Spring 2018 Math 4707, Chapter 9: Odds and ends

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slides:
https://www.cip.ifi.lmu.de/~grinberg/t/18s/4707-2018may2.pdf

9.1.

Integer sequences

References:

- The On-Line Encyclopedia of Integer Sequences (OEIS).
- for wilder sequences: the OEIS Superseeker.
- Richard Stanley, *Enumerative Combinatorics*.
- Sage Cell Server or your favorite programming language.
- FindStat for combinatorial maps.

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    Think number of $n \times n$ Latin squares.

Example: derangements that are involutions, 1

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- See https://oeis.org/A001147 for the sequence $(a_0, a_2, a_4, \ldots)$.

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  - And so on, until all σ-values are set.

  Total number of choices: $(2n-1)(2n-3)(2n-5)\cdots 1$.

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- Thus,

$$\begin{aligned} a_{2n} &= (2n-1)\,(2n-3)\,(2n-5)\cdots 1 \\ &= 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1) \\ &= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\cdots\cdot(2n-1)\cdot(2n)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} \\ &= \frac{(2n)!}{2^n n!}, \end{aligned}$$

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  qed.
- A similar argument works for a_{2n+1}, but this time the last step of the construction offers 0 choices (since there are no elements left to choose y_{2n+1} from).

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- https://oeis.org/A000085; known as the *telephone numbers*.

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- **Exercise:** Prove the non-recursive formula.
  Hint: To construct an involution in S_n, first choose its set of fixed points. On the remaining elements of $[n]$, it behaves like a derangement that is an involution.

Zen stare numbers

- n people stand in a circle. Each of them looks down at the feet of one of the $n-1$ others.
  A bell sounds, and every person (simultaneously) looks up at the eyes of the person whose feet they have been ogling.
  If two people make eye contact, they scream.
  How many possibilities are there where no one screams?

- **Mathematical restatement:**
  Let c_n be the number of all maps $f : [n] \to [n]$ such that no two elements i and j of $[n]$ satisfy $f(i) = j$ and $f(j) = i$ (simultaneously).

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- https://oeis.org/A134362.

Rook placements in a square

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- Such rook placements can also be viewed as matchings of the complete bipartite graph $K_{n,n}$.
  (A rook in row i and column j corresponds to an edge joining i with $-j$.)

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- Let's try something simpler...

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- https://oeis.org/A000170.

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- Note that $e'_6 < e'_5$, which would be unusual for a "simple" sequence.
- **Theorem:** $e'_n > 0$ for $n \geq 4$.
  See Wikipedia for explicit constructions.

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- Formula:

$$e''_n = \begin{cases} \dfrac{n(n-2)^2(2n^3-12n^2+23n-10)}{12}, & \text{if } n \text{ is even;} \\ \dfrac{(n-1)(n-3)(2n^4-12n^3+25n^2-14n+1)}{12}, & \text{if } n \text{ is odd.} \end{cases}$$

- https://oeis.org/A047659.

Symmetric rook placements in a square

- Let d'_n be the number of ways to place non-attacking rooks on an $n \times n$-chessboard in such a way that the picture is symmetric in the main diagonal (i.e., if there is a rook in cell (i, j), then there is a rook in cell (j, i)).
  (Recall: A rook attacks anyone on the same row or column.)

Symmetric rook placements in a square

- Let d'_n be the number of ways to place non-attacking rooks on an $n \times n$-chessboard in such a way that the picture is symmetric in the main diagonal (i.e., if there is a rook in cell (i, j), then there is a rook in cell (j, i)).
  (Recall: A rook attacks anyone on the same row or column.)

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d'_n	1	2	5	14	43	142	499	1850	7193	29186	123109

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- Formula:

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- https://oeis.org/A005425.

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- https://oeis.org/A000108.

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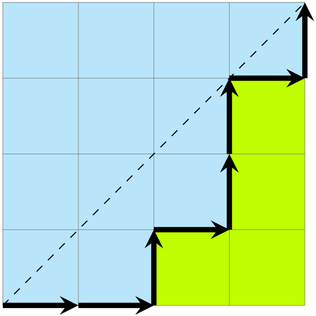
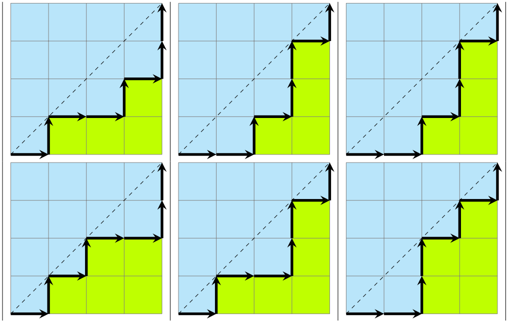
- https://oeis.org/A000108.
- This is combinatorial interpretation #79 (out of 214) from Richard Stanley's book *Catalan numbers*. He outlines bijections between all of them!

- Recall that a lattice path $(0,0) \to (n,n)$ is *Dyck* (or, as we called it on Midterm 2, *legal*) if it never reaches above the $x = y$ diagonal.
  Example:



- Recall that a lattice path $(0,0) \to (n,n)$ is *Dyck* (or, as we called it on Midterm 2, *legal*) if it never reaches above the $x = y$ diagonal.
- Fix n and k; let $N_{n,k}$ be the number of Dyck paths $(0,0) \to (n,n)$ with exactly k left turns ($=$ east-steps followed immediately by north-steps).

  Example: $N_{4,3}$ counts the following 6 Dyck paths:



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  These are the *Narayana numbers*.
- https://oeis.org/A001263.

A digression

- Let z_n be the number of positive divisors of $n!$.

n	0	1	2	3	4	5	6	7	8	9	10
z_n	1	1	2	4	8	16					

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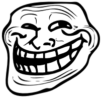

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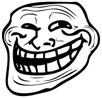

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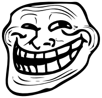

- No formula known. But fairly easy to compute.
  Also, z_n itself is a divisor of $n!$ (Luca, Young, 2012).
- https://oeis.org/A027423.

- Recall: An *(integer) partition* of n means a weakly decreasing sequence $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of positive integers whose sum is n. (See 21 March 2018, Section 4.6.)
  Let p_n be the number of partitions of n.

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p_n	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

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- No explicit formula known.
  Recursive formula (Euler's Pentagonal Number Theorem):

$$p_n = \sum_{k \text{ nonzero integer}} (-1)^{k-1} p_{n-k(3k-1)/2}$$

$$= \underbrace{\cdots + p_{n-15} - p_{n-7} + p_{n-2}}_{\text{negative } k} + \underbrace{p_{n-1} - p_{n-5} + p_{n-12} \pm \cdots}_{\text{positive } k}$$

$$= p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7} + p_{n-12} + p_{n-15} \pm \cdots .$$

(This sum is actually finite, since $p_m = 0$ for $m < 0$.)

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- `https://oeis.org/A000041`.
- Myriad properties; hundreds (thousands?) of papers written about this sequence since Euler, Sylvester, Ramanujan.

Reverse plane partitions

- A *reverse plane partition* of rectangular shape $a \times b$ is an $a \times b$-matrix of nonnegative integers such that
  - each row is weakly increasing;
  - each column is weakly increasing.

  Example ($a = 4$ and $b = 5$):

$$\begin{pmatrix} 0 & 2 & 2 & 4 & 4 \\ 1 & 2 & 3 & 7 & 8 \\ 2 & 2 & 5 & 7 & 9 \\ 3 & 5 & 8 & 8 & 9 \end{pmatrix}.$$

Reverse plane partitions

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  Case $a = b = c$:

n	0	1	2	3	4	5	6
$r_{n,n,n}$	1	2	20	980	232848	267227532	1478619421136

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  - each row is weakly increasing;
  - each column is weakly increasing.

  Let $r_{a,b,c}$ be the number of reverse plane partitions of shape $a \times b$ with entries in $\{0, 1, \ldots, c\}$.
- Formula (MacMahon):

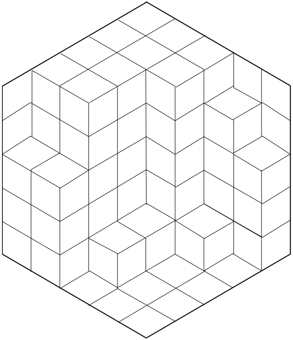
$$r_{a,b,c} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \frac{H(a)\, H(b)\, H(c)\, H(a+b+c)}{H(b+c)\, H(c+a)\, H(a+b)},$$

  where $H(m)$ is the *hyperfactorial*, defined by

$$H(m) = 0! \cdot 1! \cdot 2! \cdot \cdots \cdot (m-1)!.$$

- See https://oeis.org/A008793 for the sequence $(r_{0,0,0}, r_{1,1,1}, r_{2,2,2}, \ldots)$.

- Let $t_{a,b,c}$ be the number of tilings of a 120°-angled hexagon with sides a, b, c, a, b, c by lozenges (= rhombi with sides 1 and angles $60°, 120°, 60°, 120°$).



(Images from arXiv:math/9801111 by Saldanha and Tomei.)

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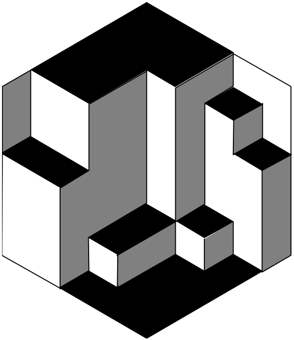

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Rhombus/lozenge tilings of a hexagon

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$$\begin{pmatrix} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 5 & 5 \\ 0 & 0 & 1 & 5 & 5 \\ 0 & 1 & 5 & 5 & 5 \\ 3 & 4 & 5 & 5 & 5 \end{pmatrix}$$

(interpret the previous picture as stacks of boxes in 3D space, and transform it into a heightmap).

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- Thus, using previous slide:

$$t_{a,b,c} = r_{a,b,c} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

$$= \frac{H(a)\,H(b)\,H(c)\,H(a+b+c)}{H(b+c)\,H(c+a)\,H(a+b)}.$$

9.2.

A glimpse of Pólya theory

References:

- Combinatorial Necklaces and Bracelets (javascript).
- Graham/Knuth/Patashnik, *Concrete Mathematics*, Section 4.9.
- Tom Davis, *Pólya's counting theory*.
- Weeks 8–9 of Padraic Bartlett's S2015M116 notes.

Counting necklaces, 1

- Let q and n be positive integers.
  Consider the set $[q]^n$ of all n-tuples of elements of $[q]$.
  We write any n-tuple $(i_1, i_2, \ldots, i_n)$ as $i_1 i_2 \cdots i_n$ (so we omit commas and parentheses).
  Examples:

$$[2]^3 = \{000, 001, 010, 011, 100, 101, 110, 111\};$$

$$[3]^2 = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}.$$

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  We write any n-tuple $(i_1, i_2, \ldots, i_n)$ as $i_1 i_2 \cdots i_n$ (so we omit commas and parentheses).
- *Rotation* is the permutation of $[q]^n$ that sends

$$i_1 i_2 \cdots i_n \mapsto i_n i_1 i_2 \cdots i_{n-1}.$$

  For example,

$$\begin{aligned} &000 \mapsto 000; \\ &001 \mapsto 100 \mapsto 010 \mapsto 001; \\ &011 \mapsto 101 \mapsto 110 \mapsto 011; \\ &111 \mapsto 111. \end{aligned}$$

- A *necklace* with n beads of q colors means a cycle of this permutation (i.e., an equivalence class of n-tuples in $[q]^n$, where we identify every n-tuple with its rotation).
  So there are 4 necklaces with 3 beads of 2 colors...

Counting necklaces, 2

- Necklaces with $n = 3$ beads of $q = 2$ colors:

$$\{000\}\,;$$
$$\{001, 100, 010\}\,;$$
$$\{011, 101, 110\}\,;$$
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- This can be used to prove Fermat's Little Theorem:

$$a^p \equiv a \mod p \qquad \text{for every prime } p \text{ and every integer } a.$$

(See, e.g., the Wikipedia, or this blog post.)

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-

$$N_q(4) = \frac{1}{4}\left(q^4 + q^2 + 2q\right);$$
$$N_q(6) = \frac{1}{6}\left(q^6 + q^3 + 2q^2 + 2q\right).$$

Counting necklaces, 4

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- For any positive integer n, we let

$$\phi(n) = (\# \text{ of } i \in [n] \text{ that are coprime to } n).$$

This defines the *Euler totient function* ϕ.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8

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  This defines the *Euler totient function* ϕ. Formula:

$$\phi(n) = n \cdot \prod_{p \text{ prime divisor of } n} \left(1 - \frac{1}{p}\right).$$

  Also, https://oeis.org/A000010.
- Now,

$$N_q(n) = \frac{1}{n} \sum_{\substack{d \text{ is a positive} \\ \text{divisor of } n}} \phi(d) \, q^{n/d} = \frac{1}{n} \sum_{k=1}^{n} q^{n/\gcd(k,n)}.$$

- Many other things can be counted similarly:
  - *aperiodic* necklaces (i.e., those of size n);
  - necklaces with a given multiplicity of each letter;
  - "multinecklaces" (multiple beads "in the same position");
  - ...

  Many things can be counted "up to cyclic rotation", and often the result will have the form

$$\frac{1}{n} \sum_{\substack{d \text{ is a positive} \\ \text{divisor of } n}} \phi(d) \cdot (\text{old result for } n/d) .$$

What's going on?

Counting necklaces, 5

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  What's going on?
- Pólya's counting theory gives the answer.
  The proper formulation needs some work to introduce (most natural to do after some abstract algebra, specifically the concept of group actions).
  It also answers questions about rotation-and-reflection and other symmetries.

9.3.

Zeckendorf family identities

References:

- Grinberg, *Zeckendorf family identities generalized*.
- Wood/Zeilberger, *A Translation Method for Finding Combinatorial Bijections*.

- Recall the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$.
- Midterm 1 Exercise 3 **(a)**:

$$7f_n = f_{n-4} + f_{n+4} \qquad \text{for all } n \geq 4.$$

Negative Fibonacci numbers

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- We can extend the Fibonacci sequence "to the left": Recursively define $f_{-1}, f_{-2}, f_{-3}, \ldots$ by using the recursion $f_n = f_{n-1} + f_{n-2}$ backwards.
  Example:

$$f_1 = f_0 + f_{-1} \implies f_{-1} = f_1 - f_0 = 1 - 0 = 1;$$
$$f_0 = f_{-1} + f_{-2} \implies f_{-2} = f_0 - f_{-1} = 0 - 1 = -1;$$
$$\ldots$$

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$$f_0 = f_{-1} + f_{-2} \implies f_{-2} = f_0 - f_{-1} = 0 - 1 = -1;$$

$\ldots$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
f_n	-8	5	-3	2	-1	1	0	1	1	2	3	5	8

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- Midterm 1 Exercise 3 **(a)**:

$$7f_n = f_{n-4} + f_{n+4} \qquad \text{for all } n \geq 4.$$

- We can extend the Fibonacci sequence "to the left": Recursively define $f_{-1}, f_{-2}, f_{-3}, \ldots$ by using the recursion $f_n = f_{n-1} + f_{n-2}$ backwards.
  Example:

$$f_1 = f_0 + f_{-1} \implies f_{-1} = f_1 - f_0 = 1 - 0 = 1;$$
$$f_0 = f_{-1} + f_{-2} \implies f_{-2} = f_0 - f_{-1} = 0 - 1 = -1;$$

$\ldots$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
f_n	-8	5	-3	2	-1	1	0	1	1	2	3	5	8

- Note the symmetry (similar to binomial coefficients):

$$f_{-n} = (-1)^{n-1} f_n.$$

Negative Fibonacci numbers

- Recall the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$.
- Midterm 1 Exercise 3 **(a)**:

$$7f_n = f_{n-4} + f_{n+4} \qquad \text{for all } n \geq 4.$$

- We can extend the Fibonacci sequence "to the left": Recursively define $f_{-1}, f_{-2}, f_{-3}, \ldots$ by using the recursion $f_n = f_{n-1} + f_{n-2}$ backwards.
  Example:

$$f_1 = f_0 + f_{-1} \implies f_{-1} = f_1 - f_0 = 1 - 0 = 1;$$
$$f_0 = f_{-1} + f_{-2} \implies f_{-2} = f_0 - f_{-1} = 0 - 1 = -1;$$
$$\ldots$$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
f_n	-8	5	-3	2	-1	1	0	1	1	2	3	5	8

- With this definition,

$$7f_n = f_{n-4} + f_{n+4} \qquad \text{for all integers } n.$$

Zeckendorf family identities, 1

- So we know that

$$7f_n = f_{n-4} + f_{n+4} \qquad \text{for all integers } n.$$

Similarly, for all integers n, we have

$$1f_n = f_n;$$
$$2f_n = f_{n-2} + f_{n+1};$$
$$3f_n = f_{n-2} + f_{n+2};$$
$$4f_n = f_{n-2} + f_n + f_{n+2};$$
$$5f_n = f_{n-4} + f_{n-1} + f_{n+3};$$
$$6f_n = f_{n-4} + f_{n+1} + f_{n+3};$$
$$7f_n = f_{n-4} + f_{n+4}.$$

Zeckendorf family identities, 1

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$$\begin{aligned}
1f_n &= f_n;\\
2f_n &= f_{n-2} + f_{n+1};\\
3f_n &= f_{n-2} + f_{n+2};\\
4f_n &= f_{n-2} + f_n + f_{n+2};\\
5f_n &= f_{n-4} + f_{n-1} + f_{n+3};\\
6f_n &= f_{n-4} + f_{n+1} + f_{n+3};\\
7f_n &= f_{n-4} + f_{n+4}.
\end{aligned}$$

Notice that the sums on the right hand side

  - never use the same f_i twice, and
  - never use two consecutive f_i's.

- So we know that

$$7f_n = f_{n-4} + f_{n+4} \qquad \text{for all integers } n.$$

Similarly, for all integers n, we have

$$\begin{aligned} 1f_n &= f_n; \\ 2f_n &= f_{n-2} + f_{n+1}; \\ 3f_n &= f_{n-2} + f_{n+2}; \\ 4f_n &= f_{n-2} + f_n + f_{n+2}; \\ 5f_n &= f_{n-4} + f_{n-1} + f_{n+3}; \\ 6f_n &= f_{n-4} + f_{n+1} + f_{n+3}; \\ 7f_n &= f_{n-4} + f_{n+4}. \end{aligned}$$

Notice that the sums on the right hand side
  - never use the same f_i twice, and
  - never use two consecutive f_i's.
- **Theorem.** For each $k \in \mathbb{N}$, there exists a unique identity "of the above form" with these two properties for kf_n.

- More generally:
  Theorem. Any sum of the form

  $$f_{n+a_1} + f_{n+a_2} + \cdots + f_{n+a_k}$$

  (where $a_1, a_2, \ldots, a_k$ are integers, which may and may not be distinct) can be "reduced" to a form

  $$f_{n+b_1} + f_{n+b_2} + \cdots + f_{n+b_\ell}$$

  in which the integers $b_1, b_2, \ldots, b_\ell$ are distinct and non-consecutive (i.e., form a lacunar set) and independent of n.
  Moreover, these $b_1, b_2, \ldots, b_\ell$ are uniquely determined.

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  Moreover, these $b_1, b_2, \ldots, b_\ell$ are uniquely determined.
- Proof idea (for existence): Reduce your expression step by step using the following two rules:
  $$\begin{aligned} f_{m-1} + f_m &\longrightarrow f_{m+1}; \\ 2f_m &\longrightarrow f_{m-2} + f_{m+1}. \end{aligned}$$

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For example,

$$4f_n = 2f_n + \underline{2f_n} \longrightarrow 2f_n + f_{n-2} + f_{n+1} = f_{n-2} + f_n + \underline{f_n + f_{n+1}}$$
$$\longrightarrow f_{n-2} + f_n + f_{n+2}.$$

(I'm underlining the terms to which I apply the reduction rules above.)

Zeckendorf family identities, 3

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- Also need to check uniqueness (easy using the Zeckendorf theorem).

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(I'm underlining the terms to which I apply the reduction rules above.)

- Need to check that this reduction eventually terminates; this is not obvious! (I use the golden ratio for this.)
- Also need to check uniqueness (easy using the Zeckendorf theorem).
- I'm currently working on generalizing this to other recurrent sequences.

9.4.

Determinant identites

References:

- Grinberg, *Notes on the combinatorial fundamentals of algebra* (aka [detnotes]).
- Prasolov, *Problems and theorems in linear algebra*.
- Zeilberger, *A combinatorial approach to matrix algebra*.

Behold the determinant

- Recall: If $A = (a_{i,j})_{1 \leq i \leq n,\ 1 \leq j \leq n}$ is an $n \times n$-matrix, then its *determinant* $\det A$ is

$$\det A = \sum_{\sigma \in S_n} (-1)^\sigma \, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

- Unsurprisingly, combinatorics of permutations can be used to prove properties of determinants. We've seen that on Homework 4.
- There is much more to say about determinants...
  A few examples:

Cauchy-Binet formula, 1

- Well-known theorem: If A and B are two $n \times n$-matrices, then

$$\det(AB) = \det A \cdot \det B.$$

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- More generally:
Cauchy-Binet theorem:
If A is an $n \times m$-matrix and if B is an $m \times n$-matrix, then
$$\det(AB) = \sum_{\substack{I \subseteq [m]; \\ |I| = n}} \det\left(A \mid^{I}\right) \cdot \det(B \mid_{I}),$$
where, for each n-element subset $I = \{i_1 < i_2 < \cdots < i_n\}$ of $[m]$, we let
  - $A \mid^{I}$ be the matrix formed by the i_1-th, i_2-th, ..., i_n-th columns of A;
  - $B \mid_{I}$ be the matrix formed by the i_1-th, i_2-th, ..., i_n-th rows of B.

- Well-known theorem: If A and B are two $n \times n$-matrices, then

$$\det(AB) = \det A \cdot \det B.$$

- More generally:
  Cauchy-Binet theorem (restated):
  If A is an $n \times m$-matrix and if B is an $m \times n$-matrix, then

$$\det(AB) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq m} \det\left(A \mid^{(i_1, i_2, \ldots, i_n)}\right) \cdot \det\left(B \mid_{(i_1, i_2, \ldots, i_n)}\right),$$

  where, for any elements $i_1, i_2, \ldots, i_n$ of $[m]$, we let
  - $A \mid^{(i_1, i_2, \ldots, i_n)}$ be the matrix formed by the i_1-th, i_2-th, ..., i_n-th columns of A;
  - $B \mid_{(i_1, i_2, \ldots, i_n)}$ be the matrix formed by the i_1-th, i_2-th, ..., i_n-th rows of B.

- Example:

$$\det\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \det\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$+ \det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} \det\begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{pmatrix}$$

$$+ \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} \det\begin{pmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

- **Desnanot-Jacobi identity.** Let A be an $n \times n$-matrix where $n \geq 2$. Let:
  - A_{NW} be A without its last row and last column;
  - A_{SE} be A without its first row and first column;
  - A_{NE} be A without its last row and first column;
  - A_{SW} be A without its first row and last column.
  - A_C be A without its first row, first column, last row and last column.

  ("NW" stands for "northwest"; "C" stands for "center", etc.)
  Then,

  $$\det A \cdot \det A_C = \det A_{NW} \cdot \det A_{SE} - \det A_{NE} \cdot \det A_{SW}.$$

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Example: $(n = 4)$

$$\det \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \cdot \det \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix}$$

$$= \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \cdot \det \begin{pmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{pmatrix}$$

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Desnanot-Jacobi identity

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$$\det A \cdot \det A_C = \det A_{NW} \cdot \det A_{SE} - \det A_{NE} \cdot \det A_{SW}.$$

- Doron Zeilberger, *Dodgson's Determinant-Evaluation Rule Proved by two-timing men and women* proves this using matchings in bipartite graphs.

- **Chio condensation identity.** Let A be an $n \times n$-matrix where $n \geq 2$.
  Let B be the $(n-1) \times (n-1)$-matrix whose (i,j)-th entry is
  $$a_{i,j} a_{n,n} - a_{i,n} a_{n,j}$$
  (where $a_{u,v}$ denotes the (u,v)-th entry of A). Then,
  $$\det B = a_{n,n}^{n-2} \det A.$$

Chio condensation

- **Chio condensation identity.** Let A be an $n \times n$-matrix where $n \geq 2$.
  Let B be the $(n-1) \times (n-1)$-matrix whose (i, j)-th entry is
  $$a_{i,j} a_{n,n} - a_{i,n} a_{n,j}$$
  (where $a_{u,v}$ denotes the (u, v)-th entry of A). Then,
  $$\det B = a_{n,n}^{n-2} \det A.$$

- Note that the entries of B are themselves little determinants:
  $$a_{i,j} a_{n,n} - a_{i,n} a_{n,j} = \det \begin{pmatrix} a_{i,j} & a_{i,n} \\ a_{n,j} & a_{n,n} \end{pmatrix}.$$

Pfaffians

- An *alternating* matrix is an $n \times n$-matrix $A = (a_{i,j})_{1 \leq i \leq n,\ 1 \leq j \leq n}$ satisfying

$$a_{i,j} = -a_{j,i} \quad \text{for all } i \text{ and } j;$$

$$a_{i,i} = 0 \quad \text{for all } i.$$

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$$a_{i,i} = 0 \qquad \text{for all } i.$$

  In other words, $A^T = -A$, and the diagonal entries of A are 0.

- Alternating 3×3-matrices look like this:

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

  Alternating 4×4-matrices look like this:

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

Pfaffians

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$$a_{i,i} = 0 \qquad \text{for all } i.$$

  In other words, $A^T = -A$, and the diagonal entries of A are 0.

- What can we say about $\det A$ if A is alternating?

$$\det \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = 0;$$

$$\det \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af + cd - be)^2 .$$

  What is the pattern?

Pfaffians

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$$a_{i,j} = -a_{j,i} \quad \text{for all } i \text{ and } j;$$
$$a_{i,i} = 0 \quad \text{for all } i.$$
In other words, $A^T = -A$, and the diagonal entries of A are 0.
- **Theorem.** Let A be an alternating $n \times n$-matrix.
  - If n is odd, then $\det A = 0$.
  - If n is even, then
$$\det A = \left(\sum_{\substack{M \text{ is a perfect matching} \\ \text{of } [n]}} \left(\pm \prod_{\{i,j\} \in M} a_{i,j} \right) \right)^2,$$
where $a_{i,j}$ are the entries of A, and the $\pm$ signs are chosen appropriately.
The sum inside the parentheses is called the *Pfaffian* of A.

9.5.

Partitions

References:

- Andrews/Eriksson, *Integer Partitions*, Cambridge 2004.
- Wilf, *Lectures on Integer Partitions*.
- Pak, *Partition bijections, a survey*.
- Sagan, *The Ubiquitous Young Tableau*.
- Fulton, *Young tableaux*, Cambridge 1997.

- Recall: An *(integer) partition* of n means a weakly decreasing sequence $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of positive integers whose sum is n. (See 21 March 2018, Section 4.6.)
  Let p_n be the number of partitions of n.
- **Euler's Pentagonal Number Theorem:**

$$p_n = \sum_{k \text{ nonzero integer}} (-1)^{k-1} p_{n-k(3k-1)/2}$$

$$= \underbrace{\cdots + p_{n-15} - p_{n-7} + p_{n-2}}_{\text{negative } k} + \underbrace{p_{n-1} - p_{n-5} + p_{n-12} \pm \cdots}_{\text{positive } k}$$

$$= p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7} + p_{n-12} + p_{n-15} \pm \cdots .$$

(This sum is actually finite, since $p_m = 0$ for $m < 0$.)

The pentagonal number theorem: proof idea, 1

- Here's a brief outline of a proof of the pentagonal number theorem.
- Define $g_k = k(3k-1)/2$ for each $k \in \mathbb{Z}$. (This is an integer, called the *k-th pentagonal number*.)

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Equivalently,

$$\sum_{k \text{ even integer}} p_{n-g_k} = \sum_{k \text{ odd integer}} p_{n-g_k}.$$

The pentagonal number theorem: proof idea, 2

- So we must prove

$$\sum_{k \text{ even integer}} p_{n-g_k} = \sum_{k \text{ odd integer}} p_{n-g_k}.$$

- For each $m \in \mathbb{Z}$, let $\operatorname{Par}(m)$ be the set of all partitions of m. Thus, we need a bijection

$$\bigcup_{k \text{ even integer}} \operatorname{Par}(n-g_k) \xrightarrow{A} \bigcup_{k \text{ odd integer}} \operatorname{Par}(n-g_k).$$

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$$\bigcup_{k \text{ even integer}} \operatorname{Par}(n - g_k) \xrightarrow{A} \bigcup_{k \text{ odd integer}} \operatorname{Par}(n - g_k).$$

- Here it is: If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ is a partition of $n - g_k$ for some even k, then

$$A(\lambda) = (p + 3k - 2, \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_p - 1) \quad \text{if } p + 3k > \lambda_1$$

  (this is a partition of $n - g_{k-1}$, where any 0 entries at the end are ignored);

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- Here it is: If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ is a partition of $n - g_k$ for some even k, then

$$A(\lambda) = \left(\lambda_2 + 1, \lambda_3 + 1, \ldots, \lambda_p + 1, \underbrace{1, 1, \ldots, 1}_{\lambda_1 - p - 3k \text{ ones}} \right)$$

$$\text{if } p + 3k \leq \lambda_1$$

(this is a partition of $n - g_{k+1}$).

The pentagonal number theorem: proof idea, 2

- For each $m \in \mathbb{Z}$, let $\operatorname{Par}(m)$ be the set of all partitions of m. Thus, we need a bijection

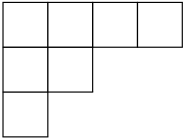
$$\bigcup_{k \text{ even integer}} \operatorname{Par}(n - g_k) \xrightarrow{A} \bigcup_{k \text{ odd integer}} \operatorname{Par}(n - g_k).$$

- Here it is: If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ is a partition of $n - g_k$ for some even k, then ...
- A is bijective, and its inverse is given by the same formula. (The proof is laborious but not difficult.)

The pentagonal number theorem: proof idea, 3

- What is the idea behind the above bijection?

- What is the idea behind the above bijection?
- Recall that any partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ can be visualized as a *Young diagram* – a table with p left-aligned rows, having $\lambda_1, \lambda_2, \ldots, \lambda_p$ cells respectively.
  Example: The partition $(4, 2, 1)$ has Young diagram



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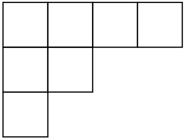

  Note that n is the total number of cells.

The pentagonal number theorem: proof idea, 3

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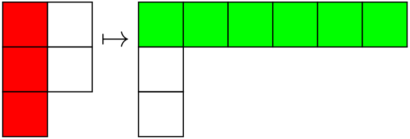
- Recall that any partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ can be visualized as a *Young diagram* – a table with p left-aligned rows, having $\lambda_1, \lambda_2, \ldots, \lambda_p$ cells respectively.
- Now, if λ is a partition of $n - g_k$, then A
  - **either** removes the first column of the Young diagram of λ, and adds a new row on top so that the new diagram is a partition of $n - g_{k-1}$;
  - **or** removes the first row of the Young diagram of λ, and adds a new column to its left so that the new diagram is a partition of $n - g_{k+1}$.

  Fortunately, for each choice of n, k and λ, exactly one of these options works (the first row cannot be shorter than the second, and likewise for columns!), so A always knows what to do.

- Now, if λ is a partition of $n - g_k$, then A
  - **either** removes the first column of the Young diagram of λ, and adds a new row on top so that the new diagram is a partition of $n - g_{k-1}$;
  - **or** removes the first row of the Young diagram of λ, and adds a new column to its left so that the new diagram is a partition of $n - g_{k+1}$.

Example for the first case (removing first column and adding a new row):

$n = 9$, $k = 2$ and $\lambda = (2, 2, 1)$:



Odd vs. distinct entries

- **Theorem (Euler again).** Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} & (\# \text{ of partitions of } n \text{ whose parts are } \textbf{odd}) \\ = & (\# \text{ of partitions of } n \text{ whose parts are } \textbf{distinct}) . \end{aligned}$$

  Example, for $n = 6$:

$$\begin{aligned} \text{odd parts: } & (5,1), (3,3), (3,1,1,1), (1,1,1,1,1,1); \\ \text{distinct parts: } & (6), (5,1), (4,2), (3,2,1). \end{aligned}$$

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- **Proof idea (Glaisher):** To construct a bijection

$$\begin{aligned} & \{\text{partitions of } n \text{ whose parts are } \textbf{odd}\} \\ \to & \{\text{partitions of } n \text{ whose parts are } \textbf{distinct}\} , \end{aligned}$$

we proceed step-by-step: Keep merging equal parts ($a, a \longrightarrow 2a$) until no more equal parts remain.

$$(5, 3, \underline{1, 1}, 1, 1) \longrightarrow (5, 3, 2, \underline{1, 1}) \longrightarrow (5, 3, \underline{2, 2}) \longrightarrow (5, 4, 3) .$$

Odd vs. distinct entries

- **Theorem (Euler again).** Let $n \in \mathbb{N}$. Then,

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Inverse map: Keep splitting even parts ($2a \longrightarrow a, a$) until no more even parts remain.

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  we proceed step-by-step: Keep merging equal parts ($a, a \longrightarrow 2a$) until no more equal parts remain.
  Inverse map: Keep splitting even parts ($2a \longrightarrow a, a$) until no more even parts remain.
  Needs proof: These two maps are well-defined (i.e., the result does not depend on choices).

- **Theorem.** Let $n \in \mathbb{N}$. Then,

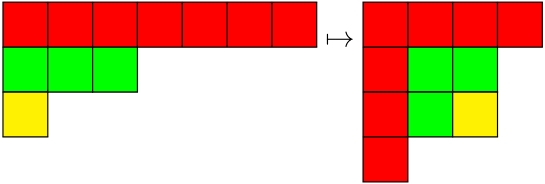
$$(\# \text{ of partitions of } n \text{ whose parts are } \textbf{odd and distinct})$$
$$= (\# \text{ of } \textbf{self-conjugate} \text{ partitions of } n).$$

Here, a partition is said to be **self-conjugate** if its Young diagram is symmetric (i.e., the lengths of its rows equal the length of its respective columns).

- **Proof idea:** To construct a bijection

$$\{\text{partitions of } n \text{ whose parts are } \textbf{odd and distinct}\}$$
$$\to \{\textbf{self-conjugate} \text{ partitions of } n\},$$

we proceed as follows:



Standard Young tableaux

- The Young diagram of a partition serves as a canvas for its "Young tableaux".

Standard Young tableaux

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- Let λ be a partition of n. A *standard Young tableau of shape* λ is a way to fill the n cells of the Young diagram of λ with the n numbers $1, 2, \ldots, n$ (each appearing once) such that
  - each row is weakly increasing;
  - each column is weakly increasing.

  Example: The standard Young tableaux of shape $(3, 2)$ are

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

1	3	4
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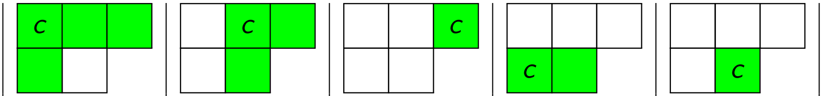
- How many are there?

- Let λ be a partition of n. Let c be a cell of λ.
  The *hook length* of c is

$$h(c) := 1 + (\# \text{ of cells of } \lambda \text{ due east of } c) \\ + (\# \text{ of cells of } \lambda \text{ due south of } c).$$

  Examples for $\lambda = (3, 2)$:
- Thus, for $\lambda = (3, 2)$:



$h(c) = 4$	$h(c) = 3$	$h(c) = 1$	$h(c) = 2$	$h(c) = 1$

The hook length formula, 1

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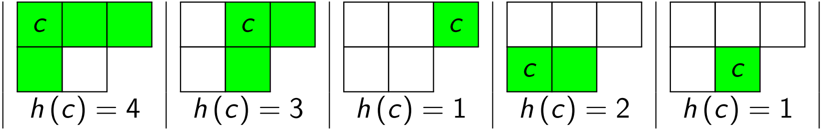

- **Theorem (hook length formula):** The number of standard Young tableaux of shape λ is

$$\frac{n!}{\prod_{c \text{ is a cell of } \lambda} h(c)}.$$

The hook length formula, 2

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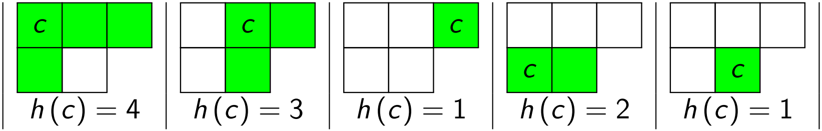

  we get that the number of standard Young tableaux of shape λ is
$$\frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5.$$

- **Theorem (hook length formula):** The number of standard Young tableaux of shape λ is

$$\frac{n!}{\prod\limits_{c \text{ is a cell of } \lambda} h(c)}.$$

- **Exercise.** If $\lambda = (m, m)$, then you the hook length formula yields the answer
$$\frac{(2m)!}{((m+1)\, m \cdots 2)\, (m\, (m-1) \cdots 1)} = \frac{1}{m+1}\binom{2m}{m},$$ which is the m-th Catalan number.
This suggests a bijection between standard Young tableaux of this shape and Dyck paths $(0,0) \to (m,m)$. Find it.

Counting all standard Young tableaux with n cells

- **Theorem (Knuth?).** Let $n \in \mathbb{N}$. The number of all standard Young tableaux (of all possible shapes) with n cells is the number of involutions in S_n. (See "telephone numbers" in Section 9.1.)