0$. Consider this a. The definition of ρ_b yields $\rho_b = f(a)$ (since $b = \overleftarrow{a}$), and thus $\rho_b = f(a) > 0$. Thus, we have shown that ρ_b is positive in Case 2.

We have now proven that ρ_b is positive in both Cases 1 and 2. Hence, ρ_b is always positive, qed.

Hence, $\rho \leq \rho_{\overleftarrow{a}} = f(a)$. Now, $(f+g)(a) = f(a) + \underbrace{g(a)}_{=-\rho} = f(a) - \rho \geq 0$ (since

$\rho \leq f(a)$). Combining this with $(f+g)(a) = f(a) + \underbrace{g(a)}_{=-\rho} = f(a) - \underbrace{\rho}_{\geq 0} \leq f(a) \leq$

$c(a)$, we obtain $0 \leq (f+g)(a) \leq c(a)$. Thus, (20) is proven in Case 2.

Let us finally consider Case 1. In this case, we have $\overrightarrow{a} \in P$. Hence, the definition of g yields $g(a) = \rho$. But the definition of ρ yields $\rho = \min\{\rho_b \mid b \in P\} \leq \rho_{\overrightarrow{a}}$ (since $\overrightarrow{a} \in P$ and thus $\rho_{\overrightarrow{a}} \in \{\rho_b \mid b \in P\}$). Also, the definition of $\rho_{\overrightarrow{a}}$ yields $\rho_{\overrightarrow{a}} = c(a) - f(a)$. Hence, $\rho \leq \rho_{\overrightarrow{a}} = c(a) - f(a)$. Now, $(f+g)(a) = f(a) + \underbrace{g(a)}_{=\rho} = f(a) + \rho \leq c(a)$ (since $\rho \leq c(a) - f(a)$). Combining this with $(f+g)(a) = f(a) + \underbrace{g(a)}_{=\rho} = f(a) + \underbrace{\rho}_{\geq 0} \geq f(a) \geq 0$, we obtain $0 \leq (f+g)(a) \leq c(a)$. Thus, (20) is proven in Case 1.

We have thus proven (20) in all three Cases 1, 2 and 3. Thus, (20) is proven.]

From (20), we conclude in particular that $(f+g)(a) \geq 0$ for each $a \in A$. Thus, $f + g$ is a map $A \to \mathbb{Q}_+$. We have shown that this map $f + g : A \to \mathbb{Q}_+$ satisfies both the capacity constraints and the conservation constraints. In other words, $f + g$ is a flow (by the definition of a flow). Furthermore, this flow has value $|f+g| = |f| + \rho > |f|$ (since ρ is positive). In other words, it has a larger value than f. Thus, there is a flow f' with a larger value than f (namely, $f' = f + g$). This proves Lemma 2.16 **(a)**.

(b) Assume that the multidigraph D_f has no path from s to t. We must prove that there exists a subset S of V satisfying $s \in V$ and $t \notin V$ and $c(S, \overline{S}) = |f|$.

Indeed, define a subset S of V by

$$S = \{v \in V \mid \text{the multidigraph } D_f \text{ has a path from } s \text{ to } v\}.$$

We shall show that $s \in S$ and $t \notin S$ and $c(S, \overline{S}) = |f|$. This will clearly complete the proof of Lemma 2.16 **(b)**.

First of all, the multidigraph D_f clearly has a path from s to s (namely, the trivial path (s)). In other words, $s \in S$ (by the definition of S).

Furthermore, the multidigraph D_f has no path from s to t (by assumption). In other words, $t \notin S$ (by the definition of S).

It thus remains to show that $c(S, \overline{S}) = f$.

We first notice the following:

> *Observation 1:* Let $b \in A_f$ be an arc whose source belongs to S. Then, the target of b also belongs to S.

[*Proof of Observation 1:* Let u be the source of the arc b. Then, $u \in S$ (since the source of b belongs to S). In other words, the multidigraph D_f has a path from s to u (by the definition of S). Fix such a path, and denote it by $\mathbf{p}$.

Also, let v be the target of the arc b. The arc b is an arc of the multidigraph D_f (since $b \in A_f$), and its source is the ending point of the path $\mathbf{p}$ (namely, the point u). Thus, we can extend the path $\mathbf{p}$ by the arc b (that is, we append the arc b and the vertex v to the end of the path $\mathbf{p}$) to obtain a walk from s to v in D_f. Hence, there exists a walk from s to v in D_f. Thus, Proposition 1.11 (applied to D_f and v instead of D and t) yields that there exists a path from s to v in D_f. In other words, the multidigraph D_f has a path from s to v. In other words, $v \in S$ (by the definition of S). In other words, the target of b belongs to S (since v is the target of b). This proves Observation 1.]

Next, we observe that

$$f(a) = 0 \qquad \text{for each } a \in \left[\overline{S}, S\right]. \tag{21}$$

[*Proof of (21):* Let $a \in \left[\overline{S}, S\right]$. Thus, $a \in A$ is an arc whose source belongs to $\overline{S}$ and whose target belongs to S (by the definition of $\left[\overline{S}, S\right]$).

We must prove that $f(a) = 0$. Assume the contrary. Thus, $f(a) \neq 0$, so that $f(a) > 0$ (since $f(a) \geq 0$). By the definition of A_f, we thus have $\overleftarrow{a} \in A_f$. But the source of the arc $\overleftarrow{a}$ is the target of a, and therefore belongs to S (since the target of a belongs to S). Hence, Observation 1 (applied to $b = \overleftarrow{a}$) yields that the target of $\overleftarrow{a}$ also belongs to S. Since the target of $\overleftarrow{a}$ is the source of a, this means that the source of a belongs to S. This contradicts the fact that the source of a belongs to $\overline{S}$. This contradiction shows that our assumption was false. Hence, $f(a) = 0$ holds, and (21) is proven.]

Similarly, we have

$$f(a) = c(a) \qquad \text{for each } a \in \left[S, \overline{S}\right]. \tag{22}$$

[*Proof of (22):* Let $a \in \left[S, \overline{S}\right]$. Thus, $a \in A$ is an arc whose source belongs to S and whose target belongs to $\overline{S}$ (by the definition of $\left[S, \overline{S}\right]$).

We must prove that $f(a) = c(a)$. Assume the contrary. Thus, $f(a) \neq c(a)$, so that $f(a) < c(a)$ (since $f(a) \leq c(a)$ (by the capacity constraints)). By the definition of A_f, we thus have $\overrightarrow{a} \in A_f$. But the source of the arc $\overrightarrow{a}$ is the source of a, and therefore belongs to S (since the source of a belongs to S). Hence, Observation 1 (applied to $b = \overrightarrow{a}$) yields that the target of $\overrightarrow{a}$ also belongs to S. Since the target of $\overrightarrow{a}$ is the target of a, this means that the target of a belongs to S. This contradicts the fact that the target of a belongs to $\overline{S}$. This contradiction shows that our assumption was false. Hence, $f(a) = c(a)$ holds, and (22) is proven.]

Now, the definition of $f\left(\overline{S}, S\right)$ yields $f\left(\overline{S}, S\right) = \sum_{a \in \left[\overline{S}, S\right]} \underbrace{f(a)}_{\substack{=0 \\ \text{(by (21))}}} = 0$.

Also, the definition of $f\left(S, \overline{S}\right)$ yields $f\left(S, \overline{S}\right) = \sum_{a \in \left[S, \overline{S}\right]} \underbrace{f(a)}_{\substack{=c(a) \\ \text{(by (22))}}} = \sum_{a \in \left[S, \overline{S}\right]} c(a) =$

$c\left(S, \overline{S}\right)$ (by the definition of $c\left(S, \overline{S}\right)$).

Now, Proposition 2.7 **(b)** yields

$$|f| = \underbrace{f\left(S,\overline{S}\right)}_{=c\left(S,\overline{S}\right)} - \underbrace{f\left(\overline{S},S\right)}_{=0} = c\left(S,\overline{S}\right).$$

In other words, $c\left(S,\overline{S}\right) = |f|$. This completes our proof of Lemma 2.16 **(b)**. □

An analogue of Lemma 2.16 holds for integer flows (assuming integer capacities):

> **Lemma 2.20.** Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Assume that $c(a) \in \mathbb{N}$ for each $a \in A$.
> Let $f : A \to \mathbb{Q}_+$ be an integer flow. Then,
> **(a)** If the multidigraph D_f has a path from s to t, then there is an integer flow f' with a larger value than f.
> **(b)** If the multidigraph D_f has no path from s to t, then there exists a subset S of V satisfying $s \in V$ and $t \notin V$ and $c\left(S,\overline{S}\right) = |f|$.

Proof of Lemma 2.20 (sketched). The same argument we gave above for Lemma 2.16 works here, but we need to notice that all the ρ_b are integers (since the $f(a)$ and the $c(a)$ are integers), and therefore ρ is an integer. □

2.7. The Ford-Fulkerson algorithm

We next claim the following:

> **Lemma 2.21.** Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Assume that $c(a) \in \mathbb{N}$ for each $a \in A$. Then, there exist an integer flow $f : A \to \mathbb{Q}_+$ and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c\left(S,\overline{S}\right) = |f|$.

Proof of Lemma 2.21 (sketched). Let us first recall that if we are given a multidigraph and two of its vertices, we can easily check whether there is a path from one vertex to the other. (A stupid way to do this would be to try all possible paths[12]; but there are fast algorithms, such as Dijkstra's algorithm.)

We claim that the following algorithm always terminates and constructs an integer flow $f : A \to \mathbb{Q}_+$ and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c\left(S,\overline{S}\right) = |f|$:

> **Ford-Fulkerson algorithm:**
>
> **Input:** N, V, A, ϕ, s, t and c as in Definition 2.2, with the property that $c(a) \in \mathbb{N}$ for each $a \in A$.
>
> **Output:** an integer flow $f : A \to \mathbb{Q}_+$ and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c\left(S,\overline{S}\right) = |f|$.
>
> **Algorithm:**

---

[12]This is doable, since Proposition 1.13 **(c)** shows that there are only finitely many paths (and the proof of Proposition 1.13 **(c)** tells us how to find them all).

- Define an integer flow $f : A \to \mathbb{Q}_+$ by ($f(a) = 0$ for all $a \in A$). (This is clearly a flow; it is called the *zero flow*.)

- **While**[13] the multidigraph D_f has a path from s to t [14], **do the following:**

  - Apply Lemma 2.20 **(a)** to construct an integer flow f' with a larger value than f. (Strictly speaking, what we mean is to apply the construction given in the proof of Lemma 2.20 **(a)**.)

  - Replace f by f'. (Thus, the flow f now has a larger value than it had before.)

  Endwhile.

- Now, the multidigraph D_f has no path from s to t. Therefore, Lemma 2.20 **(b)** shows that there exists a subset S of V satisfying $s \in V$ and $t \notin V$ and $c\left(S, \overline{S}\right) = |f|$. Consider this S. (Again, the actual construction of S is hidden in the proof of Lemma 2.20 **(b)**.)

- Output (the current values of) f and S.

Why does this algorithm work? In other words, why does it necessarily terminate (i.e., why don't we get stuck in the while loop forever?), and why does the output satisfy the requirements (that $f : A \to \mathbb{Q}_+$ is an integer flow and that S is a subset of V satisfying $s \in S$ and $t \notin S$ and $c\left(S, \overline{S}\right) = |f|$) ?

The second question has already been answered in the description of the algorithm; so all we need to do is answer the first question. In other words, we need to prove the following claim:

> *Claim 1:* The algorithm cannot get stuck in the while loop (i.e., it cannot happen that it keeps running this loop forever).

[*Proof of Claim 1:* The value of any integer flow is an integer. Thus, $|f|$ is an integer throughout the execution of the algorithm (since f is an integer flow throughout the execution of the algorithm).

On the other hand, Proposition 2.7 **(c)** (applied to $\{s\}$ instead of S) shows that $|f| \leq c\left(\{s\}, \overline{\{s\}}\right)$ throughout the execution of the algorithm (since $s \in \{s\}$ and $t \notin \{s\}$).

---

[13]This word "**While**" marks the beginning of a while loop. The end of this loop is marked by the word "**Endwhile**". Roughly speaking, the instructions inside a while loop need to be performed over and over until the condition is no longer true (i.e., in our case, until the multidigraph D_f no longer has a path from s to t). So we need to check whether D_f has a path from s to t, then – if there is such a path – perform whatever is inside the while loop, then again check whether D_f has a path from s to t, then – if there is such a path – again perform whatever is inside the while loop, and so on, until we finally fail the check (i.e., until D_f no longer has a path from s to t). Only then do we proceed beyond the while loop.

[14]This can be checked, e.g., using Dijkstra's algorithm, as mentioned above.

Each iteration of the while loop increases the value $|f|$ (because f is replaced by a new integer flow f' with a larger value than the previous f), and thus increases it by at least 1 (since $|f|$ is an integer throughout the execution of the algorithm, but if an integer increases, then it increases by at least 1). Hence, if the while loop is traversed K times (for some $K \in \mathbb{N}$), then the value $|f|$ increases by at least K, and thus this value $|f|$ must be at least K afterwards (because the value $|f|$ at the beginning of the algorithm is 0). Applying this to $K = c\left(\{s\}, \overline{\{s\}}\right) + 1$, we conclude that if the while loop is traversed $c\left(\{s\}, \overline{\{s\}}\right) + 1$ times, then the value $|f|$ must be at least $c\left(\{s\}, \overline{\{s\}}\right) + 1$ afterwards. Since the value $|f|$ can never be at least $c\left(\{s\}, \overline{\{s\}}\right) + 1$ (because we have $|f| \leq c\left(\{s\}, \overline{\{s\}}\right) < c\left(\{s\}, \overline{\{s\}}\right) + 1$ throughout the execution of the algorithm), we thus conclude that the while loop cannot be traversed $c\left(\{s\}, \overline{\{s\}}\right) + 1$ times. Thus, the while loop can only be traversed at most $c\left(\{s\}, \overline{\{s\}}\right)$ times. In particular, the algorithm must eventually leave the while loop. This proves Claim 1.]

So we have shown that the algorithm works. Clearly, its output gives us precisely the f and the S whose existence is claimed by Lemma 2.21. Thus, Lemma 2.21 is proven. $\qquad\square$

Finally, we can prove Theorem 2.12:

Proof of Theorem 2.12 (sketched). Lemma 2.21 shows that there exist an integer flow $f : A \to \mathbb{Q}_+$ and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c\left(S, \overline{S}\right) = |f|$. Denote these f and S by g and Q. Thus, $g : A \to \mathbb{Q}_+$ is an integer flow, and Q is a subset of V satisfying $s \in Q$, $t \notin Q$ and $c\left(Q, \overline{Q}\right) = |g|$.

The set $\left\{c\left(S, \overline{S}\right) \mid S \subseteq V;\ s \in S;\ t \notin S\right\}$ is nonempty and finite. Hence, its minimum $\min\left\{c\left(S, \overline{S}\right) \mid S \subseteq V;\ s \in S;\ t \notin S\right\}$ is well-defined.

We have $|g| = c\left(Q, \overline{Q}\right) \in \left\{c\left(S, \overline{S}\right) \mid S \subseteq V;\ s \in S;\ t \notin S\right\}$ (since $Q \subseteq V$ and $s \in Q$ and $t \notin Q$) and thus

$$|g| \geq \min\left\{c\left(S, \overline{S}\right) \mid S \subseteq V;\ s \in S;\ t \notin S\right\}. \tag{23}$$

On the other hand, $|f| \leq c\left(Q, \overline{Q}\right)$ for every integer flow f (by Proposition 2.7 **(c)**, applied to $S = Q$). In other words, $|f| \leq |g|$ for every integer flow f (since $c\left(Q, \overline{Q}\right) = |g|$). In other words, any element of the set $\{|f| \mid f \text{ is an integer flow}\}$ is $\leq |g|$. Since we also know that $|g|$ is an element of this set (because g is an integer flow), we thus conclude that $|g|$ is the maximum of this set. In other words,

$$|g| = \max\{|f| \mid f \text{ is an integer flow}\} \tag{24}$$

(and in particular, $\max\{|f| \mid f \text{ is an integer flow}\}$ exists). Now, (23) yields

$$\min\left\{c\left(S, \overline{S}\right) \mid S \subseteq V;\ s \in S;\ t \notin S\right\} \leq |g| = \max\{|f| \mid f \text{ is an integer flow}\}.$$

Combining this with

$$\max \left\{ |f| \ \mid \ f \text{ is an integer flow} \right\} \leq \min \left\{ c\left(S, \overline{S} \right) \ \mid \ S \subseteq V; \ s \in S; \ t \notin S \right\}$$

(because every subset S of V satisfying $s \in S$ and $t \notin S$ and every integer flow f satisfy $|f| \leq c\left(S, \overline{S} \right)$ (by Proposition 2.7 **(c)**)), we obtain

$$\max \left\{ |f| \ \mid \ f \text{ is an integer flow} \right\} = \min \left\{ c\left(S, \overline{S} \right) \ \mid \ S \subseteq V; \ s \in S; \ t \notin S \right\}.$$

This proves Theorem 2.12. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

> **Remark 2.22.** The Ford-Fulkerson algorithm that we used in our proof of Lemma 2.21 constructs an integer flow f and a subset S of V. Denote these f and S by g and S. Then, g is an integer flow of maximum value (among all integer flows on N). This follows immediately from (24). Thus, we have found an algorithm to construct an integer flow of maximum value.

The proof of Theorem 2.10 is no longer far away. We first need an analogue of Lemma 2.21:

> **Lemma 2.23.** Let N, V, A, ϕ, s, t and c be as in Definition 2.2. Then, there exist a flow $f : A \to \mathbb{Q}_+$ and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c\left(S, \overline{S} \right) = |f|$.

Proof of Lemma 2.23 (sketched). The numbers $c\left(a \right)$ for $a \in A$ are finitely many non-negative rational numbers (since c is a map $A \to \mathbb{Q}_+$). Thus, they can be brought to a common denominator: i.e., there exists a positive integer p such that $p \cdot c\left(a \right) \in \mathbb{N}$ for each $a \in A$. Consider this p. Define the function $c_p : A \to \mathbb{Q}_+$ by

$$c_p\left(a \right) = p \cdot c\left(a \right) \qquad \text{for each } a \in A.$$

Thus, $c_p\left(a \right) = p \cdot c\left(a \right) \in \mathbb{N}$ for each $a \in A$.

Let N_p be the network which is the same as N, except that the capacity function c has been replaced by c_p. Thus, Lemma 2.21 (applied to N_p and c_p instead of N and c) yields that there exist an integer flow $f : A \to \mathbb{Q}_+$ on the network N_p and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c_p\left(S, \overline{S} \right) = |f|$. Consider these f and S, and denote them by f_p and S. Thus, f_p is an integer flow on the network N_p, and S is a subset of V satisfying $s \in S$ and $t \notin S$ and $c_p\left(S, \overline{S} \right) = |f_p|$.

It is easy to see that $c_p\left(P, Q \right) = p \cdot c\left(P, Q \right)$ for any two subsets P and Q of V. Thus, $c_p\left(S, \overline{S} \right) = p \cdot c\left(S, \overline{S} \right)$, so that $c\left(S, \overline{S} \right) = \underbrace{c_p\left(S, \overline{S} \right)}_{=|f_p|} / p = |f_p| / p$.

Now, define a map $f : A \to \mathbb{Q}_+$ (not an integer flow, in general) by setting

$$f\left(a \right) = f_p\left(a \right) / p \qquad \text{for each } a \in A.$$

This map f is obtained by rescaling the flow f_p on the network N_p by the factor $1/p$; thus, it is easy to see that f is a flow on the network N. (Roughly speaking,

the capacity constraints for f are obtained from the capacity constraints for f_p by dividing by p; the same holds for the conservation constraints.)

Also, it is easy to see that $|f| = |f_p| / p$. Comparing this with $c(S, \overline{S}) = |f_p| / p$, we obtain $c(S, \overline{S}) = |f|$.

We thus have found a flow $f : A \to \mathbb{Q}_+$ on the network N and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c(S, \overline{S}) = |f|$. This proves Lemma 2.23. $\qquad\square$

Proof of Theorem 2.10 (sketched). The proof of Theorem 2.10 is analogous to the proof of Theorem 2.12, but we need to use Lemma 2.23 instead of Lemma 2.21. $\qquad\square$

As we said, we will not prove Theorem 2.13 here; its proof is harder. An analogue of Lemma 2.23 for $\mathbb{R}_+$ instead of $\mathbb{Q}_+$ indeed holds, but the way we proved Lemma 2.23 does not generalize to $\mathbb{R}_+$ (since there are no common denominators for incommensurable irrational numbers). Notice also that the Ford-Fulkerson algorithm (which we showed in the proof of Lemma 2.21) works for flows $f : A \to \mathbb{Q}_+$, but may fail when $\mathbb{Q}_+$ is replaced by $\mathbb{R}_+$; the paper [Zwick95] gives an example where it gets stuck in the while loop due to an arc with irrational capacity (the golden ratio). The good news is that Theorem 2.13 is almost entirely useless in combinatorics, to my knowledge at least. If you want to see a proof of Theorem 2.13, check out texts on linear programming.

> **Exercise 1.** Assume that we generalize the concept of a network by allowing the capacity function c to take the value ∞ as well (where ∞ is a symbol that is understood to satisfy $\infty + q = \infty$ and $\infty + \infty = \infty$ and $\infty > q$ for all $q \in \mathbb{Q}$). In other words, some arcs a may have "infinite capacity", which means that the capacity constraints for these arcs a simply say that $0 \leq f(a)$ (without requiring $f(a)$ to be $\leq$ to anything specific). Prove the following:
>
> **(a)** Theorem 2.10 and Theorem 2.12 still hold in this generality, if we additionally assume that there exists **some** subset S of V satisfying $s \in S$ and $t \notin S$ and $c(S, \overline{S}) < \infty$.
>
> **(b)** If every subset S of V satisfying $s \in S$ and $t \notin S$ satisfies $c(S, \overline{S}) = \infty$, then prove that, for every $n \in \mathbb{N}$, there exists an integer flow f of value $|f| = n$.

3. Application: Bipartite matching

3.1. Simple graphs and multigraphs

As we said, the max-flow-min-cut theorems (particularly, Theorem 2.12) have multiple applications. In many of these applications, the network isn't visible right away, but instead has to be constructed. Let me present one such application: the bipartite matching problem.

This problem is concerned with undirected graphs, so let us first define the latter. Again, there are two types of undirected graphs: the simple graphs and the

multigraphs. We will use the latter, but let us define both of them. We begin by defining simple graphs.

Definition 3.1. If W is a set, then $\mathcal{P}_2(W)$ shall mean the set of all 2-element subsets of W. For example,

$$\mathcal{P}_2(\{1,2,3,4\}) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

Definition 3.2. A *simple graph* is defined to be a pair (W, E) consisting of a finite set W and a subset E of $\mathcal{P}_2(W)$. The elements of W are called the *vertices* of this simple graph; the elements of E are called its *edges*. If e is an edge of a simple graph, then the two vertices contained in e are called the *endpoints* of e. Moreover, if u and v are the two endpoints of an edge e, then we say that the edge e *joins* u with v.

Example 3.3. (a) The pair

$$(\{1,2,3\},\{\{1,3\},\{2,3\}\})$$

is a simple graph. Its vertices are $1,2,3$. Its edges are $\{1,3\}$ and $\{2,3\}$. The edge $\{1,3\}$ has endpoints 1 and 3, and can also be written as $\{3,1\}$.
   (b) The pair

$$(\{1,3,5\},\varnothing)$$

is a simple graph. Its vertices are $1,3,5$. It has no edges.

Note that a 1-element set $\{v\}$ can never be the edge of a simple graph, at least according to our definition of a simple graph. Thus, the two endpoints of an edge must always be distinct. (Some authors use a slightly different definition of a simple graph, which uses $W \cup \mathcal{P}_2(W)$ instead of $\mathcal{P}_2(W)$; with this definition, the two endpoints of an edge could be equal.)

A simple graph (W, E) can be visually represented as follows:

- For each vertex $v \in W$, choose a point in the plane and label it with a "v".

- For each edge $\{u, v\} \in E$, draw a curve from the point labelled "u" to the point labelled "v".

There are many ways to represent a given graph.

Example 3.4. (a) The simple graph $(\{1,2,3\},\{\{1,3\},\{2,3\}\})$ from Example 3.3 **(a)** can be represented as follows:
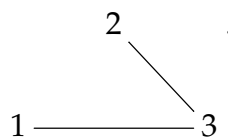
It can also be represented as follows:
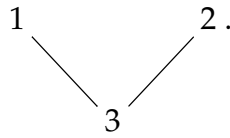
(b) The simple graph $(\{1, 3, 5\}, \varnothing)$ from Example 3.3 **(b)** can be represented as follows:

$$1 \qquad 5 \qquad 3 .$$

Note that an edge of a simple graph is uniquely determined by its two endpoints: indeed, it is the set consisting of these two endpoints. Multigraphs are similar to simple graphs, except that this is no longer true: their edges are not uniquely determined by their endpoints any more, but rather have "their own identities". Here is how multigraphs are defined:

Definition 3.5. A *multigraph* is a triple (W, E, ψ), where W and E are finite sets and where ψ is a map from E to $\mathcal{P}_2(W)$. The elements of W are called the *vertices* of this multigraph; the elements of E are called its *edges*. If e is an edge of a multigraph (W, E, ψ), then the two elements of the set $\psi(e) \in \mathcal{P}_2(W)$ are called the *endpoints* of this edge e. If (W, E, ψ) is a multigraph, and if $w \in W$ and $e \in E$, then we say that the edge e *contains* the vertex w if and only if $w \in \psi(e)$ (that is, if and only if w is an endpoint of e).

Example 3.6. Let $\alpha, \beta, \gamma, \delta$ be any four distinct objects (it doesn't matter which objects we take; for example, $10, 11, 12, 13$ do the job). Let W be the set $\{1, 2, 3\}$, and let E be the set $\{\alpha, \beta, \gamma, \delta\}$. Let $\psi : E \to \mathcal{P}_2(W)$ be the map given by

$$\begin{aligned} \psi(\alpha) &= \{1, 3\}, & \psi(\beta) &= \{2, 3\}, \\ \psi(\gamma) &= \{1, 3\}, & \psi(\delta) &= \{2, 1\}. \end{aligned}$$

Then, the triple (W, E, ψ) is a multigraph. Its vertices are $1, 2, 3$; its edges are $\alpha, \beta, \gamma, \delta$. The edge α has endpoints 1 and 3; so does the edge γ.

A multigraph (W, E, ψ) is visually represented in the same way as a simple graph (W, E), with one difference: An edge $e \in E$ is now drawn as a curve from the point labelled by its one endpoint to the point labelled by its other endpoint, and we furthermore label this curve with an "e".

Example 3.7. The multigraph (W, E, ψ) from Example 3.6 can be represented as follows:
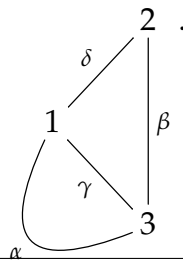
Let us summarize the difference between a simple graph and a multigraph: An edge of a simple graph (W, E) is merely a set consisting of its two endpoints, whereas an edge of a multigraph (W, E, ψ) can be an arbitrary object (so it "has its own identity") whose endpoints are assigned to it by the map ψ. Thus, we can regard multigraphs as a refined version of simple graphs. Every simple graph gives rise to a multigraph as follows:

Definition 3.8. Let (W, E) be a simple graph. Let $\iota : E \to \mathcal{P}_2(W)$ be the inclusion map (i.e., the map that sends each $e \in E$ to e itself); this is well-defined because E is a subset of $\mathcal{P}_2(W)$ (since (W, E) is a simple graph). Then, (W, E, ι) is a multigraph. This multigraph (W, E, ι) is called the *multigraph induced by* (W, E); we will often just identify it with the simple graph (W, E) (so that each simple graph becomes a multigraph in this way).

Example 3.9. The simple graph
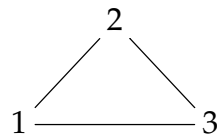
becomes identified with the multigraph
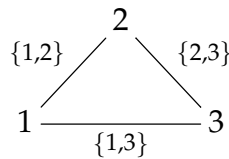
in this way.

Both simple graphs and multigraphs are subsumed under the concept of a *graph*, or, more precisely, *undirected graph*.

3.2. Bipartite matching and Hall's marriage theorem

We now define bipartite graphs.

Definition 3.10. A *bipartite graph* means a triple $(G; X, Y)$ (the semicolon means the same thing as a comma), where $G = (W, E, \psi)$ is a multigraph, and where X and Y are two subsets of W with the following properties:

- We have $X \cap Y = \varnothing$ and $X \cup Y = W$.

- Each edge of G contains exactly one vertex in X and exactly one vertex in Y.

For example, if G is the following simple graph:

$$
\begin{array}{ccc}
1 \!-\!\!-\! 2 & 5 \\
| \quad | & | \\
4 \!-\!\!-\! 3 & 6
\end{array}
\tag{25}
$$

(regarded as a multigraph)[15], then $(G; \{1,3,5\}, \{2,4,6\})$ is a bipartite graph, and $(G; \{2,4,6\}, \{1,3,5\})$ is another bipartite graph, and $(G; \{1,3,6\}, \{2,4,5\})$ is yet another bipartite graph, but $(G; \{1,2,3\}, \{4,5,6\})$ is not a bipartite graph (because the edge $\{1,2\}$ of G contains two vertices in $\{1,2,3\}$, rather than one in $\{1,2,3\}$ and one in $\{4,5,6\}$).

We often draw bipartite graphs in a rather special way. Namely, in order to draw a bipartite graph $(G; X, Y)$, we draw the graph G, but making sure that all vertices are aligned in two columns, where the left column contains all the vertices in X and the right column contains all the vertices in Y. For example, if G is the graph shown in (25), then the bipartite graph $(G; \{1,3,5\}, \{2,4,6\})$ is drawn as

$$
\begin{array}{cc}
1 \!-\!\!-\! 2 & , \\
\times \\
3 \!-\!\!-\! 4 \\
\\
5 \!-\!\!-\! 6
\end{array}
$$
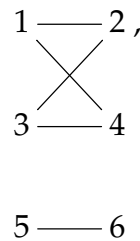
whereas the bipartite graph $(G; \{2,4,6\}, \{1,3,5\})$ is drawn as

$$
\begin{array}{cc}
2 \!-\!\!-\! 1 & . \\
\times \\
4 \!-\!\!-\! 3 \\
\\
6 \!-\!\!-\! 5
\end{array}
$$
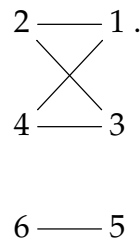
Note that the graph G will be a simple graph in all our examples, but it can be an arbitrary multigraph in general.

Definition 3.11. Let $G = (W, E, \psi)$ be a multigraph. A *matching* in G means a subset M of E such that no two distinct edges in M have an endpoint in common.

For example, the set $\{\{1,2\}, \{5,6\}\}$ is a matching in the graph G shown in (25); so is the set $\{\{1,2\}, \{3,4\}, \{5,6\}\}$ (but not the set $\{\{1,2\}, \{2,3\}, \{5,6\}\}$, because

---

[15]Thus, formally speaking, G is the simple graph whose vertices are $1,2,3,4,5,6$, and whose edges are $\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{5,6\}$.

its two edges $\{1, 2\}$ and $\{2, 3\}$ have the endpoint 2 in common). Also, the empty set $\varnothing$ is a matching in any graph.

In graph theory, we are often interested in matchings that contain as many edges as possible. There are, of course, simple bounds on how many edges a matching can contain: For example, a matching in a multigraph $G = (W, E, \psi)$ can never have more than $|W| / 2$ edges (since each edge "uses up" two vertices). Also, if $(G; X, Y)$ is a bipartite graph, then a matching in G can never have more than $|X|$ edges (since each edge "uses up" a vertex in X). How can we find a maximum-sized matching in G ? This is known as the *bipartite matching problem* (when $(G; X, Y)$ is a bipartite graph). It turns out that the Ford-Fulkerson algorithm (developed in the above proof of Lemma 2.21) gives a way to solve this problem in polynomial time.

Let us introduce a couple more concepts:

> **Definition 3.12.** Let M be a matching in a multigraph $G = (W, E, \psi)$.
>
> **(a)** A vertex v of G is said to be *matched in M* if there exists an edge $e \in M$ such that v is an endpoint of e. In this case, this edge is unique (since M is a matching), and the other endpoint of this edge (i.e., the endpoint distinct from v) is called the *M-partner* of v.
>
> **(b)** Let S be a subset of W. The matching M is said to be *S-complete* if each vertex $v \in S$ is matched in M.

For example, if G is the graph shown in (25), then the matching $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ is $\{1, 3, 6\}$-complete (since all three vertices $1, 3, 6$ are matched in it[16]), but the matching $\{\{1, 2\}, \{5, 6\}\}$ is not (since the vertex 3 is not matched in it).

> **Definition 3.13.** Let $G = (W, E, \psi)$ be a multigraph.
>
> **(a)** If v is a vertex of G, then a *neighbor* of v means any vertex w of G such that $\{v, w\} \in \psi(E)$. (Note that the condition $\{v, w\} \in \psi(E)$ simply says that there exists an edge of G whose endpoints are v and w.)
>
> **(b)** Let U be a subset of W. Then, $N(U)$ shall denote the subset
>
> $$\{v \in W \mid v \text{ has a neighbor in } U\}$$
>
> of W.

For example, if G is the graph shown in (25), then the neighbors of the vertex 1 are 2 and 4, and we have $N(\{1, 2\}) = \{1, 2, 3, 4\}$ and $N(\{1, 3\}) = \{2, 4\}$ and $N(\{2, 5\}) = \{1, 3, 6\}$ and $N(\varnothing) = \varnothing$.

The following is almost trivial:

> **Proposition 3.14.** Let $(G; X, Y)$ be a bipartite graph. Let U be a subset of X. Then, $N(U) \subseteq Y$.

---

[16]Their $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$-partners are $2, 4, 5$, respectively.

Proof of Proposition 3.14. Write the multigraph G in the form $G = (W, E, \psi)$. Recall that $(G; X, Y)$ is a bipartite graph; hence, $X \cap Y = \varnothing$ and $X \cup Y = W$. Thus, $Y = W \setminus X$ and $X = W \setminus Y$.

Now, let $p \in N(U)$ be arbitrary. Thus, $p \in N(U) = \{v \in W \mid v \text{ has a neighbor in } U\}$ (by the definition of $N(U)$). In other words, p is an element of W that has a neighbor in U.

The vertex p has a neighbor in U. Fix such a neighbor, and denote it by q. Thus, $q \in U \subseteq X = W \setminus Y$, so that $q \notin Y$. We know that q is a neighbor of p; in other words, $\{p, q\} \in \psi(E)$. In other words, $\{p, q\} = \psi(e)$ for some $e \in E$. Consider this e. Thus, e is an edge of G (since $e \in E$), and has endpoints p and q (since $\{p, q\} = \psi(e)$).

But $(G; X, Y)$ is a bipartite graph. Hence, each edge of G contains exactly one vertex in X and exactly one vertex in Y. Thus, in particular, each edge of G contains a vertex in Y. Applying this to the edge e, we conclude that the edge e contains a vertex in Y. In other words, one of the two endpoints of e lies in Y. In other words, one of the two vertices p and q lies in Y (since the endpoints of e are p and q). This vertex cannot be q (since $q \notin Y$), and thus must be p. Hence, p lies in Y. In other words, $p \in Y$.

Now, forget that we fixed p. We have thus shown that $p \in Y$ for each $p \in N(U)$. In other words, $N(U) \subseteq Y$. This proves Proposition 3.14. $\qquad\square$

We shall now study matchings in bipartite graphs. The most important result about such matchings is the following fact, known as *Hall's marriage theorem*:

Theorem 3.15. Let $(G; X, Y)$ be a bipartite graph. Then, G has an X-complete matching if and only if each subset U of X satisfies $|N(U)| \geq |U|$.

Theorem 3.15 has applications throughout mathematics, and several equivalent versions; it also has fairly elementary (but tricky) proofs (see, e.g., [LeLeMe17, §12.5.2]). We shall derive it from Theorem 2.12.

In order to do so (and also, in order to reduce the bipartite matching problem to the Ford-Fulkerson algorithm), we need to construct a network from a given bipartite graph such that the integer flows in the network shall correspond to the matchings in the graph. Let us do this.

Convention 3.16. For the rest of Section 3.2, we fix a bipartite graph $(G; X, Y)$. Write the multigraph G in the form $G = (W, E, \psi)$.

We know that $(G; X, Y)$ is a bipartite graph. Thus, each edge of G contains exactly one vertex in X and exactly one vertex in Y. In other words, if e is an edge of G, then exactly one endpoint of e lies in X, and exactly one endpoint of e lies in Y.

Now, we shall construct a network N out of our bipartite graph $(G; X, Y)$. Before we give the rigorous definition, let us show it on an example:

Example 3.17. For this example, let $G = (W, E)$ be the simple graph with vertices $1, 2, 3, 4, 5, 6, 7$ and edges $\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 7\}, \{3, 5\}, \{3, 6\}$. Set $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6, 7\}$; then, $(G; X, Y)$ is a bipartite graph which can be drawn as follows:
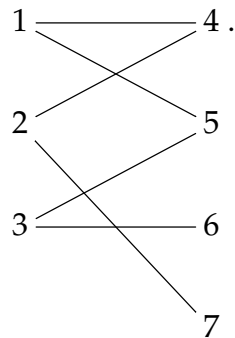
Regard G as a multigraph. Now, we shall transform this multigraph G into a multidigraph by replacing each edge e by an arc $\vec{e}$. The source of this arc $\vec{e}$ shall be the unique endpoint of e that lies in X; the target of this arc $\vec{e}$ shall be the unique endpoint of e that lies in Y. Thus, our multigraph G has been transformed into the following multidigraph:
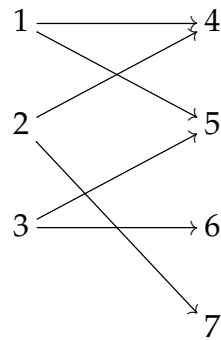
(where we are omitting the labels on the arcs). The arcs of this multidigraph will be called the *G-arcs* (to stress that they come directly from the edges of G, as opposed to the next arcs that we are going to add).

Next, we add a new vertex, which we call s and which we draw on the very left. This s will be the source of our network. For each $x \in X$, we add an arc (s, x) to our multidigraph; this arc shall have source s and target x. These new arcs (a total of $|X|$ arcs, one for each $x \in X$) will be called the *s-arcs*. Here is how our multidigraph now looks like:
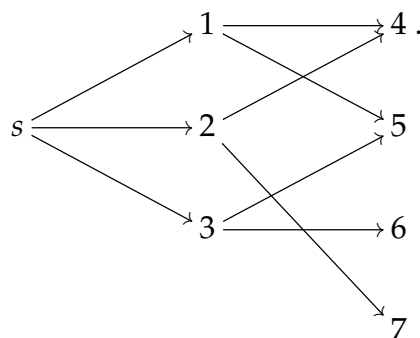
Next, we add a new vertex, which we call t and which we draw on the very right. This t will be the sink of our network. For each $y \in Y$, we add an arc (y, t) to our multidigraph; this arc shall have source y and target t. These new arcs (a total of $|Y|$ arcs, one for each $y \in Y$) will be called the *t-arcs*. Here is how our multidigraph now looks like:

.
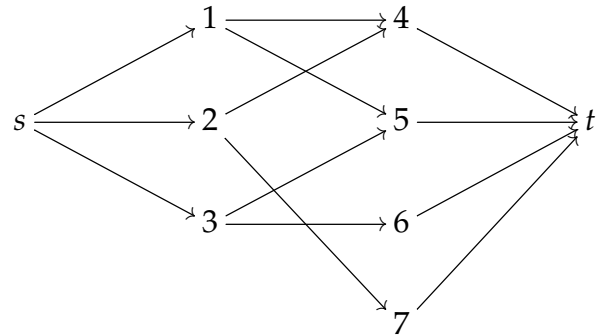
This 9-vertex multidigraph will be the underlying multidigraph of our network N. The source and the target of N shall be s and t, respectively. The capacity function c is determined by setting $c(a) = 1$ for each arc of the network.

We claim that the integer flows on the network N are in bijection with the matchings in G. Again, let us show how this bijection acts on an example. Consider the following integer flow on N:

$$\tag{26}$$
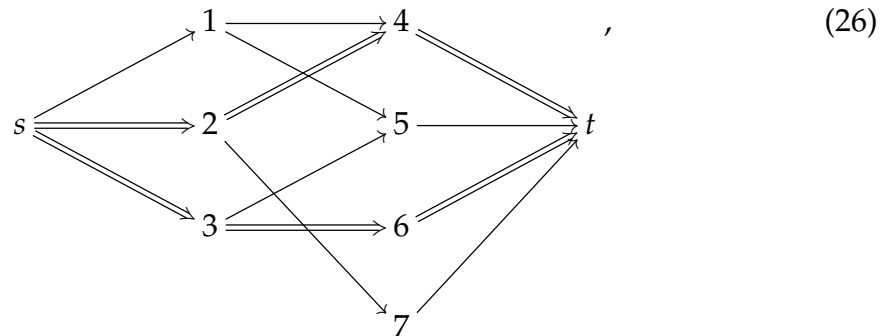
where a simple arrow ($\longrightarrow$) stands for an arc which the flow sends to 0, and where a double arrow ($\Longrightarrow$) stands for an arc which the flow sends to 1. (This is just an alternative way of drawing an integer flow when each arc has capacity 1.) Consider the arcs that are sent to 1 by this flow. Two of these arcs are G-arcs, namely $\overrightarrow{\{2, 4\}}$ and $\overrightarrow{\{3, 6\}}$. The corresponding edges $\{2, 4\}$ and $\{3, 6\}$ of G form a matching: the matching $\{\{2, 4\}, \{3, 6\}\}$. This matching has size 2, which is exactly the value of our integer flow.

Let us now formalize what we did in this example. First, here is the general definition of the network N:

Definition 3.18. (a) Pick two distinct new objects s and t. Let V be the finite set $W \cup \{s, t\}$.

The elements of V will be the vertices of our network, with s being the source and t being the sink.

Next, we shall introduce three sets A_s, A_t and A_G, whose elements will be the arcs of our network. (More precisely, the elements of A_G will be the G-arcs, the elements of A_s will be the s-arcs, and the elements of A_t be the t-arcs.)

(b) Define two finite sets A_s and A_t by

$$A_s = \{s\} \times X = \{(s, x) \mid x \in X\} \qquad \text{and}$$
$$A_t = Y \times \{t\} = \{(y, t) \mid y \in Y\}.$$

Note that A_s and A_t are disjoint (since each element of A_s is a pair whose first entry is s, whereas no element of A_t has this property).

(c) For each edge $e \in E$, we let $\overrightarrow{e}$ be the triple (x, y, e), where x is the unique endpoint of e that lies in X, and where y is the unique endpoint of e that lies in Y. (Here, we are using the fact that exactly one endpoint of e lies in X, and exactly one endpoint of e lies in Y.) Note that the triples $\overrightarrow{e}$ for $e \in E$ are pairwise distinct (i.e., if two edges e and f satisfy $e \neq f$, then $\overrightarrow{e} \neq \overrightarrow{f}$), because the third entry of the triple $\overrightarrow{e}$ is always the original edge $e \in E$.

(d) Define a finite set A_G by

$$A_G = \{\overrightarrow{e} \mid e \in E\}.$$

Thus, the set A_G is in bijection with E (since the triples $\overrightarrow{e}$ for $e \in E$ are pairwise distinct), and is disjoint from both A_s and A_t (because the elements of A_G are triples, while the elements of A_s and of A_t are pairs).

(e) Let A be the union $A_G \cup A_s \cup A_t$; this is a finite set. Define a map $\phi : A \to V \times V$ by the following three equalities

$$\begin{aligned}
\phi((x, y, e)) &= (x, y) &\quad \text{for each } (x, y, e) \in A_G; \\
\phi((s, x)) &= (s, x) &\quad \text{for each } (s, x) \in A_s; \\
\phi((y, t)) &= (y, t) &\quad \text{for each } (y, t) \in A_t.
\end{aligned}$$

Thus, (V, A, ϕ) is a multidigraph. The elements of A_G shall be called the G-arcs; the elements of A_s shall be called the s-arcs; the elements of A_t shall be called the t-arcs. Thus, the arcs of the multidigraph (V, A, ϕ) are the G-arcs, the s-arcs and the t-arcs.

(f) Define the function $c : A \to \mathbb{Q}_+$ by ($c(a) = 1$ for each $a \in A$). Thus, c is a constant function.

(g) Let N be the network consisting of the multidigraph (V, A, ϕ), the source s, the sink t and the capacity function c.

As we have seen, the sets A_G, A_s and A_t are mutually disjoint; i.e., there is

no overlap between the *G*-arcs, the *s*-arcs and the *t*-arcs. All *s*-arcs have source *s*, whereas none of the other types of arcs do. Likewise, all *t*-arcs have target *t*, whereas none of the other types of arcs do. The *G*-arcs have their sources lie in *X* and their targets lie in *Y*.

We notice that each *s*-arc is literally the pair of its source and its target, as in a simple digraph. The same is true for the *t*-arcs. However, the *G*-arcs are not pairs; thus, (V, A, ϕ) is not a simple digraph.

Our specific choice of capacity function *c* ensures that integer flows on *N* have a very simple form: If *f* is an integer flow on *N*, and if $a \in A$ is any arc, then $f(a)$ is either 0 or 1. (Indeed, $f(a)$ must be an integer since *f* is an integer flow; but the capacity constraints enforce $0 \leq f(a) \leq 1$, so that this integer $f(a)$ must be either 0 or 1.) Thus, an integer flow *f* on *N* is uniquely determined by knowing which of the arcs $a \in A$ it sends to 1 (because then, it has to send all the other arcs to 0).

Another consequence of our definition of capacity function *c* is the following:

Lemma 3.19. Let *P* and *Q* be two subsets of *V*. Then, $c(P, Q) = |[P, Q]|$.

Proof of Lemma 3.19. The definition of $c(P, Q)$ yields

$$c(P, Q) = \sum_{a \in [P,Q]} \underbrace{c(a)}_{\substack{=1 \\ \text{(by the definition of } c)}} = \sum_{a \in [P,Q]} 1 = |[P, Q]| \cdot 1 = |[P, Q]|.$$

This proves Lemma 3.19. $\qquad\qquad\square$

Now, we want to formulate the bijection between integer flows on *N* and matchings in *G*. First, we need a simple property of integer flows on our specific network *N*: [17]

Proposition 3.20. Let *f* be any integer flow on *N*. Let *M* be the subset $\{e \in E \mid f(\overrightarrow{e}) = 1\}$ of *E*.
  (a) We have $f((s, x)) = \sum_{\substack{e \in E; \\ x \in e}} f(\overrightarrow{e})$ for each $x \in X$.
  (b) We have $f((y, t)) = \sum_{\substack{e \in E; \\ y \in e}} f(\overrightarrow{e})$ for each $y \in Y$.
  (c) We have $M = \{e \mid (x, y, e) \in A_G; \ f((x, y, e)) = 1\}$.
  (d) The set $M = \{e \in E \mid f(\overrightarrow{e}) = 1\}$ is a matching in *G*.
  (e) We have $|M| = |f|$.
  (f) For any vertex $x \in X$, we have $[x \text{ is matched in } M] = f((s, x))$.
  (g) For any vertex $y \in Y$, we have $[y \text{ is matched in } M] = f((y, t))$.

For example, if *f* is the flow shown in (26), then the set *M* in Proposition 3.20 is the matching $\{\{2, 4\}, \{3, 6\}\}$ in *G*.

---

[17]We are using the *Iverson bracket notation*: If $\mathcal{A}$ is any logical statement, then $[\mathcal{A}]$ denotes the integer
$$\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}.$$

Proof of Proposition 3.20 (sketched). We shall only prove the parts of Proposition 3.20 used in the below proof of Theorem 3.15 (namely, parts **(a)**, **(b)**, **(d)** and **(e)**); the rest is left to the reader.

We first make the following observations:

> *Observation 1:* Let $x \in X$ and $e \in E$. Then, the arc $\overrightarrow{e}$ has source x if and only if $x \in e$.

[*Proof of Observation 1:* Recall that $(G; X, Y)$ is a bipartite graph. Thus, exactly one endpoint of e lies in X, and exactly one endpoint of e lies in Y. Let u be the unique endpoint of e that lies in X, and let v be the unique endpoint of e that lies in Y. Then, $\overrightarrow{e} = (u, v, e)$ (by the definition of $\overrightarrow{e}$). Hence, $\phi\left(\overrightarrow{e}\right) = \phi((u, v, e)) = (u, v)$ (by the definition of ϕ). In other words, the source of the arc $\overrightarrow{e}$ is u, and the target of the arc $\overrightarrow{e}$ is v.

We now shall prove the $\Longrightarrow$ and $\Longleftarrow$ directions of Observation 1 separately:

$\Longrightarrow$: Assume that the arc $\overrightarrow{e}$ has source x. We must prove that $x \in e$.

The arc $\overrightarrow{e}$ has source x. In other words, the source of the arc $\overrightarrow{e}$ is x. In other words, $u = x$ (since the source of the arc $\overrightarrow{e}$ is u). But u is an endpoint of e; thus, $u \in e$. Hence, $x = u \in e$. This proves the $\Longrightarrow$ direction of Observation 1.

$\Longleftarrow$: Assume that $x \in e$. We must show that the arc $\overrightarrow{e}$ has source x.

Recall that u is the unique endpoint of e that lies in X. The vertex x is an endpoint of e (since $x \in e$) and lies in X (since $x \in X$). Thus, the unique endpoint of e that lies in X must be x. In other words, u must be x (since u is the unique endpoint of e that lies in X). Thus, $u = x$. Now, the source of the arc $\overrightarrow{e}$ is $u = x$. In other words, the arc $\overrightarrow{e}$ has source x. Thus, the $\Longleftarrow$ direction of Observation 1 is proven.]

> *Observation 2:* Let $y \in Y$ and $e \in E$. Then, the arc $\overrightarrow{e}$ has target y if and only if $y \in e$.

[*Proof of Observation 2:* Analogous to Observation 1.]

> *Observation 3:* The arcs $\overrightarrow{e}$ for $e \in E$ are pairwise distinct.

[*Proof of Observation 3:* We have already shown this in Definition 3.18 **(c)**.]

Now, we observe that $s \notin W$, so that $s \notin X$ and $s \notin Y$. Also, $X \cap Y = \varnothing$ (since $(G; X, Y)$ is a bipartite graph), so that $X \subseteq V \setminus Y$.

> *Observation 4:* Let $x \in X$. Then, the arcs $a \in A$ having source x are exactly the arcs $\overrightarrow{e}$ for $e \in E$ satisfying $x \in e$.

[*Proof of Observation 4:* We have $x \in X$ but $s \notin X$ (since $s \notin W$). Thus, $x \neq s$. Hence, there are no s-arcs having source x (since all s-arcs have source $s \neq x$). Also, $x \notin Y$ (since $x \in X \subseteq V \setminus Y$). Thus, there are no t-arcs having source x (since all t-arcs have source lying in Y).

Recall that there are three types of arcs $a \in A$: the G-arcs, the s-arcs and the t-arcs. Among these three types, only G-arcs can have source x (because we have

seen that there are no *s*-arcs having source x, and that there are no *t*-arcs having source x). Hence,

$$\{\text{the arcs } a \in A \text{ having source } x\}$$
$$= \{\text{the } G\text{-arcs having source } x\}$$
$$= \{\text{the arcs } \overrightarrow{e} \text{ (with } e \in E) \text{ having source } x\}$$
$$\qquad (\text{since the } G\text{-arcs are precisely the arcs } \overrightarrow{e} \text{ (with } e \in E))$$
$$= \left\{ \overrightarrow{e} \;\middle|\; e \in E; \underbrace{\text{the arc } \overrightarrow{e} \text{ has source } x}_{\substack{\Longleftrightarrow \; (x \in e) \\ (\text{by Observation 1})}} \right\}$$
$$= \{ \overrightarrow{e} \mid e \in E; \; x \in e \} = \{\text{the arcs } \overrightarrow{e} \text{ for } e \in E \text{ satisfying } x \in e\}.$$

In other words, the arcs $a \in A$ having source x are exactly the arcs $\overrightarrow{e}$ for $e \in E$ satisfying $x \in e$. This proves Observation 4.]

Observation 5: Let $y \in Y$. Then, the arcs $a \in A$ having target y are exactly the arcs $\overrightarrow{e}$ for $e \in E$ satisfying $y \in e$.

[*Proof of Observation 5:* Analogous to Observation 4.]

Observation 6: Let $x \in X$. Then, there is a unique arc $a \in A$ having target x, namely the *s*-arc (s, x).

[*Proof of Observation 6:* We have $x \in X \subseteq W$ but $t \notin W$. Thus, $x \neq t$. Hence, there are no *t*-arcs having target x (since all *t*-arcs have target $t \neq x$). Also, $x \notin Y$ (since $x \in X \subseteq V \setminus Y$). Thus, there are no *G*-arcs having target x (since all *G*-arcs have target lying in Y).

Recall that there are three types of arcs $a \in A$: the *G*-arcs, the *s*-arcs and the *t*-arcs. Among these three types, only *s*-arcs can have target x (because we have seen that there are no *t*-arcs having target x, and that there are no *G*-arcs having target x). Hence,

$$\{\text{the arcs } a \in A \text{ having target } x\} = \{\text{the } s\text{-arcs having target } x\}.$$

But the *s*-arcs are simply the pairs of the form (s, x') for all $x' \in X$ (by the definition of the *s*-arcs), and their respective targets are x'. Thus, there is a unique *s*-arc having target x, namely (s, x) (since $x \in X$). In other words, $\{\text{the } s\text{-arcs having target } x\} = \{(s, x)\}$. Hence,

$$\{\text{the arcs } a \in A \text{ having target } x\} = \{\text{the } s\text{-arcs having target } x\} = \{(s, x)\}.$$

In other words, there is a unique arc $a \in A$ having target x, namely the *s*-arc (s, x). This proves Observation 6.]

Observation 7: Let $y \in Y$. Then, there is a unique arc $a \in A$ having source y, namely the t-arc (y, t).

[*Proof of Observation 7:* Analogous to Observation 6.]

Observation 8: The arcs $a \in A$ having source s are exactly the arcs (s, x) for $x \in X$, and these arcs are pairwise distinct.

[*Proof of Observation 8:* We have $s \notin W$ and thus $s \notin X$ (since $X \subseteq W$). Hence, there are no G-arcs having source s (since all G-arcs have source lying in X). Also, $s \notin W$ and thus $s \notin Y$ (since $Y \subseteq W$). Thus, there are no t-arcs having source s (since all t-arcs have source lying in Y).

Recall that there are three types of arcs $a \in A$: the G-arcs, the s-arcs and the t-arcs. Among these three types, only s-arcs can have source s (because we have seen that there are no G-arcs having source s, and that there are no t-arcs having source s). Hence,

$$\{\text{the arcs } a \in A \text{ having source } s\}$$
$$= \{\text{the } s\text{-arcs having source } s\}$$
$$= \{\text{the } s\text{-arcs}\} \qquad (\text{since \textbf{all} } s\text{-arcs have source } s)$$
$$= A_s = \{(s, x) \mid x \in X\}.$$

In other words, the arcs $a \in A$ having source s are exactly the arcs (s, x) for $x \in X$. Furthermore, these arcs are pairwise distinct (since the targets x of these arcs are pairwise distinct). This proves Observation 8.]

Observation 9: There are no arcs $a \in A$ having target s.

[*Proof of Observation 9:* We have $s \notin W$ and thus $s \notin X$ (since $X \subseteq W$). Hence, there are no s-arcs having target s (since all s-arcs have target lying in X). Also, $s \notin W$ and thus $s \notin Y$ (since $Y \subseteq W$). Hence, there are no G-arcs having target s (since all G-arcs have target lying in Y). Finally, $s \neq t$. Thus, there are no t-arcs having target s (since all t-arcs have target $t \neq s$).

Recall that there are three types of arcs $a \in A$: the G-arcs, the s-arcs and the t-arcs. Among these three types, none can have target s (since we have seen that there are no G-arcs having target s, no s-arcs having target s, and no t-arcs having target s). Thus, there are no arcs $a \in A$ having target s. This proves Observation 9.]

Recall that f is a flow; thus, f satisfies the capacity constraints and the conservation constraints.

(a) Let $x \in X$. Then, $x \in X \subseteq W = V \setminus \{s, t\}$. Hence, $f^-(x) = f^+(x)$ (since f satisfies the conservation constraints).

Observation 6 shows that there is a unique arc $a \in A$ having target x, namely the s-arc (s, x). Thus, $\displaystyle\sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } x}} f(a) = f((s, x))$.

The definition of $f^-(x)$ yields

$$f^-(x) = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } x}} f(a) = f((s,x)). \tag{27}$$

Observation 4 shows that the arcs $a \in A$ having source x are exactly the arcs $\overrightarrow{e}$ for $e \in E$ satisfying $x \in e$. Since these arcs $\overrightarrow{e}$ are pairwise distinct (by Observation 3), we thus conclude that $\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } x}} f(a) = \sum_{\substack{e \in E; \\ x \in e}} f(\overrightarrow{e})$.

The definition of $f^+(x)$ yields

$$f^+(x) = \sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } x}} f(a) = \sum_{\substack{e \in E; \\ x \in e}} f(\overrightarrow{e}). \tag{28}$$

Now, recall that $f^-(x) = f^+(x)$. In view of (27) and (28), this rewrites as $f((s,x)) = \sum_{\substack{e \in E; \\ x \in e}} f(\overrightarrow{e})$. This proves Proposition 3.20 **(a)**.

(b) The proof of Proposition 3.20 **(b)** is analogous to that of Proposition 3.20 **(a)**.

(d) Clearly, $M = \left\{ e \in E \mid f(\overrightarrow{e}) = 1 \right\}$ is a subset of E. We must prove that this subset M is a matching in G. Since we already know that M is a subset of E, we only need to verify that no two distinct edges in M have an endpoint in common (by the definition of a matching).

Let us assume the contrary. Thus, there exist two distinct edges g and h in M that have an endpoint in common. Consider such g and h.

The edges g and h have an endpoint in common. In other words, there exists a $w \in W$ such that $w \in g$ and $w \in h$. Consider this w.

We have $g \in M \subseteq E$ and $w \in g$. Thus, g is an edge $e \in E$ satisfying $w \in e$. Similarly, h is an edge $e \in E$ satisfying $w \in e$.

From $g \in M = \left\{ e \in E \mid f(\overrightarrow{e}) = 1 \right\}$, we conclude that $f(\overrightarrow{g}) = 1$. Similarly, $f(\overrightarrow{h}) = 1$.

We have $w \in W = X \cup Y$ (since $(G; X, Y)$ is a bipartite graph); thus, we have either $w \in X$ or $w \in Y$. So we are in one of the following two cases:

Case 1: We have $w \in X$.

Case 2: We have $w \in Y$.

Let us consider Case 1. In this case, we have $w \in X$. Proposition 3.20 **(a)** (applied

to $x = w$) thus yields

$$f\left((s,w)\right) = \sum_{\substack{e\in E;\\ w\in e}} f\left(\overrightarrow{e}\right) = \underbrace{f\left(\overrightarrow{g}\right)}_{=1} + \underbrace{f\left(\overrightarrow{h}\right)}_{=1} + \sum_{\substack{e\in E;\\ w\in e;\\ e\notin\{g,h\}}} \underbrace{f\left(\overrightarrow{e}\right)}_{\geq 0 \text{ (since } f \text{ is a map } A\to\mathbb{Q}_+)}$$

$$\left(\begin{array}{c}\text{here, we have split off the addends for } e = g \text{ and } e = h,\\ \text{since } g \text{ and } h \text{ are two distinct edges } e \in E \text{ satisfying } w \in e\end{array}\right)$$

$$\geq 1 + 1 + \sum_{\substack{e\in E;\\ w\in e;\\ e\notin\{g,h\}}} 0 = 2.$$

But recall that f satisfies the capacity constraints. Thus, $0 \leq f(a) \leq c(a)$ for each arc $a \in A$. Applying this to $a = (s, w)$, we conclude that $0 \leq f((s, w)) \leq c((s, w))$. Hence, $f((s, w)) \leq c((s, w)) = 1$ (by the definition of c). This contradicts $f((s, w)) \geq 2 > 1$. Thus, we have found a contradiction in Case 1.

Similarly, we obtain a contradiction in Case 2 (using Proposition 3.20 **(b)** instead of Proposition 3.20 **(a)**).

Hence, we have found a contradiction in both Cases 1 and 2. Thus, we always get a contradiction. This completes our proof that M is a matching in G. Thus, Proposition 3.20 **(d)** is established.

(e) For any edge $e \in E$, we have either $f\left(\overrightarrow{e}\right) = 0$ or $f\left(\overrightarrow{e}\right) = 1$ [18].

Observation 8 shows that the arcs $a \in A$ having source s are exactly the arcs (s, x) for $x \in X$, and these arcs are pairwise distinct. Hence,

$$\sum_{\substack{a\in A \text{ is an arc}\\ \text{with source } s}} f(a) = \sum_{x\in X} \underbrace{f((s,x))}_{\substack{=\sum_{\substack{e\in E;\\ x\in e}} f(\overrightarrow{e})\\ \text{(by Proposition 3.20 (a))}}} = \underbrace{\sum_{x\in X}\sum_{\substack{e\in E;\\ x\in e}} f\left(\overrightarrow{e}\right)}_{=\sum_{e\in E}\sum_{\substack{x\in X;\\ x\in e}}}$$

$$= \sum_{e\in E}\sum_{\substack{x\in X;\\ x\in e}} f\left(\overrightarrow{e}\right). \tag{29}$$

But each $e \in E$ satisfies

$$\sum_{\substack{x\in X;\\ x\in e}} f\left(\overrightarrow{e}\right) = f\left(\overrightarrow{e}\right) \tag{30}$$

---

[18] *Proof.* Let $e \in E$. Then, $\overrightarrow{e}$ is an arc in A (namely, a G-arc). But recall that f satisfies the capacity constraints. Thus, $0 \leq f(a) \leq c(a)$ for each arc $a \in A$. Applying this to $a = \overrightarrow{e}$, we conclude that $0 \leq f\left(\overrightarrow{e}\right) \leq c\left(\overrightarrow{e}\right)$. Hence, $f\left(\overrightarrow{e}\right) \leq c\left(\overrightarrow{e}\right) = 1$ (by the definition of c). Thus, $0 \leq f\left(\overrightarrow{e}\right) \leq 1$.

But f is an integer flow; thus, $f\left(\overrightarrow{e}\right)$ is an integer. Combining this with $0 \leq f\left(\overrightarrow{e}\right) \leq 1$, we conclude that $f\left(\overrightarrow{e}\right) \in \{0,1\}$. In other words, either $f\left(\overrightarrow{e}\right) = 0$ or $f\left(\overrightarrow{e}\right) = 1$. Qed.

[19]. Hence, (29) becomes

$$
\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } s}} f(a) = \underbrace{\sum_{e \in E} \sum_{\substack{x \in X; \\ x \in e}} f(\overrightarrow{e})}_{\substack{= f(\overrightarrow{e}) \\ \text{(by (30))}}} = \sum_{e \in E} f(\overrightarrow{e})
$$

$$
= \sum_{\substack{e \in E; \\ f(\overrightarrow{e})=0}} \underbrace{f(\overrightarrow{e})}_{=0} + \sum_{\substack{e \in E; \\ f(\overrightarrow{e})=1}} \underbrace{f(\overrightarrow{e})}_{=1}
$$

$$
\left(\begin{array}{c} \text{because for any edge } e \in E, \text{ we have} \\ \text{either } f(\overrightarrow{e}) = 0 \text{ or } f(\overrightarrow{e}) = 1 \text{ (but not both)} \end{array} \right)
$$

$$
= \underbrace{\sum_{\substack{e \in E; \\ f(\overrightarrow{e})=0}} 0}_{=0} + \sum_{\substack{e \in E; \\ f(\overrightarrow{e})=1}} 1 = \sum_{\substack{e \in E; \\ f(\overrightarrow{e})=1}} 1
$$

$$
= \left| \underbrace{\left\{ e \in E \ | \ f(\overrightarrow{e}) = 1 \right\}}_{=M} \right| \cdot 1 = |M| \cdot 1 = |M|. \tag{31}
$$

On the other hand, Observation 9 shows that there are no arcs $a \in A$ having target s. Hence,

$$
\sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } s}} f(a) = (\text{empty sum}) = 0. \tag{32}
$$

But the definition of $|f|$ yields

$$
|f| = \underbrace{f^+(s)}_{\substack{= \sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with source } s}} f(a) \\ \text{(by the definition of } f^+(s))}} - \underbrace{f^-(s)}_{\substack{= \sum\limits_{\substack{a \in A \text{ is an arc} \\ \text{with target } s}} f(a) \\ \text{(by the definition of } f^-(s))}}
$$

$$
= \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with source } s}} f(a)}_{\substack{=|M| \\ \text{(by (31))}}} - \underbrace{\sum_{\substack{a \in A \text{ is an arc} \\ \text{with target } s}} f(a)}_{\substack{=0 \\ \text{(by (32))}}}
$$

$$
= |M| - 0 = |M|.
$$

---

[19] *Proof of (30):* Let $e \in E$. Each edge of G contains exactly one vertex in X and exactly one vertex in Y (since $(G; X, Y)$ is a bipartite graph). Thus, in particular, each edge of G contains exactly one vertex in X. Applying this to the edge e, we conclude that e contains exactly one vertex in X. In other words, there is exactly one $x \in X$ satisfying $x \in e$. Thus, the sum $\sum\limits_{\substack{x \in X; \\ x \in e}} f(\overrightarrow{e})$ has exactly one addend. Hence, this sum simplifies as follows: $\sum\limits_{\substack{x \in X; \\ x \in e}} f(\overrightarrow{e}) = f(\overrightarrow{e})$. This proves (30).

In other words, $|M| = |f|$. This proves Proposition 3.20 **(e)**. $\qquad\square$

As already mentioned, we omit the proof of the rest of Proposition 3.20; it is an easy exercise on bookkeeping and understanding the definitions of flows and matchings.

Proposition 3.20 allows us to make the following definition:

Definition 3.21. We define a map

$$\Phi : \{\text{integer flows on } N\} \to \{\text{matchings in } G\},$$
$$f \mapsto \left\{ e \in E \mid f\left(\overrightarrow{e}\right) = 1 \right\}.$$

This map is well-defined, because Proposition 3.20 **(d)** shows that if f is an integer flow on N, then $\left\{ e \in E \mid f\left(\overrightarrow{e}\right) = 1 \right\}$ is a matching in G.

We aim to show that this map Φ is a bijection. In order to do so, we will construct its inverse, which of course will be a map transforming each matching in G into an integer flow on N. This requires the following lemma:[20]

Lemma 3.22. Let M be any matching in G. Define a map $f : A \to \mathbb{Q}_+$ by setting

$$f(a) = \begin{cases} \left[\text{there is an } e \in M \text{ such that } a = \overrightarrow{e}\right], & \text{if } a \in A_G; \\ \left[x \text{ is matched in } M\right], & \text{if } a = (s, x) \text{ for some } x \in X; \\ \left[y \text{ is matched in } M\right], & \text{if } a = (y, t) \text{ for some } y \in Y \end{cases}$$

for each $a \in A$.

Then, f is an integer flow on N.

Again, the proof of Lemma 3.22 is easy (just check that the capacity and conservation constraints are satisfied).

Lemma 3.22 allows us to define the following:

Definition 3.23. We define a map

$$\Psi : \{\text{matchings in } G\} \to \{\text{integer flows on } N\}$$

by the requirement that Ψ map any matching M in G to the integer flow f constructed in Lemma 3.22.

Proposition 3.24. (a) The maps Φ and Ψ are mutually inverse bijections between $\{\text{integer flows on } N\}$ and $\{\text{matchings in } G\}$.
  (b) For any integer flow f on N, we have $|\Phi(f)| = |f|$.

---

[20]We are again using the Iverson bracket notation.

We leave the proof of Proposition 3.24 to the reader again. (It is fairly simple: Part **(b)** follows from Proposition 3.20 **(e)**. Part **(a)** requires proving that $\Phi \circ \Psi = \text{id}$ and $\Psi \circ \Phi = \text{id}$. The proof of $\Phi \circ \Psi = \text{id}$ is obvious; the proof of $\Psi \circ \Phi = \text{id}$ relies on the conservation constraints.)

We now see how to find a matching in G of maximum size. Indeed, we can find an integer flow on N of maximum value (see Remark 2.22 for how this is done). Then, the bijection Φ transforms this integer flow into a matching in G of maximum size (because Proposition 3.24 **(b)** shows that the value of an integer flow equals the size of the matching corresponding to this flow under the bijection Φ). Hence, we obtain a matching in G of maximum size.

We are also close to proving Theorem 3.15 now. Before we do this, let us prove a really simple lemma about matchings:

> **Lemma 3.25.** Let $(G; X, Y)$ be a bipartite graph. Let M be a matching in G.
> **(a)** We have $|M| \leq |X|$.
> **(b)** If $|M| \geq |X|$, then the matching M is X-complete.

Proof of Lemma 3.25 (sketched). Recall that $(G; X, Y)$ is a bipartite graph. Hence, each edge of G contains exactly one vertex in X and exactly one vertex in Y. Thus, we can define a map $\mathbf{x} : M \to X$ that sends each edge $e \in M$ to the unique vertex in X contained in e. Consider this map $\mathbf{x}$.

The map $\mathbf{x}$ is injective[21]. Hence, $|\mathbf{x}(M)| = |M|$. Thus, $|M| = |\mathbf{x}(M)| \leq |X|$ (since $\mathbf{x}(M) \subseteq X$). This proves Lemma 3.25 **(a)**.

(b) Assume that $|M| \geq |X|$. Combining this with $|M| \leq |X|$, we obtain $|M| = |X|$.

Now, $\mathbf{x}(M)$ is a subset of X that has the same size as X (because $|\mathbf{x}(M)| = |M| = |X|$). But the only such subset is X itself (since X is a finite set). Thus, $\mathbf{x}(M)$ must be X. In other words, the map $\mathbf{x}$ is surjective. This shows that the matching M is X-complete[22]. This proves Lemma 3.25 **(b)**. $\square$

---

[21]*Proof.* Let m and n be two distinct elements of M. We want to prove that $\mathbf{x}(m) \neq \mathbf{x}(n)$.

   Assume the contrary. Thus, $\mathbf{x}(m) = \mathbf{x}(n)$. Define an $x \in X$ by $x = \mathbf{x}(m) = \mathbf{x}(n)$. Now, $\mathbf{x}(m)$ is the unique vertex in X contained in m (by the definition of $\mathbf{x}$). In other words, x is the unique vertex in X contained in m (since $x = \mathbf{x}(m)$). Thus, x is contained in m. In other words, $x \in m$. Similarly, $x \in n$. The edges m and n are two distinct edges in M, and have the endpoint x in common (since $x \in m$ and $x \in n$).

   But M is a matching. Hence, no two distinct edges in M have an endpoint in common (by the definition of a matching). This contradicts that the edges m and n are two distinct edges in M that have an endpoint in common (namely, the vertex x). This contradiction shows that our assumption was wrong; hence, $\mathbf{x}(m) \neq \mathbf{x}(n)$ is proven.

   Now, forget that we fixed m and n. We thus have shown that if m and n are two distinct elements of M, then $\mathbf{x}(m) \neq \mathbf{x}(n)$. In other words, the map $\mathbf{x}$ is injective.

[22]*Proof.* Let $v \in X$ be any vertex. Then, there exists some $m \in M$ such that $v = \mathbf{x}(m)$ (since the map $\mathbf{x}$ is surjective). Consider this m. The vertex $\mathbf{x}(m)$ is the unique vertex in X contained in m (by the definition of $\mathbf{x}$). Hence, $\mathbf{x}(m)$ is contained in m. In other words, $\mathbf{x}(m) \in m$. Hence, $v = \mathbf{x}(m) \in m$. In other words, v is an endpoint of m. Hence, there exists an edge $e \in M$ such that v is an endpoint of e (namely, $e = m$). In other words, the vertex v is matched in M.

Now, we can prove the following crucial lemma:

> **Lemma 3.26.** Let $(G; X, Y)$ be a bipartite graph. Then, there exist a matching M in G and a subset U of X satisfying $|M| \geq |N(U)| + |X| - |U|$.

Proof of Lemma 3.26. Consider the network N. It clearly has the property that $c(a) \in \mathbb{N}$ for each $a \in A$ (since $c(a) = 1$ for each $a \in A$). Hence, Lemma 2.21 shows that there exist an integer flow $f : A \to \mathbb{Q}_+$ and a subset S of V satisfying $s \in S$ and $t \notin S$ and $c(S, \overline{S}) = |f|$. Consider these f and S.

Let U be the subset $X \cap S$ of X. Then, $N(U) \subseteq Y$ (by Proposition 3.14), so that $N(U) \cap S \subseteq Y \cap S$ and thus $|N(U) \cap S| \leq |Y \cap S|$. Also, $U = X \cap S \subseteq S$.

However, the set $N(U)$ is clearly the union of its two disjoint subsets $N(U) \cap S$ and $N(U) \setminus S$ (indeed, the former contains all the elements of $N(U)$ that belong to S, whereas the latter contains all those that don't). Thus,

$$|N(U)| = \underbrace{|N(U) \cap S|}_{\leq |Y \cap S|} + |N(U) \setminus S| \leq |Y \cap S| + |N(U) \setminus S|.$$

Hence,

$$|Y \cap S| + |N(U) \setminus S| \geq |N(U)|. \tag{33}$$

But the set X is the union of its two disjoint subsets $X \cap S$ and $X \setminus S$. Hence,

$$|X| = \Big| \underbrace{X \cap S}_{=U} \Big| + |X \setminus S| = |U| + |X \setminus S|.$$

Thus,

$$|X \setminus S| = |X| - |U|. \tag{34}$$

Lemma 3.19 (applied to $P = S$ and $Q = \overline{S}$) yields $c(S, \overline{S}) = |[S, \overline{S}]|$. Thus,

$$|[S, \overline{S}]| = c(S, \overline{S}) = |f|. \tag{35}$$

Let us now analyze the set $[S, \overline{S}]$. This set consists of all arcs $a \in A$ whose source lies in S and whose target lies in $\overline{S}$. Recall that the multidigraph (V, A, ϕ) has three kinds of arcs: the G-arcs, the s-arcs and the t-arcs. Some of these arcs belong to $[S, \overline{S}]$:

- For every vertex $x \in X \setminus S$, the s-arc (s, x) belongs to $[S, \overline{S}]$ [23]. Of course, these s-arcs (s, x) are distinct (since their targets x are distinct). Thus, we have found a total of $|X \setminus S|$ different s-arcs belonging to $[S, \overline{S}]$ (one for each $x \in X \setminus S$).

Now, forget that we fixed v. We thus have shown that each vertex $v \in X$ is matched in M. In other words, the matching M is X-complete (by the definition of "X-complete").

[23]*Proof.* Let $x \in X \setminus S$. Then, $x \in X \setminus S \subseteq X$, so that $(s, x) \in A_s$ (by the definition of A_s). Thus, the arc (s, x) is an s-arc, and has source s and target x. Hence, the source of this arc lies in S (since $s \in S$), but the target of this arc lies in $\overline{S}$ (since $x \in \underbrace{X}_{\subseteq V} \setminus S \subseteq V \setminus S = \overline{S}$). In other words, this arc

(s, x) belongs to $[S, \overline{S}]$. Qed.

- For every vertex $y \in N(U) \setminus S$, there is at least one G-arc with target y that belongs to $[S, \overline{S}]$ [24]. If we pick one such G-arc for each vertex $y \in N(U) \setminus S$, then we obtain a total of $|N(U) \setminus S|$ distinct G-arcs belonging to $[S, \overline{S}]$ (indeed, they are distinct because their targets y are distinct).

- For every vertex $y \in Y \cap S$, the t-arc (y, t) belongs to $[S, \overline{S}]$ [25]. Of course, these t-arcs (y, t) are distinct (since their sources y are distinct). Thus, we have found a total of $|Y \cap S|$ different t-arcs belonging to $[S, \overline{S}]$ (one for each $y \in Y \cap S$).

Now, recall that there is no overlap between the G-arcs, the s-arcs and the t-arcs.

---

[24]*Proof.* Let $y \in N(U) \setminus S$. We must prove that there is at least one G-arc with target y that belongs to $[S, \overline{S}]$.

We know that $y \in N(U) \setminus S \subseteq N(U)$. In other words, y has a neighbor in U (by the definition of $N(U)$). In other words, there exists some $x \in U$ such that x is a neighbor of y. Consider this x. Also, $y \in N(U) \subseteq Y$.

We know that x is a neighbor of y. In other words, $\{y, x\} \in \psi(E)$ (by the definition of "neighbor"). In other words, $\{y, x\} = \psi(e)$ for some $e \in E$. Consider this e.

Now, e is an edge of G (since $e \in E$) and has endpoints y and x (since $\psi(e) = \{y, x\}$). Hence, the unique endpoint of e that lies in X is x (since $x \in U \subseteq X$), whereas the unique endpoint of e that lies in Y is y (since $y \in Y$). Hence, the definition of $\overrightarrow{e}$ yields $\overrightarrow{e} = (x, y, e)$.

Now, $e \in E$ (since e is an edge of G), and thus $\overrightarrow{e} \in A_G$ (by the definition of A_G). Hence, $(x, y, e) = \overrightarrow{e} \in A_G$. In other words, (x, y, e) is a G-arc. This G-arc (x, y, e) has source x and target y. Thus, its source lies in S (since $x \in U \subseteq S$) and its target lies in $\overline{S}$ (since $y \in \underbrace{N(U)}_{\subseteq V} \setminus S \subseteq$ $V \setminus S = \overline{S}$). In other words, this G-arc (x, y, e) belongs to $[S, \overline{S}]$. Thus, there is at least one G-arc with target y that belongs to $[S, \overline{S}]$ (namely, the G-arc (x, y, e)). Qed.

[25]*Proof.* Let $y \in Y \cap S$. Then, $y \in Y \cap S \subseteq Y$, so that $(y, t) \in A_t$ (by the definition of A_t). Thus, (y, t) is a t-arc having source y and target t. Hence, the source of this arc lies in S (since $y \in Y \cap S \subseteq S$), but the target of this arc lies in $\overline{S}$ (since $t \in \overline{S}$ (because $t \notin S$)). In other words, this arc (y, t) belongs to $[S, \overline{S}]$. Qed.

Hence,

$$
\left| \left[S, \overline{S} \right] \right| = \underbrace{\left(\text{the number of all } G\text{-arcs belonging to } \left[S, \overline{S} \right] \right)}_{\substack{\geq |N(U) \backslash S| \\ (\text{since we have found } |N(U) \backslash S| \text{ different } G\text{-arcs belonging to } [S,\overline{S}])}}
$$

$$
+ \underbrace{\left(\text{the number of all } s\text{-arcs belonging to } \left[S, \overline{S} \right] \right)}_{\substack{\geq |X \backslash S| \\ (\text{since we have found } |X \backslash S| \text{ different } s\text{-arcs belonging to } [S,\overline{S}])}}
$$

$$
+ \underbrace{\left(\text{the number of all } t\text{-arcs belonging to } \left[S, \overline{S} \right] \right)}_{\substack{\geq |Y \cap S| \\ (\text{since we have found } |Y \cap S| \text{ different } t\text{-arcs belonging to } [S,\overline{S}])}}
$$

$$
\geq |N(U) \setminus S| + |X \setminus S| + |Y \cap S|
$$

$$
= \underbrace{|Y \cap S| + |N(U) \setminus S|}_{\substack{\geq |N(U)| \\ (\text{by } (33))}} + \underbrace{|X \setminus S|}_{\substack{= |X| - |U| \\ (\text{by } (34))}}
$$

$$
\geq |N(U)| + |X| - |U|.
$$

Hence, (35) yields $|f| = \left| \left[S, \overline{S} \right] \right| \geq |N(U)| + |X| - |U|$.

But let M be the subset $\left\{ e \in E \mid f\left(\overrightarrow{e} \right) = 1 \right\}$ of E. Then, Proposition 3.20 **(d)** yields that the set $M = \left\{ e \in E \mid f\left(\overrightarrow{e} \right) = 1 \right\}$ is a matching in G. Also, Proposition 3.20 **(e)** shows that

$$
|M| = |f| \geq |N(U)| + |X| - |U|.
$$

We have thus found a matching M in G and a subset U of X satisfying $|M| \geq |N(U)| + |X| - |U|$. This proves Lemma 3.26. $\qquad \square$

We can now easily prove Theorem 3.15:

Proof of Theorem 3.15 (sketched). $\Longrightarrow$: Assume that G has an X-complete matching. We must prove that each subset U of X satisfies $|N(U)| \geq |U|$.

This is the so-called "easy part" of Theorem 3.15, and can be proven without any reference to flows and cuts. Just fix an X-complete matching M in G (such a matching exists, by assumption). Let U be a subset of X. We must prove that $|N(U)| \geq |U|$.

The matching M is X-complete; thus, each vertex $x \in X$ has an M-partner. Let $i : X \to Y$ be the map that sends each vertex $x \in X$ to its M-partner. Any two distinct elements of X must have distinct M-partners (because otherwise, the edges joining them to their common M-partner would be two distinct edges in M that have an endpoint in common; but this is not allowed for a matching). In other words, the map i is injective. Hence, every subset Z of X satisfies $|i(Z)| = |Z|$. Applying this to $Z = U$, we obtain $|i(U)| = |U|$.

But each $x \in U$ satisfies $i(x) \in N(U)$ (because the vertex $i(x)$ has a neighbor in U (namely, the vertex x)). In other words, $i(U) \subseteq N(U)$. Hence, $|i(U)| \leq |N(U)|$, so that $|N(U)| \geq |i(U)| = |U|$.

Now, forget that we fixed U. We thus have shown that each subset U of X satisfies $|N(U)| \geq |U|$. This proves the $\implies$ direction of Theorem 3.15.

$\impliedby$: Assume that

$$\text{each subset } U \text{ of } X \text{ satisfies } |N(U)| \geq |U|. \tag{36}$$

We must prove that G has an X-complete matching.

Lemma 3.26 shows that there exist a matching M in G and a subset U of X satisfying $|M| \geq |N(U)| + |X| - |U|$. Consider these M and U. Then,

$$|M| \geq \underbrace{|N(U)|}_{\substack{\geq |U| \\ \text{(by (36))}}} + |X| - |U| \geq |U| + |X| - |U| = |X|.$$

Hence, Lemma 3.25 **(b)** yields that the matching M is X-complete. Hence, G has an X-complete matching (namely, M). This proves the $\impliedby$ direction of Theorem 3.15. Thus, the proof of Theorem 3.15 is complete. $\square$

3.3. König's vertex cover theorem

Lemma 3.26 puts us at a vantage point to prove not just Hall's marriage theorem, but also its close relative, *König's vertex cover theorem*. Before we state the latter theorem, we need to define the concept of a vertex cover:

Definition 3.27. Let $G = (W, E, \psi)$ be a multigraph. Then, a *vertex cover* of G means a subset C of W such that each edge $e \in E$ contains at least one vertex in C.

For example, if $G = (W, E, \psi)$ is the simple graph

$$\begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \rule{2cm}{0.4pt} 3 \end{array} \tag{37}$$
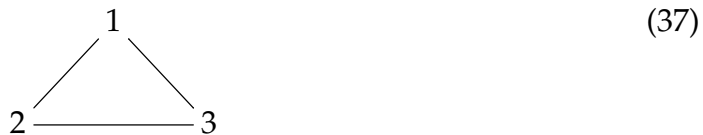
(regarded as a multigraph), then every 2-element subset of W is a vertex cover (but no smaller subset of W is). Clearly, any multigraph $G = (W, E, \psi)$ has at least one vertex cover (because the whole set W is always a vertex cover). A classical problem in computer science is to find a vertex cover of a given multigraph whose size is minimum.

Now, *König's theorem on vertex covers* states the following:

Theorem 3.28. Let $(G; X, Y)$ be a bipartite graph. Then, the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Notice that the claim of Theorem 3.28 isn't true for arbitrary graphs G; for example, if G is the simple graph in (37), then the maximum size of a matching in G is 1, but the minimum size of a vertex cover of G is 2. Nevertheless, the claim is true when G is part of a bipartite graph $(G; X, Y)$.

What is true for arbitrary graphs G is the following inequality:

> **Proposition 3.29.** Let G be a multigraph. Then, the maximum size of a matching in G is $\leq$ to the minimum size of a vertex cover of G.

Proof of Proposition 3.29. Let m be the maximum size of a matching in G. Let c be the minimum size of a vertex cover of G. We must prove that $m \leq c$.

There exists a matching M in G such that $|M| = m$ (since m is the size of a matching in G). Consider this M.

There exists a vertex cover C of G such that $|C| = c$ (since c is the size of a vertex cover of G). Consider this C.

Write the multigraph G in the form $G = (W, E, \psi)$. Recall that C is a vertex cover of G. In other words, C is a subset of W such that each edge $e \in E$ contains at least one vertex in C (by the definition of a "vertex cover").

Each edge $e \in E$ contains at least one vertex in C. In other words, for each edge $e \in E$, there is at least one vertex $v \in C$ such that $v \in e$. In other words, for each edge $e \in E$, we have

$$\text{(the number of } v \in C \text{ such that } v \in e) \geq 1. \tag{38}$$

On the other hand, M is a matching in G. In other words, M is a subset of E such that no two distinct edges in M have an endpoint in common (by the definition of a "matching"). No two distinct edges in M have an endpoint in common. In other words, no vertex of G is contained in more than one edge in M. In other words, each vertex of G is contained in at most one edge in M. In other words, for each vertex v of G, there is at most one edge $e \in M$ such that $v \in e$. In other words, for each vertex v of G, we have

$$\text{(the number of edges } e \in M \text{ such that } v \in e) \leq 1. \tag{39}$$

Now,

$$\text{(the number of pairs } (v, e) \in C \times M \text{ such that } v \in e)$$
$$= \sum_{v \in C} \underbrace{\text{(the number of edges } e \in M \text{ such that } v \in e)}_{\substack{\leq 1 \\ \text{(by (39))}}} \leq \sum_{v \in C} 1$$
$$= |C| \cdot 1 = |C| = c.$$

Thus,

$$c \geq \text{(the number of pairs } (v,e) \in C \times M \text{ such that } v \in e)$$
$$= \sum_{e \in M} \underbrace{\text{(the number of } v \in C \text{ such that } v \in e)}_{\substack{\geq 1 \\ \text{(by (38))}}} \geq \sum_{e \in M} 1$$
$$= |M| \cdot 1 = |M| = m.$$

In other words, $m \leq c$. This completes our proof of Proposition 3.29. $\qquad\square$

Proof of Theorem 3.28 (sketched). Let m be the maximum size of a matching in G. Let c be the minimum size of a vertex cover of G. We must prove that $m = c$.

Proposition 3.29 yields $m \leq c$.

Write the multigraph G in the form $G = (W, E, \psi)$.

Lemma 3.26 shows that there exist a matching M in G and a subset U of X satisfying $|M| \geq |N(U)| + |X| - |U|$. Consider these M and U.

The size $|M|$ of the matching M is clearly $\leq$ to the maximum size of a matching in G. In other words, $|M| \leq m$ (since m is the maximum size of a matching in G). Thus,

$$m \geq |M| \geq |N(U)| + |X| - |U|. \tag{40}$$

On the other hand, let C be the subset $(X \setminus U) \cup N(U)$ of W. Then, we have the following:

> *Observation 1:* The set C is a vertex cover of G.

[*Proof of Observation 1:* Let $e \in E$ be any edge. We shall show that e contains at least one vertex in C.

Indeed, assume the contrary. Thus, e contains no vertex in C.

Each edge of G contains exactly one vertex in X and exactly one vertex in Y (since $(G; X, Y)$ is a bipartite graph). Thus, in particular, each edge of G contains exactly one vertex in X. Applying this to the edge e, we conclude that the edge e contains exactly one vertex in X. Let x be this vertex. Thus, $x \in X$, and the edge e contains the vertex x. If we had $x \in C$, then e would contain a vertex in C (namely, the vertex x), which would contradict the fact that e contains no vertex in C. Hence, we cannot have $x \in C$. Thus, we have $x \notin C$. Hence, $x \in X \setminus C$.

But $C = (X \setminus U) \cup N(U) \supseteq X \setminus U$, so that $X \setminus \underbrace{C}_{\supseteq X \setminus U} \subseteq X \setminus (X \setminus U) = U$ (since $U \subseteq X$). Hence, $x \in X \setminus C \subseteq U$.

Now, let y be the endpoint of the edge e distinct from x. (This is well-defined, since we already know that e contains x.) Then, x and y are the two endpoints of the edge e. Hence, $\psi(e) = \{x, y\}$. Thus, $\{y, x\} = \{x, y\} = \psi\left(\underbrace{e}_{\in E}\right) \in \psi(E)$; in other words, x is a neighbor of y. Hence, the vertex y has a neighbor in U (namely,

the neighbor x). In other words, $y \in N(U)$ (by the definition of $N(U)$). Hence, $y \in N(U) \subseteq (X \setminus U) \cup N(U) = C$. In other words, y is a vertex in C. Thus, the edge e contains at least one vertex in C (namely, the vertex y), because e contains y. This contradicts the fact that e contains no vertex in C.

This contradiction proves that our assumption was wrong. Hence, we have shown that e contains at least one vertex in C.

Now, forget that we fixed e. We thus have proven that each edge $e \in E$ contains at least one vertex in C. In other words, C is a vertex cover of G (by the definition of "vertex cover"). This proves Observation 1.]

Now, $C = (X \setminus U) \cup N(U)$, so that

$$|C| = |(X \setminus U) \cup N(U)| \leq \underbrace{|X \setminus U|}_{\substack{=|X|-|U| \\ (\text{since } U \subseteq X)}} + |N(U)|$$

$$(\text{this is in fact an equality, but we don't need this})$$

$$= |X| - |U| + |N(U)| = |N(U)| + |X| - |U| \leq m \qquad (\text{by (40)}).$$

But the set C is a vertex cover of G (by Observation 1). Hence, its size $|C|$ is $\geq$ to the minimum size of a vertex cover of G. In other words, $|C| \geq c$ (since c is the minimum size of a vertex cover of G). Thus, $c \leq |C| \leq m$. Combining this with $m \leq c$, we obtain $m = c$. This completes the proof of Theorem 3.28. $\qquad \square$

References

[ForFul62]   L. R. Ford (Jr.) and D. R. Fulkerson, *Flows in Networks*, Princeton University Press 1962.
A preprint is freely available at `https://www.rand.org/content/dam/rand/pubs/reports/2007/R375.pdf` .

[Goeman17] Michel X. Goemans, *18.453 Combinatorial Optimization*, course at MIT, Spring 2017.
`http://math.mit.edu/~goemans/18453S17/18453.html`

[Grinbe17b] Darij Grinberg, *UMN, Spring 2017, Math 5707: Lecture 16 (flows and cuts in networks)*,
`http://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec16.pdf` .

[LeLeMe17] Eric Lehman, F. Thomson Leighton, Albert R. Meyer, *Mathematics for Computer Science*, version 15 November 2017.
`https://courses.csail.mit.edu/6.042/fall17/mcs.pdf`

[Martin17]   Jeremy L. Martin, *Lecture Notes on Algebraic Combinatorics*, 8 June 2017.
`http://people.ku.edu/~jlmartin/LectureNotes.pdf`

[Schrij12]   Alexander Schrijver, *On the History of the Transportation and Maximum Flow Problems*, Documenta Mathematica, Extra Volume ISMP (2012), pp. 169–180.

[Schrij17]   Alexander (Lex) Schrijver, *A Course in Combinatorial Optimization*, 23 March 2017.
https://homepages.cwi.nl/~lex/files/dict.pdf

[Thalwi08]   Timon Thalwitzer, *Max-Flow Min-Cut*, diploma thesis at Universität Wien.
http://www.mat.univie.ac.at/~kratt/theses/thalwitz.pdf

[Zwick95]   Uri Zwick, *The smallest networks on which the Ford-Fulkerson maximum flow procedure may fail to terminate*, Theoretical Computer Science **148**, Issue 1, 21 August 1995, pp. 165–170.