University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Midterm 3 (preliminary version)

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Notations

Here is a list of notations that are used in this homework:

- As usual, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$ of all nonnegative integers.
- We shall use the Iverson bracket notation as well as the notation $[n]$ for the set $\{1, 2, \ldots, n\}$ (when $n \in \mathbb{Z}$).
- If $n \in \mathbb{N}$, then S_n denotes the set of all permutations of $[n]$.
- A *point* shall mean an element of $\mathbb{Z}^2$, that is, a pair of integers. We depict these points as lattice points on the Cartesian plane, and add and subtract them as vectors. Recall the notion of a *lattice path*, defined in §6.1 (class notes from 2018-11-12) and (equivalently) in UMN Spring 2018 Math 4707 Midterm 1. (Lattice paths have up-steps and right-steps.) We abbreviate "lattice path" as "*LP*".
- A *formal power series* (short *FPS*) shall always mean a formal power series in the indeterminate x with rational coefficients (as defined in class).

  If f is an FPS and if $n \in \mathbb{N}$, then $[x^n] f$ shall denote the coefficient of x^n in f.

Let us recall five fundamental properties of binomial coefficients (all of which appear in [Grinbe16, §3.1]):

Proposition 0.1. *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $m \geq n$. Then,*

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!}.$$

Proposition 0.2. *Let $m \in \mathbb{Q}$ and $n \in \mathbb{N}$. Then,*

$$\binom{m}{n} = (-1)^n \binom{n-m-1}{n}.$$

Proposition 0.3. *Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then,*

$$\binom{m}{n} \in \mathbb{Z}.$$

Proposition 0.4. *Let $n \in \{1, 2, 3, \ldots\}$ and $m \in \mathbb{Q}$. Then,*

$$\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}.$$

Proposition 0.5. *Let $m \in \mathbb{Q}$, $a \in \mathbb{N}$ and $i \in \mathbb{N}$ be such that $i \geq a$. Then,*

$$\binom{m}{i}\binom{i}{a} = \binom{m}{a}\binom{m-a}{i-a}.$$

Let us also prove two more simple identities for binomial coefficients:

Lemma 0.6. *Let $k \in \mathbb{N}$ and $a \in \mathbb{N}$.*

(a) *We have*

$$k\binom{a}{a-k} = a\binom{a-1}{a-k}. \qquad (1)$$

(b) *Let $n \in \mathbb{Q}$ be such that that $n \neq a$. Then,*

$$\binom{n-a-1}{k-1} = \frac{k}{n-a}\binom{n-a}{k}. \qquad (2)$$

Proof of Lemma 0.6. **(a)** We must prove the equality (1). This equality is obvious in the case when $a-k<0$ (because both of its sides are 0 in this case); thus, we WLOG assume that $a-k \geq 0$. Hence, the definition of $\binom{a}{a-k}$ yields

$$\binom{a}{a-k} = \frac{a(a-1)(a-2)\cdots(a-(a-k)+1)}{(a-k)!} = \frac{a(a-1)(a-2)\cdots(k+1)}{(a-k)!}.$$

Multiplying this equality by k, we find

$$k\binom{a}{a-k} = k \cdot \frac{a(a-1)(a-2)\cdots(k+1)}{(a-k)!} = \frac{a(a-1)(a-2)\cdots(k+1)\cdot k}{(a-k)!}$$
$$= \frac{a(a-1)(a-2)\cdots k}{(a-k)!}. \qquad (3)$$

On the other hand, the definition of $\binom{a-1}{a-k}$ yields

$$\binom{a-1}{a-k} = \frac{(a-1)(a-2)(a-3)\cdots((a-1)-(a-k)+1)}{(a-k)!} = \frac{(a-1)(a-2)(a-3)\cdots k}{(a-k)!}.$$

Multiplying this equality by a, we find

$$\begin{aligned} a\binom{a-1}{a-k} &= a\cdot\frac{(a-1)(a-2)(a-3)\cdots k}{(a-k)!} = \frac{a\cdot(a-1)(a-2)(a-3)\cdots k}{(a-k)!} \\ &= \frac{a(a-1)(a-2)\cdots k}{(a-k)!}. \end{aligned} \qquad (4)$$

Comparing the equalities (3) and (4), we obtain $k\binom{a}{a-k} = a\binom{a-1}{a-k}$. This proves (1). Thus, Lemma 0.6 **(a)** is proven.

(b) We have $n-a \neq 0$ (since $n \neq a$). We must prove the equality (2). This equality is obvious in the case when $k=0$ (because both of its sides are 0 in this case); thus, we WLOG assume that $k \neq 0$. Hence, $k \in \{1,2,3,\ldots\}$ (since $k \in \mathbb{N}$). Hence, Proposition 0.4 (applied to $n-a$ and k instead of m and n) yields $\binom{n-a}{k} = \frac{n-a}{k}\binom{n-a-1}{k-1}$. Solving this for $\binom{n-a-1}{k-1}$, we find $\binom{n-a-1}{k-1} = \frac{k}{n-a}\binom{n-a}{k}$ (indeed, we are allowed to divide by $n-a$, since $n-a \neq 0$). This proves (2). Thus, Lemma 0.6 **(b)** is proven. $\square$

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Let $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$.

(a) Prove that

$$\sum_{k=0}^{n}\binom{n}{k}(x+k)^k(y-k)^{n-k} = \sum_{t=0}^{n}\frac{n!}{t!}(x+y)^t.$$

(b) Prove that

$$\sum_{k=0}^{n}\binom{n}{k}x(x+k)^{k-1}(y-k)^{n-k} = (x+y)^n.$$

(Here, the "$x(x+k)^{k-1}$" expression should be understood as 1 when $k=0$; this gives it meaning even if $x=0$.)

[**Hint:** **(a)** Expand $(x+k)^k$ and $(y-k)^{n-k}$ by the binomial theorem, then try using [Grinbe18b, Exercise 2].

(b) Rewrite $x(x+k)^{k-1}$ as $(x+k)^k - k(x+k)^{k-1}$, thus splitting the left hand side into two sums. Apply part **(a)** to both of them.]

1.2 Remark

Both identities above are classical. The claim of part **(a)** is ascribed to Cauchy in Riordan's text [Riorda68, §1.5, Cauchy's identity]. The claim of part **(b)** was found by Abel in 1826 [Abel26]. More precisely, Abel found the slightly more general formula

$$\sum_{k=0}^{n}\binom{n}{k}x(x+kz)^{k-1}(y-kz)^{n-k} = (x+y)^n$$

for any $x, y, z \in \mathbb{Q}$. But this formula can be easily derived from our part **(b)**: When z is nonzero, it follows by applying our part **(b)** to x/z and y/z instead of x and y; otherwise it boils down to the binomial formula. See [Grinbe18a] and the references therein for further generalizations.

Of course, the assumptions "$x \in \mathbb{Q}$ and $y \in \mathbb{Q}$" in the problem can be replaced by "$x \in R$ and $y \in R$", where R is any commutative ring.

1.3 Solution sketch

The solution I alluded to in the Hint is presented in full detail in [QEDMO09]. In fact, part **(a)** of the exercise is [QEDMO09, the problem], while part **(b)** is [QEDMO09, Theorem 4]. Note that [QEDMO09, Corollary 2] is precisely [Grinbe18b, Theorem 0.2 + Theorem 0.4][1].

However, Tomoya Imaizumi found a shorter solution along the same lines; I am thus going to show his argument instead.

We shall use the following fact ([Grinbe18b, Theorem 0.4]):

Lemma 1.1. *Let $a \in \mathbb{N}$ and $b \in \mathbb{Q}$. Then,*

$$\sum_{k=0}^{a} (-1)^k \binom{a}{k} (b-k)^a = a!.$$

We now come to the solution of the problem.

(a) Define $z \in \mathbb{Q}$ by $z = x + y$. For each $k \in \{0, 1, \ldots, n\}$, we have

$$x + k = \underbrace{(x+y)}_{=z} + (k-y) = z + (k-y)$$

and thus

$$\Big(\underbrace{x+k}_{=z+(k-y)}\Big)^k = (z + (k-y))^k = \sum_{t=0}^{k} \binom{k}{t} z^t \Big(\underbrace{k-y}_{=-(y-k)}\Big)^{k-t}$$

(by the binomial theorem)

$$= \sum_{t=0}^{k} \binom{k}{t} z^t \underbrace{(-(y-k))^{k-t}}_{=(-1)^{k-t}(y-k)^{k-t}} = \sum_{t=0}^{k} \binom{k}{t} z^t (-1)^{k-t} (y-k)^{k-t}.$$

[1] except that the latter is stated for rational numbers, while the former is stated for elements of a commutative ring; but this does not matter to us, since we are applying this result to rational numbers only

Hence,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\underbrace{(x+k)^k}_{=\sum_{t=0}^{k}\binom{k}{t}z^t(-1)^{k-t}(y-k)^{k-t}}(y-k)^{n-k} \\
&=\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{t=0}^{k}\binom{k}{t}z^t(-1)^{k-t}(y-k)^{k-t}\right)(y-k)^{n-k} \\
&=\underbrace{\sum_{k=0}^{n}\sum_{t=0}^{k}}_{=\sum_{\substack{(k,t)\in\{0,1,\ldots,n\}^2;\\ t\le k}}=\sum_{t=0}^{n}\sum_{k=t}^{n}}\underbrace{\binom{n}{k}\binom{k}{t}}_{\substack{=\binom{n}{t}\binom{n-t}{k-t}\\ \text{(by Proposition 0.5,}\\ \text{applied to } m=n,\ i=k \text{ and } a=t)}}z^t(-1)^{k-t}\underbrace{(y-k)^{k-t}(y-k)^{n-k}}_{\substack{=(y-k)^{(k-t)+(n-k)}=(y-k)^{n-t}\\ \text{(since } (k-t)+(n-k)=n-t)}} \\
&=\sum_{t=0}^{n}\sum_{k=t}^{n}\binom{n}{t}\binom{n-t}{k-t}z^t(-1)^{k-t}(y-k)^{n-t} \\
&=\sum_{t=0}^{n}\binom{n}{t}z^t\underbrace{\sum_{k=t}^{n}\binom{n-t}{k-t}(-1)^{k-t}(y-k)^{n-t}}_{\substack{=\sum_{k=0}^{n-t}\binom{n-t}{(k+t)-t}(-1)^{(k+t)-t}(y-(k+t))^{n-t}\\ \text{(here, we have substituted } k+t \text{ for } k \text{ in the sum)}}} \\
&=\sum_{t=0}^{n}\binom{n}{t}z^t\sum_{k=0}^{n-t}\underbrace{\binom{n-t}{(k+t)-t}}_{=\binom{n-t}{k}}\underbrace{(-1)^{(k+t)-t}}_{=(-1)^k}\left(\underbrace{y-(k+t)}_{=(y-t)-k}\right)^{n-t} \\
&=\sum_{t=0}^{n}\underbrace{\binom{n}{t}}_{\substack{=\frac{n!}{t!\,(n-t)!}\\ \text{(by Proposition 0.1, applied to } n \text{ and } t\\ \text{instead of } m \text{ and } n)}}z^t\underbrace{\sum_{k=0}^{n-t}\binom{n-t}{k}(-1)^k((y-t)-k)^{n-t}}_{\substack{=\sum_{k=0}^{n-t}(-1)^k\binom{n-t}{k}((y-t)-k)^{n-t}=(n-t)!\\ \text{(by Lemma 1.1, applied to } a=n-t \text{ and } b=y-t)}} \\
&=\sum_{t=0}^{n}\underbrace{\frac{n!}{t!\,(n-t)!}z^t(n-t)!}_{=\frac{n!}{t!}z^t}=\sum_{t=0}^{n}\frac{n!}{t!}z^t=\sum_{t=0}^{n}\frac{n!}{t!}(x+y)^t
\end{aligned}$$

(since $z=x+y$). This solves part **(a)** of the exercise.

(b) We can easily prove our claim in the case when $n=0$. Hence, we WLOG assume that $n\neq 0$ for the rest of this solution. Thus, $n\geq 1$. Hence, $n-1\in\mathbb{N}$. Thus, we can apply part **(a)** of the exercise to $n-1$, $x+1$ and $y-1$ instead of n, x and y. We thus obtain

$$\sum_{k=0}^{n-1} \binom{n-1}{k} ((x+1)+k)^k ((y-1)-k)^{(n-1)-k}$$
$$= \sum_{t=0}^{n-1} \frac{(n-1)!}{t!} \left(\underbrace{(x+1)+(y-1)}_{=x+y} \right)^t$$
$$= \sum_{t=0}^{n-1} \frac{(n-1)!}{t!} (x+y)^t . \qquad (5)$$

We have agreed to interpret the expression "$x(x+k)^{k-1}$" as 1 when $k=0$ (even when $(x+k)^{k-1}$ may be undefined). Likewise, let us agree to interpret the expression "$k(x+k)^{k-1}$" as 0 when $k=0$. Thus, every $k \in \mathbb{N}$ satisfies

$$x(x+k)^{k-1} = (x+k)^k - k(x+k)^{k-1} . \qquad (6)$$

[*Proof of* (6)*:* Let $k \in \mathbb{N}$. If $k \neq 0$, then (6) follows from

$$(x+k)^k = (x+k)(x+k)^{k-1} = x(x+k)^{k-1} + k(x+k)^{k-1} .$$

Hence, it suffices to prove (6) in the case when $k=0$. But in this case, we have agreed to interpret the expressions "$x(x+k)^{k-1}$" and "$k(x+k)^{k-1}$" as 1 and 0, respectively, whereas the expression "$(x+k)^k$" simply evaluates to $(x+0)^0 = 1$. Thus, in this case, the equality (6) boils down to $1 = 1-0$. This is clearly true. Hence, the proof of (6) is complete.]

Now,

$$\sum_{k=0}^{n} \binom{n}{k} k (x+k)^{k-1} (y-k)^{n-k}$$
$$= \underbrace{\binom{n}{0} 0 (x+0)^{0-1} (y-0)^{n-0}}_{=0} + \sum_{k=1}^{n} \underbrace{\binom{n}{k}}_{\substack{=\frac{n}{k}\binom{n-1}{k-1} \\ \text{(by Proposition 0.4,} \\ \text{applied to } n \text{ and } k \text{ instead of } m \text{ and } n)}} k (x+k)^{k-1} (y-k)^{n-k}$$

(here, we have split off the addend for $k=0$ from the sum)

$$= \sum_{k=1}^{n} \underbrace{\frac{n}{k}\binom{n-1}{k-1} k}_{=n\binom{n-1}{k-1}} (x+k)^{k-1} (y-k)^{n-k}$$
$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\underbrace{x+k}_{=(x+1)+(k-1)} \right)^{k-1} \underbrace{(y-k)^{n-k}}_{\substack{=((y-1)-(k-1))^{(n-1)-(k-1)} \\ \text{(since } y-k=(y-1)-(k-1) \\ \text{and } n-k=(n-1)-(k-1))}}$$
$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} ((x+1)+(k-1))^{k-1} ((y-1)-(k-1))^{(n-1)-(k-1)}$$
$$= n \underbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} ((x+1)+k)^{k} ((y-1)-k)^{(n-1)-k}}_{\substack{=\sum_{t=0}^{n-1} \frac{(n-1)!}{t!}(x+y)^t \\ \text{(by (5))}}}$$

(here, we have substituted k for $k-1$ in the sum)

$$= n \sum_{t=0}^{n-1} \frac{(n-1)!}{t!} (x+y)^t = \sum_{t=0}^{n-1} \frac{n \cdot (n-1)!}{t!} (x+y)^t$$
$$= \sum_{t=0}^{n-1} \frac{n!}{t!} (x+y)^t \tag{7}$$

(since $n \cdot (n-1)! = n!$).

Now,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\underbrace{x\left(x+k\right)^{k-1}}_{\substack{=(x+k)^k-k(x+k)^{k-1}\\ \text{(by (6))}}}\left(y-k\right)^{n-k}\\
&=\sum_{k=0}^{n}\binom{n}{k}\left(\left(x+k\right)^{k}-k\left(x+k\right)^{k-1}\right)\left(y-k\right)^{n-k}\\
&=\underbrace{\sum_{k=0}^{n}\binom{n}{k}\left(x+k\right)^{k}\left(y-k\right)^{n-k}}_{\substack{=\sum_{t=0}^{n}\frac{n!}{t!}(x+y)^t\\ \text{(by part (a) of the exercise)}}}-\underbrace{\sum_{k=0}^{n}\binom{n}{k}k\left(x+k\right)^{k-1}\left(y-k\right)^{n-k}}_{\substack{=\sum_{t=0}^{n-1}\frac{n!}{t!}(x+y)^t\\ \text{(by (7))}}}\\
&=\sum_{t=0}^{n}\frac{n!}{t!}\left(x+y\right)^{t}-\sum_{t=0}^{n-1}\frac{n!}{t!}\left(x+y\right)^{t}=\underbrace{\frac{n!}{n!}}_{=1}\left(x+y\right)^{n}=\left(x+y\right)^{n}.
\end{aligned}$$

This solves part **(b)** of the exercise.

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2 Exercise 2

2.1 Problem

Let n be a positive integer.

(a) Let A be the $n\times n$-matrix $\left([i\neq j]\right)_{i,j\in[n]}$. (This is the $n\times n$-matrix whose diagonal entries are 0 while all its other entries are 1. For example, for $n=3$, it is $\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$.)

Prove that $\det A=\left(-1\right)^{n-1}\left(n-1\right)$.

(b) Prove that

$$\sum_{\substack{\sigma\in S_n \text{ is a}\\ \text{derangement}}}\left(-1\right)^{\sigma}=\left(-1\right)^{n-1}\left(n-1\right).$$

(c) Let $b_1,b_2,\ldots,b_n$ be any n numbers. Let B be the $n\times n$-matrix $\left([i\neq j]\prod_{h\in[n]\setminus\{i,j\}}b_h\right)_{i,j\in[n]}$.

(For example, for $n=4$, we have

$$B=\begin{pmatrix}0&b_3b_4&b_2b_4&b_2b_3\\ b_3b_4&0&b_1b_4&b_1b_3\\ b_2b_4&b_1b_4&0&b_1b_2\\ b_2b_3&b_1b_3&b_1b_2&0\end{pmatrix}.\;)$$

Prove that

$$\det B=\left(-1\right)^{n-1}\left(n-1\right)\prod_{h\in[n]}b_h^{n-2}.$$

(Here, the "$(-1)^{n-1}(n-1)\prod_{h\in[n]} b_h^{n-2}$" expression should be understood as 0 if $n=1$, even if $\prod_{h\in[n]} b_h^{n-2}$ may be undefined in this case when some of the b_h are 0.)

[**Hint:** **(a)** If you need a reminder on the basic properties of determinants, see, e.g., [Grinbe16, Exercises 6.7 and 6.8].

(c) If you divide by some b_h in your proof, make sure to argue why this is legitimate, or separately treat the case when some of the b_h are 0. (There is a combinatorial proof that does not require any division.)]

2.2 Solution sketch

(a) There are several ways to do this. Here is one of the most elementary:

We shall use the following simple fact:

Lemma 2.1. *Consider an $n\times n$-matrix.*

***(a)** If we add a multiple of one of its rows to another of its rows, then its determinant does not change.*

In particular:

***(b)** If we add one of its rows to another of its rows, then its determinant does not change.*

***(c)** If we subtract one of its rows from another of its rows, then its determinant does not change.*

Proof of Lemma 2.1. **(a)** See [Grinbe16, Exercise 6.8 **(a)**] for a proof of Lemma 2.1 **(a)**.

(b) This follows from Lemma 2.1 **(a)**, because adding a row r is the same as adding 1 times the row r.

(c) This follows from Lemma 2.1 **(a)**, because subtracting a row r is the same as adding -1 times the row r. □

Now, let us solve the problem. The $n\times n$-matrix A looks as follows:

$$A=\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

(with the diagonal entries all being 0, while all the other entries are 1).

Subtract the first row of A from each of the other rows; let A' be the resulting $n\times n$-matrix. Then, A' looks as follows:

$$A'=\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

(with the diagonal entries being $0,-1,-1,\ldots,-1$, while the off-diagonal entries in the first row and first column are 1, and all the remaining entries are 0). The matrix A' was obtained from A by several steps of the form "subtract a row from another row"; thus, $\det(A') = \det A$ (since any such step preserves the determinant[2]).

Next, add each of the last $n-1$ rows of A' to the first row; let A'' be the resulting $n \times n$-matrix. Then, A'' looks as follows:

$$A'' = \begin{pmatrix} n-1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix} \qquad (8)$$

(with the diagonal entries being $n-1, -1, -1, \ldots, -1$, while the off-diagonal entries in the first column are 1, and all the remaining entries are 0). The matrix A'' was obtained from A' by several steps of the form "add a row to another row"; thus, $\det(A'') = \det(A')$ (since any such step preserves the determinant[3]).

But (8) shows that the matrix A'' is lower-triangular. Hence, its determinant equals the product of its diagonal entries. Since the diagonal entries of A'' are $n-1, -1, -1, \ldots, -1$, we can rewrite this as follows:

$$\det(A'') = (n-1) \cdot \underbrace{(-1) \cdot (-1) \cdot \cdots \cdot (-1)}_{n-1 \text{ times}} = (n-1) \cdot (-1)^{n-1} = (-1)^{n-1} (n-1).$$

Comparing this with $\det(A'') = \det(A') = \det A$, we obtain $\det A = (-1)^{n-1}(n-1)$. This solves part **(a)** of the exercise.

(b) This is just the claim of part **(a)** in disguise. Indeed, if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are any n statements, then $\prod_{i \in [n]} [\mathcal{A}_i] = [\mathcal{A}_i \text{ for all } i \in [n]]$. Hence, for each permutation $\sigma \in S_n$, we have

$$\prod_{i \in [n]} [i \neq \sigma(i)] = [i \neq \sigma(i) \text{ for all } i \in [n]] = [\sigma(i) \neq i \text{ for all } i \in [n]]$$
$$= [\sigma \text{ is a derangement}] \qquad (9)$$

(since $\sigma(i) \neq i$ holds for all $i \in [n]$ if and only if σ is a derangement). Now, recall that $A = ([i \neq j])_{i,j \in [n]}$. Hence, the definition of $\det A$ yields

$$\begin{aligned}
\det A &= \sum_{\sigma \in S_n} (-1)^{\sigma} \underbrace{\prod_{i \in [n]} [i \neq \sigma(i)]}_{\substack{=[\sigma \text{ is a derangement}] \\ (\text{by } (9))}} \\
&= \sum_{\sigma \in S_n} (-1)^{\sigma} [\sigma \text{ is a derangement}] \\
&= \sum_{\substack{\sigma \in S_n \text{ is a} \\ \text{derangement}}} (-1)^{\sigma} \underbrace{[\sigma \text{ is a derangement}]}_{\substack{=1 \\ (\text{since } \sigma \text{ is a derangement})}} + \sum_{\substack{\sigma \in S_n \text{ is not a} \\ \text{derangement}}} (-1)^{\sigma} \underbrace{[\sigma \text{ is a derangement}]}_{\substack{=0 \\ (\text{since } \sigma \text{ is not a derangement})}} \\
&\qquad (\text{since each } \sigma \in S_n \text{ is either a derangement or not}) \\
&= \sum_{\substack{\sigma \in S_n \text{ is a} \\ \text{derangement}}} (-1)^{\sigma} + \underbrace{\sum_{\substack{\sigma \in S_n \text{ is not a} \\ \text{derangement}}} (-1)^{\sigma} 0}_{=0} = \sum_{\substack{\sigma \in S_n \text{ is a} \\ \text{derangement}}} (-1)^{\sigma}.
\end{aligned}$$

[2]by Lemma 2.1 **(c)**

[3]by Lemma 2.1 **(b)**

Thus,

$$\sum_{\substack{\sigma\in S_n \text{ is a} \\ \text{derangement}}} (-1)^\sigma = \det A = (-1)^{n-1} (n-1)$$

(by part **(a)** of the exercise). This solves part **(b)** of the exercise.

(c) We have

$$B = \left([i \neq j] \prod_{h\in[n]\setminus\{i,j\}} b_h \right)_{i,j\in[n]} .$$

Hence, the definition of $\det B$ yields

$$\begin{aligned}
\det B &= \sum_{\sigma\in S_n} (-1)^\sigma \underbrace{\prod_{i\in[n]} \left([i \neq \sigma(i)] \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h \right)}_{= \left(\prod_{i\in[n]} [i\neq\sigma(i)] \right) \left(\prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h \right)} \\
&= \sum_{\sigma\in S_n} (-1)^\sigma \underbrace{\left(\prod_{i\in[n]} [i \neq \sigma(i)] \right)}_{\substack{=[\sigma \text{ is a derangement}] \\ (\text{by } (9))}} \left(\prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h \right) \\
&= \sum_{\sigma\in S_n} (-1)^\sigma [\sigma \text{ is a derangement}] \prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h \\
&= \sum_{\substack{\sigma\in S_n \text{ is a} \\ \text{derangement}}} (-1)^\sigma \underbrace{[\sigma \text{ is a derangement}]}_{\substack{=1 \\ (\text{since } \sigma \text{ is a derangement})}} \prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h \\
&\qquad + \sum_{\substack{\sigma\in S_n \text{ is not a} \\ \text{derangement}}} (-1)^\sigma \underbrace{[\sigma \text{ is a derangement}]}_{\substack{=0 \\ (\text{since } \sigma \text{ is not a derangement})}} \prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h \\
&\qquad (\text{since each } \sigma \in S_n \text{ is either a derangement or not}) \\
&= \sum_{\substack{\sigma\in S_n \text{ is a} \\ \text{derangement}}} (-1)^\sigma \prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h + \underbrace{\sum_{\substack{\sigma\in S_n \text{ is not a} \\ \text{derangement}}} (-1)^\sigma 0 \prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h}_{=0} \\
&= \sum_{\substack{\sigma\in S_n \text{ is a} \\ \text{derangement}}} (-1)^\sigma \prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h . \qquad (10)
\end{aligned}$$

Now, we aim to simplify the right hand side of this equality. Let $\sigma \in S_n$ be a derangement. Then,

$$\prod_{i\in[n]} \underbrace{\prod_{h\in[n]\setminus\{i,\sigma(i)\}}}_{=\prod_{\substack{h\in[n];\\ h\neq i \text{ and } h\neq\sigma(i)}}} b_h = \underbrace{\prod_{i\in[n]} \prod_{\substack{h\in[n];\\ h\neq i \text{ and } h\neq\sigma(i)}}}_{=\prod_{h\in[n]} \prod_{\substack{i\in[n];\\ h\neq i \text{ and } h\neq\sigma(i)}}} b_h = \prod_{h\in[n]} \prod_{\substack{i\in[n];\\ h\neq i \text{ and } h\neq\sigma(i)}} b_h. \qquad (11)$$

Now, let $h \in [n]$. Then, $\sigma(h) \neq h$ (since σ is a derangement) and thus $\sigma^{-1}(h) \neq h$. Hence, $\left|\{\sigma^{-1}(h), h\}\right| = 2$. Thus,

$$\left|[n] \setminus \left\{\sigma^{-1}(h), h\right\}\right| = \underbrace{|[n]|}_{=n} - \underbrace{\left|\left\{\sigma^{-1}(h), h\right\}\right|}_{=2} \qquad \left(\text{since } \left\{\sigma^{-1}(h), h\right\} \subseteq [n]\right)$$
$$= n - 2. \qquad (12)$$

Furthermore, for any $i \in [n]$, we have the following chain of logical equivalences:

$$(h \neq i \text{ and } h \neq \sigma(i))$$
$$\Longleftrightarrow \left(i \neq h \text{ and } \underbrace{\sigma(i) \neq h}_{\Longleftrightarrow (i\neq\sigma^{-1}(h))}\right) \Longleftrightarrow \left(i \neq h \text{ and } i \neq \sigma^{-1}(h)\right) \Longleftrightarrow \left(i \notin \underbrace{\left\{h, \sigma^{-1}(h)\right\}}_{=\{\sigma^{-1}(h),h\}}\right)$$
$$\Longleftrightarrow \left(i \notin \left\{\sigma^{-1}(h), h\right\}\right).$$

Hence, we have the following equality between product signs:

$$\prod_{\substack{i\in[n];\\ h\neq i \text{ and } h\neq\sigma(i)}} = \prod_{\substack{i\in[n];\\ i\notin\left\{\sigma^{-1}(h),h\right\}}} = \prod_{i\in[n]\setminus\{\sigma^{-1}(h),h\}}.$$

Therefore,

$$\prod_{\substack{i\in[n];\\ h\neq i \text{ and } h\neq\sigma(i)}} b_h = \prod_{i\in[n]\setminus\{\sigma^{-1}(h),h\}} b_h = b_h^{\left|[n]\setminus\left\{\sigma^{-1}(h),h\right\}\right|} = b_h^{n-2} \qquad (\text{by } (12)). \qquad (13)$$

Now, forget that we fixed h. We thus have proven that (13) holds for each $h \in [n]$. Thus, (11) becomes

$$\prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h = \prod_{h\in[n]} \underbrace{\prod_{\substack{i\in[n];\\ h\neq i \text{ and } h\neq\sigma(i)}} b_h}_{\substack{=b_h^{n-2}\\ (\text{by } (13))}} = \prod_{h\in[n]} b_h^{n-2}. \qquad (14)$$

Now, forget that we fixed σ. We thus have proven the equality (14) for each derangement $\sigma \in S_n$. Now, (10) becomes

$$\det B = \sum_{\substack{\sigma \in S_n \text{ is a} \\ \text{derangement}}} (-1)^\sigma \underbrace{\prod_{i\in[n]} \prod_{h\in[n]\setminus\{i,\sigma(i)\}} b_h}_{\substack{=\prod_{h\in[n]} b_h^{n-2} \\ \text{(by (14))}}} = \sum_{\substack{\sigma \in S_n \text{ is a} \\ \text{derangement}}} (-1)^\sigma \prod_{h\in[n]} b_h^{n-2}$$

$$= \underbrace{\left(\sum_{\substack{\sigma \in S_n \text{ is a} \\ \text{derangement}}} (-1)^\sigma \right)}_{\substack{=(-1)^{n-1}(n-1) \\ \text{(by part (b) of the exercise)}}} \prod_{h\in[n]} b_h^{n-2} = (-1)^{n-1} (n-1) \prod_{h\in[n]} b_h^{n-2}.$$

This solves part **(c)** of the exercise.

[The above solution of part **(c)** may have made you somewhat dizzy, as it seems to involve negative powers of the b_h's when $n = 1$. But it's all right: If $n = 1$, then there are no derangements $\sigma \in S_n$, and thus the equalities (13) and (14) are vacuously true (since they both depend on a derangement $\sigma \in S_n$), whereas the expression $(-1)^{n-1} (n-1) \prod_{h\in[n]} b_h^{n-2}$ should be understood as 0 due to the vanishing $n-1$ factor.]

2.3 Remark

(a) There are various other ways to solve part **(a)** of the exercise, particularly if you know some linear algebra. For example, it is easy to find the eigenvalues of A: Namely, A has the eigenvalue $n-1$ with multiplicity 1 (with eigenvector $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$), and the eigenvalue -1 with multiplicity $n-1$ (with eigenspace consisting of all vectors $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{Q}^n$ satisfying $v_1 + v_2 + \cdots + v_n = 0$). Here, "multiplicity" can mean either algebraic multiplicity or geometric multiplicity (our matrix A is sufficiently nice that these are the same). These two eigenvalues are distinct and their multiplicities add up to $1 + (n-1) = n$, so that we know that we have found **all** the eigenvalues of A, and therefore we conclude that $\det A = (n-1) \cdot \underbrace{(-1) \cdot (-1) \cdot \cdots \cdot (-1)}_{n-1 \text{ times}}$ (since the determinant of a matrix is the product of its eigenvalues). This yields a new solution to part **(a)**.

(c) A different way to solve part **(c)** proceeds as follows: Assume first that all n numbers $b_1, b_2, \ldots, b_n$ are nonzero. For each $i \in [n]$, multiply the i-th row of the matrix B by b_i. Next, for each $j \in [n]$, multiply the j-th column of the resulting matrix by b_j. Denote the resulting $n \times n$-matrix by C. Then, it is easy to see that

$$C = \left([i \neq j] \prod_{h\in[n]} b_h \right)_{i,j\in[n]} = \left(\prod_{h\in[n]} b_h \right) \cdot A.$$

Hence,

$$\det C = \det\left(\left(\prod_{h\in[n]} b_h\right)\cdot A\right) = \left(\prod_{h\in[n]} b_h\right)^n \cdot \underbrace{\det A}_{\substack{=(-1)^{n-1}(n-1) \\ \text{(by part (a) of the exercise)}}}$$
$$= \left(\prod_{h\in[n]} b_h\right)^n \cdot (-1)^{n-1}(n-1) = (-1)^{n-1}(n-1)\cdot\left(\prod_{h\in[n]} b_h\right)^n .$$

But if we recall how C was constructed from B, we easily observe that

$$\det C = \underbrace{\left(\prod_{j\in[n]} b_j\right)}_{=\prod_{h\in[n]} b_h}\cdot\underbrace{\left(\prod_{i\in[n]} b_i\right)}_{=\prod_{h\in[n]} b_h}\cdot\det B$$
$$= \left(\prod_{h\in[n]} b_h\right)\cdot\left(\prod_{h\in[n]} b_h\right)\cdot\det B = \left(\prod_{h\in[n]} b_h\right)^2\cdot\det B.$$

Comparing these two equalities, we find

$$\left(\prod_{h\in[n]} b_h\right)^2\cdot\det B = (-1)^{n-1}(n-1)\cdot\left(\prod_{h\in[n]} b_h\right)^n .$$

Dividing this equality by $\left(\prod_{h\in[n]} b_h\right)^2$ (this is allowed, since all n numbers $b_1, b_2, \ldots, b_n$ are nonzero), we obtain

$$\det B = (-1)^{n-1}(n-1)\cdot\left(\prod_{h\in[n]} b_h\right)^{n-2} = (-1)^{n-1}(n-1)\cdot\prod_{h\in[n]} b_h^{n-2}.$$

Thus, part **(c)** of the exercise is solved under the assumption that all n numbers $b_1, b_2, \ldots, b_n$ are nonzero. How do we get rid of this assumption? Using the polynomial identity trick. Indeed, we have to apply the polynomial identity trick n times (or, more formally, use induction), each time arguing that our claim is an equality between two polynomials in the variable b_i (while all other $n-1$ variables $b_1, b_2, \ldots, b_{i-1}, b_{i+1}, b_{i+2}, \ldots, b_n$ are kept constant) that holds for infinitely many values of b_i (indeed, for all nonzero values) and therefore must also hold for **all** values of b_i (including 0). We leave the exact details to the reader. Lots of identities can be proven this way.

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3 Exercise 3

3.1 Problem

Let n be a positive integer. If $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ and $k \in [n]$, then

- we say that k is a *1-position* of $\mathbf{i}$ if $i_k = 1$;
- we say that k is a *10-position* of $\mathbf{i}$ if $k < n$, $i_k = 1$ and $i_{k+1} = 0$;
- we say that k is a *cyclic 10-position* of $\mathbf{i}$ if $i_k = 1$ and $i_{k+1} = 0$, where i_{n+1} is understood to be i_1.

(The first two of these concepts have already been defined in Homework set #2 Exercise 5. The concept of a "cyclic 10-position" differs from that of a "10-position" only in that we consider the n-tuple to "wrap around".)

Let $k \in \mathbb{N}$ and $a \in \{0, 1, \ldots, n-1\}$. Prove the following:

(a) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly a 1-positions and exactly k 10-positions is $\dbinom{a}{k}\dbinom{n-a}{k}$.

(b) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly a 1-positions and exactly k cyclic 10-positions is $\dfrac{n}{n-a}\dbinom{a-1}{a-k}\dbinom{n-a}{k}$ (this expression should be interpreted as $[k=0]$ when $a = n$).

(c) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and having exactly a 1-positions and exactly k cyclic 10-positions is $\dbinom{a-1}{a-k}\dbinom{n-a}{k}$.

3.2 Remark

1. You can rewrite the "$\dbinom{a}{k}$" in part **(a)** as "$\dbinom{a}{a-k}$" in order to make the similarity to the other two parts more glaring. Likewise, you could rewrite the "$\dbinom{a-1}{a-k}$" in parts **(b)** and **(c)** as $\dbinom{a-1}{k-1}$ when $a > 0$, but not in the border case when $a = 0$.

2. Sanity check: By summing over all a, we conclude from part **(a)** that the number of n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly k 10-positions is

$$\sum_{a=0}^{n} \binom{a}{k}\binom{n-a}{k} = \binom{n+1}{2k+1}$$

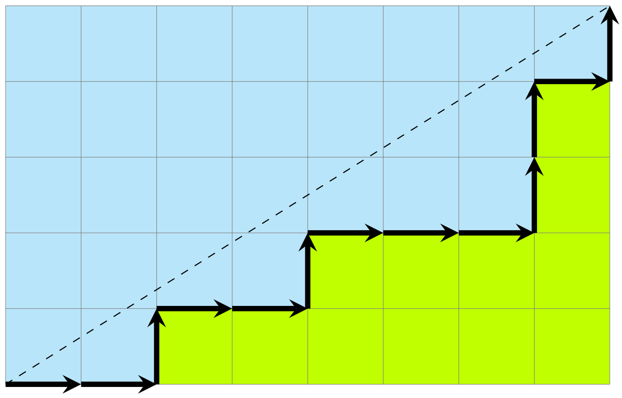
(by a simple application of Proposition 2.21 in the class from 2018-09-26). This is exactly the result of Homework set #2 Exercise 5.

3. Part **(b)** has an application to counting (a,b)-legal paths in the sense of §6.4 from class.

Indeed, let a and b be two coprime positive integers. We say that an LP $\mathbf{v}$ is (a,b)*-legal* if each $(x,y) \in \mathbf{v}$ satisfies $ax \geq by$. Proposition 6.7 in the class from 2018-11-14 shows that the number of (a,b)-legal paths from $(0,0)$ to (b,a) is $\dfrac{1}{a+b}\dbinom{a+b}{a}$. (This is a so-called *rational Catalan number*[4].)

[4] The word "rational" refers to the fact that the line $ax = by$ has a rational (not integer in general) slope; the rational Catalan number is still an integer.

Now, let us define a *left turn* of an LP $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ to be an $i \in [n-1]$ such that the i-th step of $\mathbf{v}$ is a right-step (i.e., we have $v_i - v_{i-1} = (1,0)$) but the $(i+1)$-st step of $\mathbf{v}$ is an up-step (i.e., we have $v_{i+1} - v_i = (0,1)$). For instance, the LP from $(0,0)$ to $(8,5)$ depicted in



is $(5,8)$-legal and has the left turns 2, 5, 9 and 12.

Now, given $k \in \mathbb{N}$, we claim that the number of (a,b)-legal LPs from $(0,0)$ to (b,a) having exactly k left turns is $\dfrac{1}{b}\dbinom{a-1}{a-k}\dbinom{b}{k}$ (this is a so-called *rational Narayana number*). Indeed, set $n = a+b$; then, every LP from $(0,0)$ to (b,a) can be encoded as an n-tuple $\mathbf{i} \in \{0,1\}^n$ with a 1-positions (by encoding each right-step as a 0 and each up-step as a 1). If a given LP is encoded by an n-tuple $(i_1, i_2, \ldots, i_n)$, then its shift (defined as in the proof of Proposition 6.7 in the class from 2018-11-14) is encoded by the n-tuple $(i_2, i_3, \ldots, i_n, i_1)$. As we know, each cycle of S (again, see the proof of Proposition 6.7 in the class from 2018-11-14 for the definition of S) has size $a+b$ and contains exactly one (a,b)-legal LP; this (a,b)-legal LP clearly starts with a right-step and ends with an up-step; hence it is easy to see that its left turns are in bijection with the cyclic 10-positions of the corresponding n-tuple[5]. It is now easy to conclude from part **(b)** that the number of (a,b)-legal LPs from $(0,0)$ to (b,a) having exactly k left turns is $\dfrac{1}{n-a}\dbinom{a-1}{a-k}\dbinom{n-a}{k} = \dfrac{1}{b}\dbinom{a-1}{a-k}\dbinom{b}{k}$.

4. It may be easiest to solve the problem starting with part **(c)**.

5. I have smuggled a correction into the problem: The assumption "$a \in \mathbb{N}$" has been replaced by "$a \in \{0, 1, \ldots, n-1\}$". Indeed, the problem would be false otherwise: For example, when $a > n$, there are no n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly a 1-positions at all, but the number $\dbinom{a}{k}\dbinom{n-a}{k}$ may be nonzero (so that part **(a)** of the problem would fail in this case).

3.3 Solution sketch

Forget that we fixed n, k and a.

Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ be the set of all positive integers.

Let us recall the following easy counting result (which was Theorem 2.34 in class notes from 2018-10-03):

[5]Indeed, if we define "cyclic 01-positions" in the obvious way, then the left turns of our (a,b)-legal LP are exactly the cyclic 01-positions of the corresponding n-tuple. But the latter are in bijection with the cyclic 10-positions, because the cyclic 01-positions and the cyclic 10-positions alternate.

Proposition 3.1. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,*

$$\left(\text{the number of all } (x_1, x_2, \ldots, x_k) \in \mathbb{P}^k \text{ satisfying } x_1 + x_2 + \cdots + x_k = n\right) = \begin{cases} \dbinom{n-1}{k-1}, & \text{if } n \geq 1; \\ [k=0], & \text{if } n = 0. \end{cases}$$

Let us bring this proposition into a more convenient (and slightly more general) form:

Corollary 3.2. *Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then,*

$$\left|\left\{(x_1, x_2, \ldots, x_k) \in \mathbb{P}^k \mid x_1 + x_2 + \cdots + x_k = n\right\}\right| = \binom{n-1}{n-k}.$$

Proof of Corollary 3.2 (sketched). If $n < 0$, then both sides of the claimed equality are 0 (indeed, the left hand side is 0 because there exists no $(x_1, x_2, \ldots, x_k) \in \mathbb{P}^k$ satisfying $x_1 + x_2 + \cdots + x_k = n$, whereas the right hand side is 0 because $n - \underbrace{k}_{\geq 0} \leq n < 0$), and therefore the equality holds. Hence, Corollary 3.2 is true when $n < 0$. Thus, for the rest of this proof, we WLOG assume that $n \geq 0$. Hence, $n \in \mathbb{N}$.

If $n \geq 1$, then the symmetry of the binomial coefficients yields

$$\binom{n-1}{n-k} = \binom{n-1}{(n-1)-(n-k)} = \binom{n-1}{k-1} \tag{15}$$

(since $(n-1)-(n-k) = k-1$). On the other hand, if $n = 0$, then

$$\begin{aligned} \binom{n-1}{n-k} &= \binom{0-1}{0-k} = \binom{-1}{-k} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k > 0 \end{cases} \\ &\qquad \left(\begin{array}{c} \text{this is easy to see by distinguishing cases:} \\ \text{if } k = 0, \text{ then } \dbinom{-1}{-k} = \dbinom{-1}{-0} = 1, \\ \text{whereas otherwise } \dbinom{-1}{-k} = 0 \text{ due to } -k < 0 \end{array}\right) \\ &= \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0 \end{cases} \qquad \left(\begin{array}{c} \text{since } k > 0 \text{ holds if and only if } k \neq 0 \\ \text{(because } k \in \mathbb{N}) \end{array}\right) \\ &= [k = 0]. \end{aligned}$$

Combining this equality with (15), we obtain

$$\binom{n-1}{n-k} = \begin{cases} \dbinom{n-1}{k-1}, & \text{if } n \geq 1; \\ [k=0], & \text{if } n = 0. \end{cases} \tag{16}$$

Now,

$$\begin{aligned} &\left|\left\{(x_1, x_2, \ldots, x_k) \in \mathbb{P}^k \mid x_1 + x_2 + \cdots + x_k = n\right\}\right| \\ &= \left(\text{the number of all } (x_1, x_2, \ldots, x_k) \in \mathbb{P}^k \text{ satisfying } x_1 + x_2 + \cdots + x_k = n\right) \\ &= \begin{cases} \dbinom{n-1}{k-1}, & \text{if } n \geq 1; \\ [k=0], & \text{if } n = 0 \end{cases} \qquad \text{(by Proposition 3.1)} \\ &= \binom{n-1}{n-k} \qquad \text{(by (16))}. \end{aligned}$$

This proves Corollary 3.2. □

In the following, we shall use the following notation: If $g \in \{0,1\}$ and $m \in \mathbb{N}$, then g^{*m} shall denote $\underbrace{g, g, \ldots, g}_{m \text{ times}}$. Thus, for example, $(1^{*4}) = (1,1,1,1)$ and $(1^{*0}) = ()$ and

$$\left(0^{*3}, 1^{*2}, 0^{*1}, 1^{*3}\right) = (0,0,0,1,1,0,1,1,1).$$

(Professional combinatorialists often write g^m instead of g^{*m}, but we prefer to avoid this out of an abundance of caution.)

Let n be a positive integer. The following claim is clear[6]:

Claim 1: Let $\mathbf{i} \in \{0,1\}^n$ be an n-tuple.

(a) If $\mathbf{i}$ starts with a 0 and ends with a 0, then $\mathbf{i}$ can be **uniquely** represented as

$$\mathbf{i} = \left(0^{*p_1}, 1^{*q_1}, 0^{*p_2}, 1^{*q_2}, \ldots, 0^{*p_h}, 1^{*q_h}, 0^{*p_{h+1}}\right)$$

for some $h \in \mathbb{N}$ and some positive integers $p_1, p_2, \ldots, p_{h+1}$ and $q_1, q_2, \ldots, q_h$ satisfying $(p_1 + p_2 + \cdots + p_{h+1}) + (q_1 + q_2 + \cdots + q_h) = n$. Moreover, the h in this representation is the number of cyclic 10-positions of $\mathbf{i}$, and also is the number of 10-positions of $\mathbf{i}$. Furthermore, $q_1 + q_2 + \cdots + q_h$ is the number of 1-positions of $\mathbf{i}$.

(b) If $\mathbf{i}$ starts with a 0 and ends with a 1, then $\mathbf{i}$ can be **uniquely** represented as

$$\mathbf{i} = \left(0^{*p_1}, 1^{*q_1}, 0^{*p_2}, 1^{*q_2}, \ldots, 0^{*p_h}, 1^{*q_h}\right)$$

for some $h \in \mathbb{N}$ and some positive integers $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_h$ satisfying $(p_1 + p_2 + \cdots + p_h) + (q_1 + q_2 + \cdots + q_h) = n$. Moreover, the h in this representation is the number of cyclic 10-positions of $\mathbf{i}$, whereas $h - 1$ is the number of 10-positions of $\mathbf{i}$. Furthermore, $q_1 + q_2 + \cdots + q_h$ is the number of 1-positions of $\mathbf{i}$.

(c) If $\mathbf{i}$ starts with a 1 and ends with a 0, then $\mathbf{i}$ can be **uniquely** represented as

$$\mathbf{i} = \left(1^{*q_1}, 0^{*p_1}, 1^{*q_2}, 0^{*p_2}, \ldots, 1^{*q_h}, 0^{*p_h}\right)$$

for some $h \in \mathbb{N}$ and some positive integers $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_h$ satisfying $(p_1 + p_2 + \cdots + p_h) + (q_1 + q_2 + \cdots + q_h) = n$. Moreover, the h in this representation is the number of cyclic 10-positions of $\mathbf{i}$, and also is the number of 10-positions of $\mathbf{i}$. Furthermore, $q_1 + q_2 + \cdots + q_h$ is the number of 1-positions of $\mathbf{i}$.

(d) If $\mathbf{i}$ starts with a 1 and ends with a 1, then $\mathbf{i}$ can be **uniquely** represented as

$$\mathbf{i} = \left(1^{*q_1}, 0^{*p_1}, 1^{*q_2}, 0^{*p_2}, \ldots, 1^{*q_h}, 0^{*p_h}, 1^{*q_{h+1}}\right)$$

for some $h \in \mathbb{N}$ and some positive integers $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_{h+1}$ satisfying $(p_1 + p_2 + \cdots + p_h) + (q_1 + q_2 + \cdots + q_{h+1}) = n$. Moreover, the h in this representation is the number of cyclic 10-positions of $\mathbf{i}$, and also is the number of 10-positions of $\mathbf{i}$. Furthermore, $q_1 + q_2 + \cdots + q_{h+1}$ is the number of 1-positions of $\mathbf{i}$.

[6] The idea behind it is known as "run-length encoding".

[*Proof of Claim 1 (sketched):* Intuitively this should be obvious – just record the lengths of the runs of $\mathbf{i}$ (where the *runs* of $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ are defined to be the inclusion-maximal[7] intervals $[u, v] \subseteq [n]$ such that $i_u = i_{u+1} = \cdots = i_v$). If you want to make this rigorous: The existence of the representations in all four parts of Claim 1 can be proven by induction on n, where in the induction step you remove the last entry from your n-tuple to obtain an $(n-1)$-tuple (but you have to prove parts **(a)** and **(b)** together, since you don't know what your $(n-1)$-tuple will end with after you remove the last entry; likewise, you have to prove parts **(c)** and **(d)** together). The uniqueness of these representations is easy to see (e.g., in part **(a)**, you just need to observe that $p_1, q_1, p_2, q_2, \ldots, p_h, q_h, p_{h+1}$ are the lengths of the runs of $\mathbf{i}$ from first to last). The claims about 10-positions, cyclic 10-positions and 1-positions are obvious.]

We can use Claim 1 to count the n-tuples with given first and last entries and given numbers of 1-positions and cyclic 10-positions:

Claim 2: Let $a \in \mathbb{N}$ and $k \in \mathbb{N}$.

(a) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 0 and having exactly a 1-positions and exactly k cyclic 10-positions is $\dbinom{n-a-1}{n-a-k-1}\dbinom{a-1}{a-k}$.

(b) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 1 and having exactly a 1-positions and exactly k cyclic 10-positions is $\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k}$.

(c) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 1 and ending with a 0 and having exactly a 1-positions and exactly k cyclic 10-positions is $\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k}$.

(d) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 1 and ending with a 1 and having exactly a 1-positions and exactly k cyclic 10-positions is $\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k-1}$.

[*Proof of Claim 2:* Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ be the set of all positive integers.
(a) Claim 1 **(a)** shows that

- every n-tuple $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 0 can be **uniquely** represented as
$$\mathbf{i} = \left(0^{*p_1}, 1^{*q_1}, 0^{*p_2}, 1^{*q_2}, \ldots, 0^{*p_h}, 1^{*q_h}, 0^{*p_{h+1}}\right)$$
for some $h \in \mathbb{N}$ and some positive integers $p_1, p_2, \ldots, p_{h+1}$ and $q_1, q_2, \ldots, q_h$ satisfying $(p_1 + p_2 + \cdots + p_{h+1}) + (q_1 + q_2 + \cdots + q_h) = n$;

- furthermore, the h in this representation is the number of cyclic 10-positions of $\mathbf{i}$, whereas $q_1 + q_2 + \cdots + q_h$ is the number of 1-positions of $\mathbf{i}$.

Consequently, every n-tuple $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 0 and having exactly a 1-positions and exactly k cyclic 10-positions can be **uniquely** represented as

$$\mathbf{i} = \left(0^{*p_1}, 1^{*q_1}, 0^{*p_2}, 1^{*q_2}, \ldots, 0^{*p_k}, 1^{*q_k}, 0^{*p_{k+1}}\right)$$

for some positive integers $p_1, p_2, \ldots, p_{k+1}$ and $q_1, q_2, \ldots, q_k$ satisfying $(p_1 + p_2 + \cdots + p_{k+1}) + (q_1 + q_2 + \cdots + q_k) = n$ and $q_1 + q_2 + \cdots + q_k = a$. Note that the condition "$(p_1 + p_2 + \cdots + p_{k+1}) + (q_1 + q_2 + \cdots + q_k) = n$ and $q_1 + q_2 + \cdots + q_k = a$" can be equivalently rewritten as

[7] "Inclusion-maximal" means "maximal with respect to inclusion" (i.e., not being subsets of larger such intervals).

"$p_1 + p_2 + \cdots + p_{k+1} = n - a$ and $q_1 + q_2 + \cdots + q_k = a$". Thus, every n-tuple $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 0 and having exactly a 1-positions and exactly k cyclic 10-positions can be **uniquely** represented as

$$\mathbf{i} = \left(0^{*p_1}, 1^{*q_1}, 0^{*p_2}, 1^{*q_2}, \ldots, 0^{*p_k}, 1^{*q_k}, 0^{*p_{k+1}}\right)$$

for some $(p_1, p_2, \ldots, p_{k+1}) \in \mathbb{P}^{k+1}$ satisfying $p_1+p_2+\cdots+p_{k+1} = n-a$ and some $(q_1, q_2, \ldots, q_k) \in \mathbb{P}^k$ satisfying $q_1 + q_2 + \cdots + q_k = a$.

Thus, there is a bijection

$$\begin{aligned}\Phi : &\left\{(p_1, p_2, \ldots, p_{k+1}) \in \mathbb{P}^{k+1} \ \mid \ p_1 + p_2 + \cdots + p_{k+1} = n - a\right\} \\ &\times \left\{(q_1, q_2, \ldots, q_k) \in \mathbb{P}^k \ \mid \ q_1 + q_2 + \cdots + q_k = a\right\} \\ &\to \{\mathbf{i} \in \{0,1\}^n \text{ starting with a 0 and ending with a 0 and having} \\ &\qquad \text{exactly } a \text{ 1-positions and exactly } k \text{ cyclic 10-positions}\}\end{aligned}$$

sending each $\left((p_1, p_2, \ldots, p_{k+1}), (q_1, q_2, \ldots, q_k)\right)$ to $\left(0^{*p_1}, 1^{*q_1}, 0^{*p_2}, 1^{*q_2}, \ldots, 0^{*p_k}, 1^{*q_k}, 0^{*p_{k+1}}\right)$. The existence of this bijection entails that

$$\begin{aligned}&|\{\mathbf{i} \in \{0,1\}^n \text{ starting with a 0 and ending with a 0 and having} \\ &\qquad \text{exactly } a \text{ 1-positions and exactly } k \text{ cyclic 10-positions}\}| \\ &= \left|\left\{(p_1, p_2, \ldots, p_{k+1}) \in \mathbb{P}^{k+1} \ \mid \ p_1 + p_2 + \cdots + p_{k+1} = n - a\right\}\right. \\ &\qquad \left.\times \left\{(q_1, q_2, \ldots, q_k) \in \mathbb{P}^k \ \mid \ q_1 + q_2 + \cdots + q_k = a\right\}\right| \\ &= \underbrace{\left|\left\{(p_1, p_2, \ldots, p_{k+1}) \in \mathbb{P}^{k+1} \ \mid \ p_1 + p_2 + \cdots + p_{k+1} = n - a\right\}\right|}_{\substack{=\dbinom{n-a-1}{n-a-(k+1)} \\ \text{(by Corollary 3.2, applied to } n-a \text{ and } k+1 \text{ instead of } n \text{ and } k)}} \\ &\qquad \cdot \underbrace{\left|\left\{(q_1, q_2, \ldots, q_k) \in \mathbb{P}^k \ \mid \ q_1 + q_2 + \cdots + q_k = a\right\}\right|}_{\substack{=\dbinom{a-1}{a-k} \\ \text{(by Corollary 3.2, applied to } a \text{ and } k \text{ instead of } n \text{ and } k)}} \\ &= \binom{n-a-1}{n-a-(k+1)} \binom{a-1}{a-k} = \binom{n-a-1}{n-a-k-1} \binom{a-1}{a-k}.\end{aligned}$$

This proves Claim 2 **(a)**.

(b) The proof of Claim 2 **(b)** is analogous to the proof of Claim 2 **(a)**, with the only difference that we now need to apply Claim 1 **(b)** instead of Claim 1 **(a)**.

(c) The proof of Claim 2 **(c)** is analogous to the proof of Claim 2 **(a)**, with the only difference that we now need to apply Claim 1 **(c)** instead of Claim 1 **(a)**.

(d) The proof of Claim 2 **(d)** is analogous to the proof of Claim 2 **(a)**, with the only difference that we now need to apply Claim 1 **(d)** instead of Claim 1 **(a)**.]

[*Remark:* The number in Claim 2 **(b)** equals the number in Claim 2 **(c)**. Thus, you should suspect that there is a bijection from the set

$$\begin{aligned}&\{\mathbf{i} \in \{0,1\}^n \text{ starting with a 0 and ending with a 1 and having} \\ &\qquad \text{exactly } a \text{ 1-positions and exactly } k \text{ cyclic 10-positions}\}\end{aligned}$$

to the set

$$\begin{aligned}&\{\mathbf{i} \in \{0,1\}^n \text{ starting with a 1 and ending with a 0 and having} \\ &\qquad \text{exactly } a \text{ 1-positions and exactly } k \text{ cyclic 10-positions}\}.\end{aligned}$$

And indeed, it is easy to find such a bijection: It sends each

$$\left(0^{*p_1}, 1^{*q_1}, 0^{*p_2}, 1^{*q_2}, \ldots, 0^{*p_k}, 1^{*q_k}\right) \qquad \text{to} \qquad \left(1^{*q_1}, 0^{*p_1}, 1^{*q_2}, 0^{*p_2}, \ldots, 1^{*q_k}, 0^{*p_k}\right)$$

(where $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k$ are supposed to be positive).]

Let us next state an analogue of Claim 2 in which cyclic 10-positions are replaced by (regular) 10-positions:

Claim 3: Let $a \in \mathbb{N}$ and $k \in \mathbb{N}$.

(a) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 0 and having exactly a 1-positions and exactly k 10-positions is $\dbinom{n-a-1}{n-a-k-1}\dbinom{a-1}{a-k}$.

(b) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 1 and having exactly a 1-positions and exactly k 10-positions is $\dbinom{n-a-1}{n-a-k-1}\dbinom{a-1}{a-k-1}$.

(c) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 1 and ending with a 0 and having exactly a 1-positions and exactly k 10-positions is $\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k}$.

(d) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 1 and ending with a 1 and having exactly a 1-positions and exactly k 10-positions is $\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k-1}$.

[*Proof of Claim 3:* **(a)** If an n-tuple $\mathbf{i} \in \{0,1\}^n$ starts with a 0 and ends with a 0, then its 10-positions are precisely its cyclic 10-positions[8]. Thus, the n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 0 and having exactly a 1-positions and exactly k 10-positions are **the same as** the n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 0 and having exactly a 1-positions and exactly k **cyclic** 10-positions. Hence, Claim 3 **(a)** is equivalent to from Claim 2 **(a)**, and therefore must hold (since Claim 2 **(a)** has already been proven).

(c) The proof of Claim 3 **(c)** is analogous to the proof of Claim 3 **(a)**.

(d) The proof of Claim 3 **(d)** is analogous to the proof of Claim 3 **(a)**, with the only difference that this time, the fact that n cannot be a cyclic 10-position of $\mathbf{i}$ follows from the assumption that $\mathbf{i}$ starts with 1 (rather than the assumption that $\mathbf{i}$ ends with 0).

(b) This is a bit trickier.

If an n-tuple $\mathbf{i} \in \{0,1\}^n$ starts with a 0 and ends with a 1, then n is a cyclic 10-position of $\mathbf{i}$ [9], but of course is not a 10-position of $\mathbf{i}$. Hence, if an n-tuple $\mathbf{i} \in \{0,1\}^n$ starts with a 0 and ends with a 1, then the number of its cyclic 10-positions is by 1 larger than the number of its 10-positions. Thus, such an n-tuple $\mathbf{i}$ has exactly k 10-positions **if and only if** it has exactly $k+1$ **cyclic** 10-positions. Thus, the n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 1 and having exactly a 1-positions and exactly k 10-positions are **the same as** the n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and ending with a 1 and having exactly a 1-positions and exactly $k+1$ **cyclic** 10-positions. Thus, Claim 2 **(b)** (applied to $k+1$ instead of k) shows that the number of such n-tuples is

$$\binom{n-a-1}{n-a-(k+1)}\binom{a-1}{a-(k+1)} = \binom{n-a-1}{n-a-k-1}\binom{a-1}{a-k-1}.$$

[8] In fact, the only difference between its 10-positions and its cyclic 10-positions is that n may be a cyclic 10-position without being a 10-position. But this can never happen for an n-tuple $\mathbf{i} \in \{0,1\}^n$ that ends with a 0 (because for n to be a cyclic 10-position, the n-tuple must end with a 1). Thus, there is no difference between the 10-positions and the cyclic 10-positions of $\mathbf{i}$.

[9] Indeed, if we write $\mathbf{i}$ as $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, then we have $i_n = 1$ (since $\mathbf{i}$ ends with a 1) and $i_{n+1} = i_1 = 0$ (since $\mathbf{i}$ starts with a 0); but these two equalities tell us precisely that n is a cyclic 10-position of $\mathbf{i}$.

This proves Claim 3 **(b)**.]

We can now finally solve the exercise:

Fix $k \in \mathbb{N}$ and $a \in \{0, 1, \ldots, n-1\}$. Hence, $a \geq 0$ and $a \leq n-1$, so that $n - a \geq 1 \geq 0$. Better yet, we have $n - a - 1 \geq 0$ (since $n - a \geq 1$).

(a) Recall that n is positive. Hence, any n-tuple $\mathbf{i} \in \{0, 1\}^n$ must start with either 0 or

1, and must end with either 0 or 1. Thus,

$$\begin{aligned}
&(\text{the number of } n\text{-tuples } \mathbf{i} \in \{0,1\}^n \text{ having exactly } a \text{ 1-positions and exactly } k \text{ 10-positions}) \\
&= \underbrace{(\text{the number of such } n\text{-tuples starting with a 0 and ending with a 0})}_{\substack{=\binom{n-a-1}{n-a-k-1}\binom{a-1}{a-k} \\ \text{(by Claim 3 } \mathbf{(a)}\text{)}}} \\
&\quad + \underbrace{(\text{the number of such } n\text{-tuples starting with a 0 and ending with a 1})}_{\substack{=\binom{n-a-1}{n-a-k-1}\binom{a-1}{a-k-1} \\ \text{(by Claim 3 } \mathbf{(b)}\text{)}}} \\
&\quad + \underbrace{(\text{the number of such } n\text{-tuples starting with a 1 and ending with a 0})}_{\substack{=\binom{n-a-1}{n-a-k}\binom{a-1}{a-k} \\ \text{(by Claim 3 } \mathbf{(c)}\text{)}}} \\
&\quad + \underbrace{(\text{the number of such } n\text{-tuples starting with a 1 and ending with a 1})}_{\substack{=\binom{n-a-1}{n-a-k}\binom{a-1}{a-k-1} \\ \text{(by Claim 3 } \mathbf{(d)}\text{)}}} \\
&= \binom{n-a-1}{n-a-k-1}\binom{a-1}{a-k} + \binom{n-a-1}{n-a-k-1}\binom{a-1}{a-k-1} \\
&\quad + \binom{n-a-1}{n-a-k}\binom{a-1}{a-k} + \binom{n-a-1}{n-a-k}\binom{a-1}{a-k-1} \\
&= \binom{n-a-1}{n-a-k-1} \underbrace{\left(\binom{a-1}{a-k} + \binom{a-1}{a-k-1}\right)}_{\substack{=\binom{a}{a-k} \\ \text{(by the recurrence relation of the binomial coefficients)}}} \\
&\quad + \binom{n-a-1}{n-a-k} \underbrace{\left(\binom{a-1}{a-k} + \binom{a-1}{a-k-1}\right)}_{\substack{=\binom{a}{a-k} \\ \text{(by the recurrence relation of the binomial coefficients)}}} \\
&= \binom{n-a-1}{n-a-k-1}\binom{a}{a-k} + \binom{n-a-1}{n-a-k}\binom{a}{a-k} \\
&= \underbrace{\left(\binom{n-a-1}{n-a-k} + \binom{n-a-1}{n-a-k-1}\right)}_{\substack{=\binom{n-a}{n-a-k} \\ \text{(by the recurrence relation of the binomial coefficients)}}} \binom{a}{a-k} \\
&= \binom{n-a}{n-a-k}\binom{a}{a-k}.
\end{aligned}$$

Comparing this with

$$\underbrace{\binom{a}{k}}_{\substack{=\binom{a}{a-k} \\ \text{(by the symmetry of the binomial coefficients,} \\ \text{since } a\geq 0)}} \underbrace{\binom{n-a}{k}}_{\substack{=\binom{n-a}{n-a-k} \\ \text{(by the symmetry of the binomial coefficients,} \\ \text{since } n-a\geq 0)}}$$
$$= \binom{a}{a-k}\binom{n-a}{n-a-k} = \binom{n-a}{n-a-k}\binom{a}{a-k},$$

we obtain

$$(\text{the number of } n\text{-tuples } \mathbf{i} \in \{0,1\}^n \text{ having exactly } a \text{ 1-positions and exactly } k \text{ 10-positions})$$
$$= \binom{a}{k}\binom{n-a}{k}.$$

This solves part **(a)** of the exercise.

(b) We have $a \in \{0, 1, \ldots, n-1\}$, thus $a \leq n-1 < n$, thus $a \neq n$. In other words, $n \neq a$.

Recall that n is positive. Hence, any n-tuple $\mathbf{i} \in \{0,1\}^n$ must start with either 0 or 1, and must end with either 0 or 1. Thus,

$$\begin{aligned}
&(\text{the number of } n\text{-tuples } \mathbf{i} \in \{0,1\}^n \text{ having exactly } a \text{ 1-positions} \\
&\qquad\qquad \text{and exactly } k \text{ cyclic 10-positions}) \\
&= \underbrace{(\text{the number of such } n\text{-tuples starting with a 0 and ending with a 0})}_{\substack{=\dbinom{n-a-1}{n-a-k-1}\dbinom{a-1}{a-k} \\ \text{(by Claim 2 (a))}}} \\
&\quad + \underbrace{(\text{the number of such } n\text{-tuples starting with a 0 and ending with a 1})}_{\substack{=\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k} \\ \text{(by Claim 2 (b))}}} \\
&\quad + \underbrace{(\text{the number of such } n\text{-tuples starting with a 1 and ending with a 0})}_{\substack{=\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k} \\ \text{(by Claim 2 (c))}}} \\
&\quad + \underbrace{(\text{the number of such } n\text{-tuples starting with a 1 and ending with a 1})}_{\substack{=\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k-1} \\ \text{(by Claim 2 (d))}}} \\
&= \binom{n-a-1}{n-a-k-1}\binom{a-1}{a-k} + \binom{n-a-1}{n-a-k}\binom{a-1}{a-k} \\
&\quad + \binom{n-a-1}{n-a-k}\binom{a-1}{a-k} + \binom{n-a-1}{n-a-k}\binom{a-1}{a-k-1} \\
&= \underbrace{\left(\binom{n-a-1}{n-a-k} + \binom{n-a-1}{n-a-k-1}\right)}_{\substack{=\dbinom{n-a}{n-a-k} \\ \text{(by the recurrence relation of the binomial coefficients)}}} \binom{a-1}{a-k} \\
&\quad + \binom{n-a-1}{n-a-k} \underbrace{\left(\binom{a-1}{a-k} + \binom{a-1}{a-k-1}\right)}_{\substack{=\dbinom{a}{a-k} \\ \text{(by the recurrence relation of the binomial coefficients)}}} \\
&= \underbrace{\binom{n-a}{n-a-k}}_{\substack{=\dbinom{n-a}{(n-a)-(n-a-k)} \\ \text{(by the symmetry of the binomial coefficients,} \\ \text{since } n-a\geq 0)}} \binom{a-1}{a-k} + \underbrace{\binom{n-a-1}{n-a-k}}_{\substack{=\dbinom{n-a-1}{(n-a-1)-(n-a-k)} \\ \text{(by the symmetry of the binomial coefficients,} \\ \text{since } n-a\geq 0)}} \binom{a}{a-k}
\end{aligned}$$

$$= \underbrace{\binom{n-a}{(n-a)-(n-a-k)}}_{=\binom{n-a}{k}} \binom{a-1}{a-k} + \underbrace{\binom{n-a-1}{(n-a-1)-(n-a-k)}}_{=\binom{n-a-1}{k-1} \underset{\text{(by (2))}}{=} \frac{k}{n-a}\binom{n-a}{k}} \binom{a}{a-k}$$

$$= \binom{n-a}{k}\binom{a-1}{a-k} + \frac{k}{n-a}\binom{n-a}{k}\binom{a}{a-k} = \binom{n-a}{k} \cdot \left(\binom{a-1}{a-k} + \frac{1}{n-a} \underbrace{k\binom{a}{a-k}}_{\underset{\text{(by (1))}}{=a\binom{a-1}{a-k}}} \right)$$

$$= \binom{n-a}{k} \cdot \left(\binom{a-1}{a-k} + \frac{1}{n-a} a \binom{a-1}{a-k} \right) = \binom{n-a}{k}\binom{a-1}{a-k} \underbrace{\left(1 + \frac{1}{n-a} a\right)}_{=\frac{n}{n-a}}$$

$$= \binom{n-a}{k}\binom{a-1}{a-k}\frac{n}{n-a} = \frac{n}{n-a}\binom{a-1}{a-k}\binom{n-a}{k}.$$

This solves part **(b)** of the exercise.

(c) Recall that n is positive. Hence, any n-tuple $\mathbf{i} \in \{0,1\}^n$ must end with either 0 or

1. Thus,

$$\begin{aligned}
&(\text{the number of } n\text{-tuples } \mathbf{i} \in \{0,1\}^n \text{ starting with a 0 and having} \\
&\qquad \text{exactly } a \text{ 1-positions and exactly } k \text{ cyclic 10-positions}) \\
&= \underbrace{(\text{the number of such } n\text{-tuples ending with a 0})}_{\substack{=\dbinom{n-a-1}{n-a-k-1}\dbinom{a-1}{a-k} \\ (\text{by Claim 2 } \textbf{(a)})}} \\
&\quad + \underbrace{(\text{the number of such } n\text{-tuples ending with a 1})}_{\substack{=\dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k} \\ (\text{by Claim 2 } \textbf{(b)})}} \\
&= \dbinom{n-a-1}{n-a-k-1}\dbinom{a-1}{a-k} + \dbinom{n-a-1}{n-a-k}\dbinom{a-1}{a-k} \\
&= \underbrace{\left(\dbinom{n-a-1}{n-a-k} + \dbinom{n-a-1}{n-a-k-1}\right)}_{\substack{=\dbinom{n-a}{n-a-k} \\ (\text{by the recurrence relation of the binomial coefficients})}} \dbinom{a-1}{a-k} \\
&= \underbrace{\dbinom{n-a}{n-a-k}}_{\substack{=\dbinom{n-a}{(n-a)-(n-a-k)} \\ (\text{by the symmetry of the binomial coefficients,} \\ \text{since } n-a\geq 0)}} \dbinom{a-1}{a-k} = \underbrace{\dbinom{n-a}{(n-a)-(n-a-k)}}_{=\dbinom{n-a}{k}} \dbinom{a-1}{a-k} \\
&= \dbinom{n-a}{k}\dbinom{a-1}{a-k} = \dbinom{a-1}{a-k}\dbinom{n-a}{k}.
\end{aligned}$$

This solves part **(c)** of the exercise.

3.4 Remark

Other solutions exist. Here is a sketch of one such solution to part **(a)**:

(a) Here is an algorithm that constructs each n-tuple $\mathbf{i} \in \{0,1\}^n$ having exactly a 1-positions and exactly k 10-positions:

- Begin with the a-tuple $(1, 1, \ldots, 1)$ (with a entries). This a-tuple already contains all the 1's that $\mathbf{i}$ will contain (since $\mathbf{i}$ should have exactly a 1-positions); thus, all we need to do is to insert $n-a$ many 0's into this a-tuple.
- Among the a many 1's that our a-tuple has, we choose the k many 1's that will be followed by 0's. (We know that we want exactly k many 1's to be followed by 0's, since $\mathbf{i}$ should have exactly k 10-positions.) This choice can be made in $\dbinom{a}{k}$ many ways (since we have to choose k among a many 1's). We color the k chosen 1's red.
- We insert a nonnegative amount of 0's at the beginning of our tuple. Furthermore, after each of the k red 1's, we insert a positive amount of 0's, taking care that we insert exactly $n-a$ many 0's in total (including the ones that have been inserted at

the beginning of our tuple). This can be done in $\binom{n-a}{k}$ many ways, because our choice boils down to choosing a $(k+1)$-tuple $(y, x_1, x_2, \ldots, x_k)$ of nonnegative integers satisfying ($x_i > 0$ for all $i \in [k]$) and $y + x_1 + x_2 + \cdots + x_k = n - a$ [10], and it is not hard to check that the number of such $(k+1)$-tuples is $\binom{n-a}{k}$ [11].

Thus, the total number of such n-tuples is $\binom{a}{k}\binom{n-a}{k}$. This solves part **(a)** of the problem. A similar argument can be made for part **(c)**. Part **(b)** can be derived from part **(c)** by rotating the n-tuple cyclically – do you see how this works?

---

4 Exercise 4

4.1 Problem

An *integer formal power series* (short *IFPS*) shall mean a formal power series whose coefficients all are integers. For example, $1 - 2x + 3x^2 - 4x^3 \pm \cdots$ is an IFPS, while $1 - \frac{1}{2}x$ is not.

If m is an integer, and if a and b are two IFPSs, then we say that $a \equiv b \mod m$ if and only if there exists an IFPS c such that $a - b = mc$. (This is completely analogous to the definition of congruence modulo m for integers.) The following facts hold:

(A1) Two IFPSs a and b and an integer m satisfy $a \equiv b \mod m$ if and only if each $n \in \mathbb{N}$ satisfies $[x^n]\, a \equiv [x^n]\, b \mod m$ (that is, each coefficient of a is congruent to the corresponding coefficient of b modulo m).

(A2) Each integer m and each IFPS a satisfy $a \equiv a \mod m$.

(A3) If m is an integer, and if a and b are two IFPSs satisfying $a \equiv b \mod m$, then $b \equiv a \mod m$.

(A4) If m is an integer, and if a, b and c are three IFPSs satisfying $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$.

(A5) If m is an integer, and if a, b, c and d are four IFPSs satisfying $a \equiv c \mod m$ and $b \equiv d \mod m$, then

$$a + b \equiv c + d \mod m, \qquad a - b \equiv c - d \mod m, \qquad \text{and } ab \equiv cd \mod m.$$

(A6) If m is an integer, and if a and b are two IFPSs with constant terms ± 1 (that is, $[x^0]\, a = \pm 1$ and $[x^0]\, b = \pm 1$) satisfying $a \equiv b \mod m$, then

$$a^{-1} \equiv b^{-1} \mod m.$$

[10] Indeed, in this $(k+1)$-tuple, the entry y is the number of 0's that we insert at the beginning of our tuple, whereas the entry x_i is the number of 0's that we insert after the i-th red 1 counted from the left.

[11] For example, you can achieve this by finding a bijection between these $(k+1)$-tuples and k-element subsets of $[n-a]$.

(A7) If m is an integer and n is a nonnegative integer, and if a and b are two IFPSs satisfying $a \equiv b \mod m$, then

$$a^n \equiv b^n \mod m.$$

Furthermore, if a and b have constant terms ± 1, then this also holds for negative n.

(You can use all these seven facts without proof, but if you are curious: Fact (A1) is essentially obvious; facts (A2)–(A5) are proven just as for integers. Fact (A6) follows by observing that $a^{-1} - b^{-1} = -a^{-1}b^{-1}(a-b)$, since the assumption on the constant terms forces a^{-1} and b^{-1} to be well-defined IFPSs. Finally, fact (A7) is proven by forwards induction for $n \geq 0$ and then by backwards induction for $n < 0$.)

Now, let p be a prime.

(a) Prove that $(1+x)^p \equiv 1 + x^p \mod p$.

(b) Prove *Lucas's congruence*: Any $a, b \in \mathbb{Z}$ and $c, d \in \{0, 1, \ldots, p-1\}$ satisfy

$$\binom{ap+c}{bp+d} \equiv \binom{a}{b}\binom{c}{d} \mod p. \qquad (17)$$

(c) Prove that if $m \in \mathbb{N}$, and if a and b are two IFPSs satisfying $a \equiv b \mod m$, then $a^m \equiv b^m \mod m^2$.

(d) Prove that any $a, b \in \mathbb{Z}$ satisfy

$$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^2.$$

[**Hint:** **(a)** What do you remember about $\binom{p}{k}$?

(b) $(1+x)^{ap+c} = ((1+x)^p)^a (1+x)^c$. It is helpful to define $[x^n] f = 0$ for any negative n and any FPS f.

(c) Write a as $b + mc$ for some IFPS c. The same congruence holds for integers.]

4.2 Remark

The claim of part **(a)** is a particular case of the following fundamental fact (sometimes known as "Freshman's Dream" or "Idiot's Binomial Formula"): If R is a commutative ring, and if u and v are any two elements of R, then $(u+v)^p \equiv u^p + v^p \bmod p$, where the relation "$a \equiv b \mod p$" (for two elements a and b of R) is defined exactly as you would expect (namely, it means that there exists some $c \in R$ satisfying $a - b = pc$). The solution I will give below for part **(a)** can be easily adapted to prove this fact.

Part **(b)** of the problem is the famous *Lucas's congruence*. See [Grinbe18c, proof of Theorem 1.11] for a more elementary proof of it as well as for a few references (though most sources only prove it in the case when a and b are nonnegative).

Part **(c)** of the problem is a particular case of the "exponent lifting principle". In a slightly more general form, it says that if R is a commutative ring, and if m, a and b are three elements of R satisfying $a \equiv b \bmod m$, then $a^m \equiv b^m \bmod m^2$. (Here, again, the congruence "$a \equiv b \mod m$" has to be understood as "there exists some $c \in R$ satisfying $a - b = mc$".)

Part **(d)** of the problem is another famous congruence, sometimes ascribed to Charles Babbage (though I don't know if he ever stated it). See [Grinbe18c, proof of Theorem 1.12] for an elementary proof of it and some references (which references, again, only prove it for a and b nonnegative).

4.3 Partial solution

We shall use the following fact (which was Theorem 1.24 in our classwork from 2018-09-10, and also appears in [Grinbe18d, Theorem 1.2]):

Theorem 4.1. *Let p be a prime number. Then, the binomial coefficient $\binom{p}{k}$ is divisible by p for every $k \in \{1, 2, \ldots, p-1\}$.*

Let us next make a few further general observations. Fact (A5) shows that we can add, subtract and multiply any two congruences between IFPSs modulo m (where m is an integer). Thus, we can also add, subtract and multiply any finite number of congruences between IFPSs modulo m. In other words, if $m \in \mathbb{Z}$ and $u \in \mathbb{N}$, and if $a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_u$ are any $2u$ IFPSs satisfying

$$a_k \equiv b_k \bmod m \qquad \text{for each } k \in [u],$$

then

$$\sum_{k=1}^{u} a_k \equiv \sum_{k=1}^{u} b_k \bmod m \qquad \text{and} \qquad \prod_{k=1}^{u} a_k \equiv \prod_{k=1}^{u} b_k \bmod m.$$

This can be proven by straightforward induction on u. We shall use this tacitly below.

Furthermore, facts (A2), (A3) and (A4) show that congruences between IFPSs modulo m can be "chained together": If we have an integer $m \in \mathbb{Z}$ and finitely many IFPSs $a_1, a_2, \ldots, a_u$ satisfying

$$a_i \equiv a_{i+1} \bmod m \qquad \text{for each } i \in [u-1],$$

then any two of the IFPSs $a_1, a_2, \ldots, a_u$ are congruent to each other modulo m (that is, we have $a_i \equiv a_j \bmod m$ for each $i, j \in [u]$). In such a situation, we write

$$a_1 \equiv a_2 \equiv \cdots \equiv a_u \bmod m,$$

and speak of a "chain of congruences"; so, just as for a chain of equalities, once we have shown that each two adjacent "links" of the chain are congruent, we can conclude that any two "links" of the chain are congruent. Again, this will be used tacitly in the following.

Next, let us recall *Newton's binomial theorem* (Theorem 8.8 in our classwork from 2018-11-28):

Theorem 4.2. *We have*

$$(1+x)^n = \sum_{k \in \mathbb{N}} \binom{n}{k} x^k \qquad \textit{for each } n \in \mathbb{Z}.$$

Proof of Theorem 4.2. I seem to have forgotten to prove this in class, so let me do it now. Recall that

$$(1+x)^{-n} = \sum_{k \in \mathbb{N}} (-1)^k \binom{n+k-1}{k} x^k \qquad \text{for each } n \in \mathbb{N}. \tag{18}$$

(Indeed, this is Theorem 8.9 in our classwork from 2018-11-28.)

Now, fix $n \in \mathbb{Z}$. We are in one of the following two cases:

Case 1: We have $n \geq 0$.

Case 2: We have $n < 0$.

Let us first consider Case 1. In this case, we have $n \geq 0$. Thus, $n \in \mathbb{N}$. Hence,

$$(1+x)^n = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \underbrace{1^{n-k}}_{=1} \qquad \text{(by the binomial theorem)}$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Comparing this with

$$\sum_{k\in\mathbb{N}} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k>n} \underbrace{\binom{n}{k}}_{\substack{=0 \\ (\text{since } n<k)}} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

we obtain $(1+x)^n = \sum_{k\in\mathbb{N}} \binom{n}{k} x^k$. Hence, Theorem 4.2 is proven in Case 1.

Let us next consider Case 2. In this case, we have $n < 0$. Thus, $-n > 0$, so that $-n \in \mathbb{N}$. Hence, (18) (applied to $-n$ instead of n) yields

$$(1+x)^{-(-n)} = \sum_{k\in\mathbb{N}} (-1)^k \binom{-n+k-1}{k} x^k. \tag{19}$$

But each $k \in \mathbb{N}$ satisfies

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k} \qquad \text{(by Proposition 0.2 (applied to } n \text{ and } k \text{ instead of } m \text{ and } n))$$
$$= (-1)^k \binom{-n+k-1}{k} \tag{20}$$

(since $k-n-1 = -n+k-1$). Now, $n = -(-n)$, so that

$$(1+x)^n = (1+x)^{-(-n)} = \sum_{k\in\mathbb{N}} \underbrace{(-1)^k \binom{-n+k-1}{k}}_{\substack{=\binom{n}{k} \\ (\text{by (20)})}} x^k \qquad \text{(by (19))}$$
$$= \sum_{k\in\mathbb{N}} \binom{n}{k} x^k.$$

Hence, Theorem 4.2 is proven in Case 2.

We have now proven Theorem 4.2 in both Cases 1 and 2. Thus, the proof of Theorem 4.2 is complete. □

Let us now solve parts **(a)** and **(c)** of the exercise.

(a) For each $k \in \{1, 2, \ldots, p-1\}$, we have $p \mid \binom{p}{k}$ (by Theorem 4.1) and thus $\binom{p}{k} \equiv 0 \mod p$ and thus

$$\binom{p}{k} x^k \equiv 0 \mod p$$

(indeed, fact (A5) above shows that we can multiply the congruences $\binom{p}{k} \equiv 0 \mod p$ and $x^k \equiv x^k \mod p$, thus obtaining $\binom{p}{k} x^k \equiv 0x^k = 0 \mod p$). If we sum up these congruences over all $k \in \{1, 2, \ldots, p-1\}$, then we obtain

$$\sum_{k=1}^{p-1} \binom{p}{k} x^k \equiv \sum_{k=1}^{p-1} 0 = 0 \mod p.$$

The binomial theorem yields

$$\begin{aligned}
(1+x)^p &= \sum_{k=0}^{p} \binom{p}{k} x^k = \underbrace{\binom{p}{0}}_{=1} \underbrace{x^0}_{=1} + \underbrace{\sum_{k=1}^{p-1} \binom{p}{k} x^k}_{\equiv 0 \mod p} + \underbrace{\binom{p}{p}}_{=1} x^p \\
&\qquad \left(\begin{array}{c} \text{here, we have split off the addends for } k=0 \text{ and for } k=p \\ \text{from the sum (since } 0 \neq p) \end{array} \right) \\
&\equiv 1 + 0 + x^p = 1 + x^p \mod p.
\end{aligned}$$

This solves part **(a)** of the exercise.

(c) Let $m \in \mathbb{N}$. Let a and b be two IFPSs satisfying $a \equiv b \mod m$. From $a \equiv b \mod m$, we conclude that there exists an IFPS c such that $a - b = mc$. Consider this c.

We must prove that $a^m \equiv b^m \mod m^2$. If $m = 0$, then this is obvious (because if $m = 0$, then the FPSs $a^m = a^0 = 1$ and $b^m = b^0 = 1$ are equal, and therefore congruent to each other modulo any integer). Thus, for the rest of this proof, we WLOG assume that $m \neq 0$. Hence, $m \geq 1$ (since $m \in \mathbb{N}$).

We have $a - b = mc$, thus $a = mc + b$. Hence,

$$\begin{aligned}
a^m &= (mc+b)^m = \sum_{k=0}^{m} \binom{m}{k} (mc)^k b^{m-k} \qquad \text{(by the binomial formula)} \\
&= \underbrace{\binom{m}{0}}_{=1} \underbrace{(mc)^0}_{=1} \underbrace{b^{m-0}}_{=b^m} + \underbrace{\binom{m}{1}}_{=m} \underbrace{(mc)^1}_{=mc} b^{m-1} + \sum_{k=2}^{m} \binom{m}{k} \underbrace{(mc)^k}_{=m^k c^k} b^{m-k} \\
&\qquad \left(\begin{array}{c} \text{here, we have split off the addends for } k=0 \text{ and } k=1 \\ \text{from the sum (these addends both exist, since } m \geq 1) \end{array} \right) \\
&= b^m + \underbrace{mmcb^{m-1}}_{=m^2 cb^{m-1} \equiv 0 \mod m^2} + \sum_{k=2}^{m} \binom{m}{k} \underbrace{m^k}_{\substack{=m^2 m^{k-2} \\ \equiv 0 \mod m^2 \\ \text{(since } k \geq 2 \text{ and} \\ \text{thus } m^{k-2} \in \mathbb{Z})}} c^k b^{m-k} \\
&\equiv b^m + 0 + \underbrace{\sum_{k=2}^{m} \binom{m}{k} 0 c^k b^{m-k}}_{=0} = b^m \mod m^2.
\end{aligned}$$

This solves part **(c)** of the exercise.

I have yet to write up solutions to parts **(b)** and **(d)**. Actually I am at a loss at how to solve part **(d)** using IFPSs; I had the impression such a solution was easy when I posed the problem, but this impression has revealed to be a fata morgana. Thus, for the time being, I refer to [Grinbe18c] for the solutions to both parts **(b)** and **(d)** (solutions that don't use IFPSs, however).

[...]

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5 Exercise 5

5.1 Problem

For each $n \in \mathbb{N}$, we define two FPSs P_n and S_n by

$$P_n = \prod_{s=1}^{n} (1 - x^s) \qquad \text{and} \qquad S_n = \sum_{s=-n}^{n} (-1)^s x^{s(3s+1)/2}.$$

(For example,

$$P_4 = (1-x)\left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right) = 1 - x - x^2 + 2x^5 - x^8 - x^9 + x^{10}$$

and

$$S_4 = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26}. \ )$$

(a) Show that S_n is well-defined, i.e., that all of the exponents $s(3s+1)/2$ are nonnegative integers (even when $s < 0$).

(b) Set

$$F_n = \sum_{s=0}^{n} (-1)^s \frac{P_n}{P_s} x^{sn+s(s+1)/2} \qquad \text{for each } n \in \mathbb{N}.$$

Show that $F_n = S_n$ for each $n \in \mathbb{N}$.

[**Hint:** **(b)** Induction on n. In the induction step, use $P_n = P_{n-1} - x^n P_{n-1}$ to split the sum defining F_n into two subsums after first splitting off the $s = n$ addend. This leads to $F_n - F_{n-1} = S_n - S_{n-1}$.]

5.2 Remark

If we take the limit $n \to \infty$ in the claim of part **(b)** (see [Loehr11, §7.5] for the meaning of "limit" here), then we quickly obtain

$$\prod_{s=1}^{\infty} (1 - x^s) = \sum_{s=-\infty}^{\infty} (-1)^s x^{s(3s+1)/2}.$$

(Indeed, each of the addends $(-1)^s \frac{P_n}{P_s} x^{sn+s(s+1)/2}$ for $s > 0$ tends to 0 when $n \to \infty$.) This is Euler's pentagonal number theorem again.

Part **(b)** of this exercise is the main result of [Shanks51].

5.3 Solution sketch

(a) We must prove that $s(3s+1)/2$ is a nonnegative integer for each $s \in \mathbb{Z}$.

So let $s \in \mathbb{Z}$. We must prove that $s(3s+1)/2$ is a nonnegative integer.

Proposition 0.3 (applied to s and 2 instead of m and n) yields $\dbinom{s}{2} \in \mathbb{Z}$. Thus, $\dfrac{s(s-1)}{2} = \dbinom{s}{2} \in \mathbb{Z}$. Now, a simple calculation shows that

$$s(3s+1)/2 = \underbrace{s^2}_{\in\mathbb{Z}} + \underbrace{s}_{\in\mathbb{Z}} + \underbrace{\frac{s(s-1)}{2}}_{\in\mathbb{Z}}.$$

Thus, $s(3s+1)/2$ is a sum of three integers, and hence is an integer.

Another simple calculation shows that $s(3s+1) = 3 \underbrace{\left(s+\frac{1}{6}\right)^2}_{\substack{\geq 0 \\ \text{(since squares of real numbers} \\ \text{are nonnegative)}}} - \frac{1}{12} \geq 3 \cdot 0 - \frac{1}{12} = -\frac{1}{12} > -1$. Since $s(3s+1)$ is an integer, this yields $s(3s+1) \geq -1+1 = 0$. Hence, $s(3s+1)/2 \geq 0$ as well. In other words, $s(3s+1)/2$ is nonnegative. Hence, we have shown that $s(3s+1)/2$ is a nonnegative integer. This solves part **(a)** of the problem.

[*Remark:* The solution to part **(a)** that I just gave is perhaps the shortest. There are various other approaches. For example, in order to prove that $s(3s+1)$ is nonnegative, one can show that the two numbers s and $3s+1$ have the same sign (i.e., either both are ≥ 0, or both are ≤ 0). Also, the integrality of $s(3s+1)/2$ can be proven by observing that at least one of the two integers s and $3s+1$ is even.]

(b) The following solution to part **(b)** is exactly the argument of Shanks [Shanks51] (just with more details).

Let us first observe that

$$P_n = P_{n-1} - x^n P_{n-1} \qquad \text{for each integer } n \geq 1. \tag{21}$$

[*Proof of* (21)*:* Let $n \geq 1$ be an integer. Thus, $n-1 \in \mathbb{N}$. The definition of P_{n-1} yields $P_{n-1} = \prod_{s=1}^{n-1} (1-x^s)$. Now, the definition of P_n yields

$$P_n = \prod_{s=1}^{n} (1-x^s) = (1-x^n) \cdot \underbrace{\prod_{s=1}^{n-1} (1-x^s)}_{=P_{n-1}} \qquad \left(\begin{array}{c}\text{here, we have split off the factor} \\ \text{for } s=n \text{ from the product, since } n \geq 1\end{array}\right)$$
$$= (1-x^n) \cdot P_{n-1} = P_{n-1} - x^n P_{n-1}.$$

This proves (21).]

For any $n \in \mathbb{N}$, the formal power series $P_n = \prod_{s=1}^{n} (1-x^s)$ has constant term 1 (indeed, it is a product of the n formal power series $1-x^s$ for $s \in \{1,2,\ldots,n\}$, each of which has constant term 1), and thus is invertible. In other words, all the formal power series $P_0, P_1, P_2, \ldots$ are invertible. Hence, quotients like the $\dfrac{P_n}{P_s}$ in the definition of F_n are well-defined.

We can now restate (21) as follows: If Q is any formal power series, then

$$\frac{Q}{P_{n-1}} = (1 - x^n) \cdot \frac{Q}{P_n} \qquad \text{for each integer } n \geq 1. \tag{22}$$

[*Proof of* (22)*:* Let Q be any formal power series. Let $n \geq 1$ be an integer. From (21), we obtain $P_n = P_{n-1} - x^n P_{n-1} = (1 - x^n) P_{n-1}$. Multiplying both sides of this equality by $\frac{Q}{P_n P_{n-1}}$, we obtain $\frac{Q}{P_{n-1}} = (1 - x^n) \cdot \frac{Q}{P_n}$. This proves (22).]

We must prove that

$$F_n = S_n \qquad \text{for each } n \in \mathbb{N}. \tag{23}$$

We shall prove this by induction on n:

Induction base: The definition of F_0 yields

$$F_0 = \sum_{s=0}^{0} (-1)^s \frac{P_0}{P_s} x^{s0+s(s+1)/2} = \underbrace{(-1)^0}_{=1} \underbrace{\frac{P_0}{P_0}}_{=1} \underbrace{x^{0\cdot 0+0(0+1)/2}}_{=x^0=1} = 1.$$

The definition of S_0 yields

$$S_0 = \sum_{s=-0}^{0} (-1)^s x^{s(3s+1)/2} = \underbrace{(-1)^0}_{=1} \underbrace{x^{0(3\cdot 0+1)/2}}_{=x^0=1} = 1.$$

Comparing these two equalities, we obtain $F_0 = S_0$. In other words, (23) holds for $n = 0$. This completes the induction base.

Induction step: Let m be a positive integer. Assume that (23) holds for $n = m - 1$. We must prove that (23) holds for $n = m$.

We have assumed that (23) holds for $n = m - 1$. In other words, we have $F_{m-1} = S_{m-1}$. Note that $m \geq 1$ (since m is a positive integer), thus $m - 1 \geq 0$. Also, $m \neq -m$ (since $m \geq 1 > 0$).

Applying (21) to $n = m$, we find $P_m = P_{m-1} - x^m P_{m-1}$.

The definition of F_{m-1} yields

$$\begin{aligned} F_{m-1} &= \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} \underbrace{x^{s(m-1)+s(s+1)/2}}_{\substack{=x^{sm+s(s-1)/2} \\ (\text{since } s(m-1)+s(s+1)/2 = sm+s(s-1)/2)}} \\ &= \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} x^{sm+s(s-1)/2}. \end{aligned} \tag{24}$$

The definition of F_m yields

$$\begin{aligned} F_m &= \sum_{s=0}^{m} (-1)^s \frac{P_m}{P_s} x^{sm+s(s+1)/2} \\ &= (-1)^m \underbrace{\frac{P_m}{P_m}}_{=1} \underbrace{x^{mm+m(m+1)/2}}_{\substack{=x^{m(3m+1)/2} \\ (\text{since } mm+m(m+1)/2 = m(3m+1)/2)}} + \sum_{s=0}^{m-1} (-1)^s \frac{P_m}{P_s} x^{sm+s(s+1)/2} \\ &\qquad (\text{here, we have split off the addend for } s = m \text{ from the sum}) \\ &= (-1)^m x^{m(3m+1)/2} + \sum_{s=0}^{m-1} (-1)^s \frac{P_m}{P_s} x^{sm+s(s+1)/2}. \end{aligned}$$

Subtracting $(-1)^m x^{m(3m+1)/2}$ from both sides of this equality, we find

$$\begin{aligned}
& F_m - (-1)^m x^{m(3m+1)/2} \\
&= \sum_{s=0}^{m-1} (-1)^s \underbrace{\frac{P_m}{P_s}}_{\substack{=\frac{P_{m-1} - x^m P_{m-1}}{P_s} \\ (\text{since } P_m = P_{m-1} - x^m P_{m-1})}} x^{sm+s(s+1)/2} = \sum_{s=0}^{m-1} (-1)^s \underbrace{\frac{P_{m-1} - x^m P_{m-1}}{P_s}}_{=\frac{P_{m-1}}{P_s} - \frac{P_{m-1}}{P_s} x^m} x^{sm+s(s+1)/2} \\
&= \sum_{s=0}^{m-1} (-1)^s \left(\frac{P_{m-1}}{P_s} - \frac{P_{m-1}}{P_s} x^m \right) x^{sm+s(s+1)/2} \\
&= \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} x^{sm+s(s+1)/2} - \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} x^m x^{sm+s(s+1)/2}. \qquad (25)
\end{aligned}$$

Let us rewrite the first sum on the right hand side of this equality:

$$\begin{aligned}
& \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} \underbrace{x^{sm+s(s+1)/2}}_{\substack{=x^{s+(sm+s(s-1)/2)} \\ (\text{since } sm+s(s+1)/2 = s+(sm+s(s-1)/2))}} \\
&= \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} \underbrace{x^{s+(sm+s(s-1)/2)}}_{=x^s x^{sm+s(s-1)/2}} = \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} \underbrace{x^s}_{=1-(1-x^s)} x^{sm+s(s-1)/2} \\
&= \sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} (1 - (1 - x^s)) x^{sm+s(s-1)/2} \\
&= \underbrace{\sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} x^{sm+s(s-1)/2}}_{\substack{=F_{m-1} \\ (\text{by } (24))}} - \underbrace{\sum_{s=0}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} (1 - x^s) x^{sm+s(s-1)/2}}_{\substack{=(-1)^0 \frac{P_{m-1}}{P_0} (1-x^0) x^{0m+0(0-1)/2} + \sum_{s=1}^{m-1} (-1)^s \frac{P_{m-1}}{P_s} (1-x^s) x^{sm+s(s-1)/2} \\ (\text{here, we have split off the addend for } s=0 \text{ from the sum,} \\ \text{since } m-1 \geq 0)}} \\
&= F_{m-1} - \left(\underbrace{(-1)^0 \frac{P_{m-1}}{P_0} (1 - x^0) x^{0m+0(0-1)/2}}_{\substack{=0 \\ (\text{since this product contains the factor } 1-x^0 = 1-1 = 0)}} + \sum_{s=1}^{m-1} (-1)^s \underbrace{\frac{P_{m-1}}{P_s} (1 - x^s)}_{=(1-x^s) \cdot \frac{P_{m-1}}{P_s}} x^{sm+s(s-1)/2} \right) \\
&= F_{m-1} - \sum_{s=1}^{m-1} (-1)^s (1 - x^s) \cdot \frac{P_{m-1}}{P_s} x^{sm+s(s-1)/2}. \qquad (26)
\end{aligned}$$

Let us now take a closer look at the second sum on the right hand side of (25):

$$\begin{aligned}
&\sum_{s=0}^{m-1}(-1)^s \frac{P_{m-1}}{P_s} \underbrace{x^m x^{sm+s(s+1)/2}}_{\substack{=x^{m+(sm+s(s+1)/2)}=x^{(s+1)m+s(s+1)/2} \\ (\text{since } m+(sm+s(s+1)/2)=(s+1)m+s(s+1)/2)}} \\
&= \sum_{s=0}^{m-1}(-1)^s \frac{P_{m-1}}{P_s} x^{(s+1)m+s(s+1)/2} \\
&= (-1)^{m-1} \underbrace{\frac{P_{m-1}}{P_{m-1}}}_{=1} \underbrace{x^{((m-1)+1)m+(m-1)((m-1)+1)/2}}_{\substack{=x^{m(3m-1)/2} \\ (\text{since } ((m-1)+1)m+(m-1)((m-1)+1)/2=m(3m-1)/2)}} \\
&\qquad + \sum_{s=0}^{m-2}(-1)^s \frac{P_{m-1}}{P_s} x^{(s+1)m+s(s+1)/2} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend for } s=m-1 \text{ from the sum,} \\ \text{since } m-1\geq 0\end{array}\right) \\
&= (-1)^{m-1} x^{m(3m-1)/2} + \sum_{s=0}^{m-2}(-1)^s \frac{P_{m-1}}{P_s} x^{(s+1)m+s(s+1)/2} \\
&= (-1)^{m-1} x^{m(3m-1)/2} \\
&\qquad + \sum_{s=1}^{m-1} \underbrace{(-1)^{s-1}}_{=-(-1)^s} \underbrace{\frac{P_{m-1}}{P_{s-1}}}_{\substack{=(1-x^s)\cdot\frac{P_{m-1}}{P_s} \\ (\text{by (22), applied to } Q=P_{m-1} \\ \text{and } n=s)}} \underbrace{x^{((s-1)+1)m+(s-1)((s-1)+1)/2}}_{\substack{=x^{sm+s(s-1)/2} \\ (\text{since } ((s-1)+1)m+(s-1)((s-1)+1)/2=sm+s(s-1)/2)}} \\
&\qquad (\text{here, we have substituted } s-1 \text{ for } s \text{ in the sum}) \\
&= (-1)^{m-1} x^{m(3m-1)/2} - \sum_{s=1}^{m-1}(-1)^s (1-x^s)\cdot\frac{P_{m-1}}{P_s} x^{sm+s(s-1)/2}.
\end{aligned} \tag{27}$$

Now, (25) becomes

$$\begin{aligned}
&F_m - (-1)^m x^{m(3m+1)/2} \\
&= \underbrace{\sum_{s=0}^{m-1}(-1)^s \frac{P_{m-1}}{P_s} x^{sm+s(s+1)/2}}_{\substack{=F_{m-1}-\sum_{s=1}^{m-1}(-1)^s(1-x^s)\cdot\frac{P_{m-1}}{P_s}x^{sm+s(s-1)/2} \\ (\text{by (26)})}} - \underbrace{\sum_{s=0}^{m-1}(-1)^s \frac{P_{m-1}}{P_s} x^m x^{sm+s(s+1)/2}}_{\substack{=(-1)^{m-1}x^{m(3m-1)/2}-\sum_{s=1}^{m-1}(-1)^s(1-x^s)\cdot\frac{P_{m-1}}{P_s}x^{sm+s(s-1)/2} \\ (\text{by (27)})}} \\
&= \left(F_{m-1} - \sum_{s=1}^{m-1}(-1)^s (1-x^s)\cdot\frac{P_{m-1}}{P_s} x^{sm+s(s-1)/2}\right) \\
&\qquad - \left((-1)^{m-1} x^{m(3m-1)/2} - \sum_{s=1}^{m-1}(-1)^s (1-x^s)\cdot\frac{P_{m-1}}{P_s} x^{sm+s(s-1)/2}\right) \\
&= F_{m-1} - \underbrace{(-1)^{m-1}}_{=-(-1)^m} x^{m(3m-1)/2} = F_{m-1} + (-1)^m x^{m(3m-1)/2}.
\end{aligned}$$

Adding $(-1)^m x^{m(3m+1)/2}$ to both sides of this equality, we find

$$\begin{aligned} F_m &= \underbrace{F_{m-1}}_{=S_{m-1}} + (-1)^m x^{m(3m-1)/2} + (-1)^m x^{m(3m+1)/2} \\ &= S_{m-1} + (-1)^m x^{m(3m-1)/2} + (-1)^m x^{m(3m+1)/2}. \end{aligned} \tag{28}$$

On the other hand, the definition of S_{m-1} yields

$$S_{m-1} = \sum_{s=-(m-1)}^{m-1} (-1)^s x^{s(3s+1)/2} = \sum_{s=-m+1}^{m-1} (-1)^s x^{s(3s+1)/2} \tag{29}$$

(since $-(m-1) = -m+1$). But the definition of S_m yields

$$\begin{aligned} & S_m \\ &= \sum_{s=-m}^{m} (-1)^s x^{s(3s+1)/2} \\ &= \underbrace{\sum_{s=-m+1}^{m-1} (-1)^s x^{s(3s+1)/2}}_{\substack{=S_{m-1} \\ \text{(by (29))}}} + \underbrace{(-1)^{-m}}_{\substack{=(-1)^m \\ \text{(since } -m \equiv m \bmod 2)}} \underbrace{x^{(-m)(3(-m)+1)/2}}_{\substack{=x^{m(3m-1)/2} \\ \text{(since } (-m)(3(-m)+1)=m(3m-1))}} + (-1)^m x^{m(3m+1)/2} \\ & \qquad \left(\begin{array}{c} \text{here, we have split off the addends for } s=-m \text{ and for } s=m \\ \text{from the sum (indeed, these are two distinct addends, since } m \neq -m) \end{array} \right) \\ &= S_{m-1} + (-1)^m x^{m(3m-1)/2} + (-1)^m x^{m(3m+1)/2}. \end{aligned}$$

Comparing this with (28), we obtain $F_m = S_m$. In other words, (23) holds for $n = m$. This completes the induction step.

Thus, (23) is proven by induction. In other words, part **(b)** of the exercise is solved.

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6 Exercise 6

6.1 Problem

Recall that we are using the following notations:

- A *partition* means a weakly decreasing sequence of positive integers. (Thus, a partition is the same as a partition of some $n \in \mathbb{N}$.)
- Given any $n \in \mathbb{Z}$, we let $p(n)$ denote the number of all partitions of n. (This is 0 when $n < 0$.)
- Given any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we let $p_k(n)$ denote the number of all partitions of n into k parts.

Prove the following:

(a) Any $n \in \mathbb{N}$ satisfies

$$np(n) = \sum_{k=1}^{n} \sigma(k) p(n-k),$$

where $\sigma(k)$ denotes the sum of all positive divisors of k.

(b) Any $n \in \mathbb{N}$ satisfies

$$\sum_{k=0}^{n} k p_k(n) = \sum_{k=1}^{n} \partial(k) p(n-k),$$

where $\partial(k)$ denotes the number of positive divisors of k.

(c) Let $a : \{1, 2, 3, \ldots\} \to \mathbb{Q}$ be any map. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we let $a(\lambda)$ denote the sum $a(\lambda_1) + a(\lambda_2) + \cdots + a(\lambda_k)$. Then, any $n \in \mathbb{N}$ satisfies

$$\sum_{\lambda \vdash n} a(\lambda) = \sum_{k=1}^{n} \left(\sum_{d \mid k} a(d) \right) p(n-k).$$

Here, the summation sign "$\sum_{\lambda \vdash n}$" means "sum over all partitions λ of n", whereas the summation sign "$\sum_{d \mid k}$" means "sum over all positive divisors d of k".

[**Hint:** Parts **(a)** and **(b)** are particular cases of **(c)**. Particular cases are not always easier to prove than generalizations.]

6.2 Solution

We begin by introducing a notation (which is actually standard):

Definition 6.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be any partition. Let d be a positive integer. Then, $m_d(\lambda)$ shall denote the number of all $i \in [k]$ satisfying $\lambda_i = d$. (In other words, $m_d(\lambda)$ is the number of times that the number d appears in λ.) We refer to $m_d(\lambda)$ as the *multiplicity* of d in λ. Note that $m_d(\lambda) \in \mathbb{N}$.

Example 6.2. If $\lambda = (5, 3, 3, 3, 3, 2)$, then

$$m_1(\lambda) = 0, \qquad m_2(\lambda) = 1, \qquad m_3(\lambda) = 4, \qquad m_4(\lambda) = 0, \qquad m_5(\lambda) = 1$$

and $m_d(\lambda) = 0$ for all $d > 5$.

We notice that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, then all but finitely many positive integers d satisfy $m_d(\lambda) = 0$ (because λ has only finitely many entries, and any positive integer other than these entries doesn't appear in λ at all).

Let us now solve the problem. We begin with part **(c)**, since it is the easiest part to solve.

(c) Let $n \in \mathbb{N}$. We begin with the following simple claims:

Claim 1: Let λ be any partition. Then, $a(\lambda) = \sum_{d \geq 1} m_d(\lambda) a(d)$. (Here, the infinite sum $\sum_{d \geq 1} m_d(\lambda) a(d)$ is well-defined, since all but finitely many positive integers d satisfy $m_d(\lambda) = 0$.)

[*Proof of Claim 1:* Write λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then, the definition of $a(\lambda)$ yields

$$a(\lambda) = a(\lambda_1) + a(\lambda_2) + \cdots + a(\lambda_k) = \sum_{i\in[k]} a(\lambda_i) = \sum_{d\geq 1} \sum_{\substack{i\in[k];\\ \lambda_i = d}} a\left(\underbrace{\lambda_i}_{=d}\right)$$

(since λ_i is a positive integer for each $i \in [k]$)

$$= \sum_{d\geq 1} \underbrace{\sum_{\substack{i\in[k];\\ \lambda_i=d}} a(d)}_{=|\{i\in[k] \ | \ \lambda_i=d\}|\cdot a(d)} = \sum_{d\geq 1} \underbrace{|\{i \in [k] \ | \ \lambda_i = d\}|}_{\substack{=(\text{the number of all } i\in[k] \text{ satisfying } \lambda_i=d)\\ =m_d(\lambda)\\ (\text{since this is how } m_d(\lambda) \text{ was defined})}} \cdot a(d) = \sum_{d\geq 1} m_d(\lambda)\, a(d).$$

This proves Claim 1.]

Claim 2: Let $p \in \mathbb{N}$. Then, $p = \sum_{i\geq 1} [p \geq i]$. (Here, the infinite sum $\sum_{i\geq 1} [p \geq i]$ is well-defined, since all but finitely many positive integers i satisfy $i > p$ and thus $[p \geq i] = 0$.)

[*Proof of Claim 2:* We have

$$\sum_{i\geq 1} [p \geq i] = \sum_{i=1}^{p} \underbrace{[p \geq i]}_{\substack{=1\\ (\text{since } p\geq i \text{ (because } i\leq p))}} + \sum_{i\geq p+1} \underbrace{[p \geq i]}_{\substack{=0\\ (\text{since we don't have } p\geq i\\ (\text{because } i\geq p+1>p))}} = \sum_{i=1}^{p} 1 + \underbrace{\sum_{i\geq p+1} 0}_{=0}$$
$$= \sum_{i=1}^{p} 1 = p \cdot 1 = p.$$

This proves Claim 2.]

In the following, the notation "$\lambda \vdash n$" shall stand for "λ is a partition of n". (This is in line with how we defined the summation sign "$\sum_{\lambda\vdash n}$" in the statement of the exercise.)

Claim 3: Let d and i be positive integers. Then,

$$|\{\lambda \vdash n \ | \ m_d(\lambda) \geq i\}| = p(n - di).$$

[*Proof of Claim 3:* Let λ be a partition of n satisfying $m_d(\lambda) \geq i$. Recall that $m_d(\lambda)$ is the number of times that the number d appears in λ (by the definition of $m_d(\lambda)$). This number is $\geq i$ (since $m_d(\lambda) \geq i$). Thus, the number d appears at least i times in λ. Thus, we can remove i entries equal to d from the partition λ [12]. The result will be a partition of $n - di$ (since removing i entries equal to d has the effect of decreasing the sum of all entries by di).

Now, forget that we fixed λ. We thus have shown that if λ is a partition of n satisfying $m_d(\lambda) \geq i$, then we can remove i entries equal to d from the partition λ and obtain a partition of $n - di$. Thus, we obtain a map

$$A : \{\lambda \vdash n \ | \ m_d(\lambda) \geq i\} \to \{\text{partitions of } n - di\}$$

[12] It does not matter **which** i entries equal to d we remove (in case there are many options, i.e., in case λ has more than i entries equal to d); the resulting tuple will always be the same.

which sends each λ to the result of removing i entries equal to d from λ.

On the other hand, if μ is a partition of $n-di$, then we can insert i entries equal to d into μ (in such a way that the resulting tuple will still be weakly decreasing[13]); the result will be a partition of n (because inserting i entries equal to d has the effect of increasing the sum of all entries by di). Hence, we obtain a map

$$B : \{\text{partitions of } n-di\} \to \{\lambda \vdash n \mid m_d(\lambda) \geq i\}$$

which sends each μ to the result of inserting i entries equal to d into μ.

It is straightforward to see that the maps A and B are mutually inverse. Thus, they are bijections. Hence, there is a bijection from $\{\text{partitions of } n-di\}$ to $\{\lambda \vdash n \mid m_d(\lambda) \geq i\}$ (namely, the map B). Therefore,

$$\begin{aligned} |\{\lambda \vdash n \mid m_d(\lambda) \geq i\}| &= |\{\text{partitions of } n-di\}| \\ &= (\text{the number of all partitions of } n-di) = p(n-di) \end{aligned}$$

(since $p(n-di)$ is defined as the number of all partitions of $n-di$). This proves Claim 3.]

Claim 4: Let d and i be positive integers. Then,

$$\sum_{\lambda \vdash n} [m_d(\lambda) \geq i] = p(n-di).$$

[*Proof of Claim 4:* This is an easy consequence of Claim 3 using the standard "counting by roll call" technique. In more detail: We have

$$\begin{aligned} \sum_{\lambda \vdash n} [m_d(\lambda) \geq i] &= \sum_{\substack{\lambda \vdash n; \\ m_d(\lambda) \geq i}} \underbrace{[m_d(\lambda) \geq i]}_{\substack{=1 \\ (\text{since } m_d(\lambda) \geq i)}} + \sum_{\substack{\lambda \vdash n; \\ \text{not } m_d(\lambda) \geq i}} \underbrace{[m_d(\lambda) \geq i]}_{\substack{=0 \\ (\text{since we don't have } m_d(\lambda) \geq i)}} \\ &= \sum_{\substack{\lambda \vdash n; \\ m_d(\lambda) \geq i}} 1 + \underbrace{\sum_{\substack{\lambda \vdash n; \\ \text{not } m_d(\lambda) \geq i}} 0}_{=0} = \sum_{\substack{\lambda \vdash n; \\ m_d(\lambda) \geq i}} 1 = |\{\lambda \vdash n \mid m_d(\lambda) \geq i\}| \cdot 1 \\ &= |\{\lambda \vdash n \mid m_d(\lambda) \geq i\}| = p(n-di) \qquad (\text{by Claim 3}). \end{aligned}$$

This proves Claim 4.]

Claim 5: Let k be an integer such that $k > n$. Then, $p(n-k) = 0$.

[*Proof of Claim 5:* From $k > n$, we obtain $n-k < 0$. Now, recall that $p(m) = 0$ whenever m is a negative integer (because there are no partitions of m when m is negative). Applying this to $m = n-k$, we obtain $p(n-k) = 0$. Claim 5 is proven.]

Now,

$$\begin{aligned} \sum_{\lambda \vdash n} \underbrace{a(\lambda)}_{\substack{= \sum_{d \geq 1} m_d(\lambda) a(d) \\ (\text{by Claim 1})}} &= \underbrace{\sum_{\lambda \vdash n} \sum_{d \geq 1}}_{\substack{= \sum_{d \geq 1} \sum_{\lambda \vdash n} \\ (\text{this interchange of summation signs} \\ \text{is legitimate, since one of them is} \\ \text{a finite sum (indeed, there are only} \\ \text{finitely many partitions } \lambda \text{ of } n))}} m_d(\lambda) a(d) = \sum_{d \geq 1} \sum_{\lambda \vdash n} m_d(\lambda) a(d) \\ &= \sum_{d \geq 1} a(d) \sum_{\lambda \vdash n} m_d(\lambda). \end{aligned} \tag{30}$$

[13] Again, there might be several ways to perform this insertion (in the sense that there are several spots in which we can insert these entries), but the result will always be the same.

But each positive integer d satisfies

$$\sum_{\lambda\vdash n} \underbrace{m_d(\lambda)}_{\substack{=\sum_{i\geq 1}[m_d(\lambda)\geq i] \\ \text{(by Claim 2, applied to } p=m_d(\lambda))}}$$

$$= \underbrace{\sum_{\lambda\vdash n}\sum_{i\geq 1}}_{\substack{=\sum_{i\geq 1}\sum_{\lambda\vdash n} \\ \text{(this interchange of summation signs} \\ \text{is legitimate, since one of them is} \\ \text{a finite sum (indeed, there are only} \\ \text{finitely many partitions } \lambda \text{ of } n))}} [m_d(\lambda)\geq i] = \sum_{i\geq 1} \underbrace{\sum_{\lambda\vdash n}[m_d(\lambda)\geq i]}_{\substack{=p(n-di) \\ \text{(by Claim 4)}}}$$

$$= \sum_{i\geq 1} p(n-di) = \sum_{\substack{k\geq 1;\\ d\mid k}} p(n-k) \qquad \text{(here, we have substituted } k \text{ for } di \text{ in the sum)}$$

$$= \underbrace{\sum_{\substack{k\geq 1;\\ d\mid k;\\ k\leq n}}}_{=\sum_{\substack{k\in\{1,2,\ldots,n\};\\ d\mid k}}} p(n-k) + \sum_{\substack{k\geq 1;\\ d\mid k;\\ k>n}} \underbrace{p(n-k)}_{\substack{=0\\ \text{(by Claim 5)}}}$$

(since each positive integer k satisfies either $k\leq n$ or $k>n$, but not both)

$$= \sum_{\substack{k\in\{1,2,\ldots,n\};\\ d\mid k}} p(n-k) + \underbrace{\sum_{\substack{k\geq 1;\\ d\mid k;\\ k>n}} 0}_{=0} = \sum_{\substack{k\in\{1,2,\ldots,n\};\\ d\mid k}} p(n-k).$$

Hence, (30) becomes

$$\sum_{\lambda\vdash n} a(\lambda) = \sum_{d\geq 1} a(d) \underbrace{\sum_{\lambda\vdash n} m_d(\lambda)}_{=\sum_{\substack{k\in\{1,2,\ldots,n\};\\ d\mid k}} p(n-k)} = \sum_{d\geq 1} a(d) \sum_{\substack{k\in\{1,2,\ldots,n\};\\ d\mid k}} p(n-k)$$

$$= \underbrace{\sum_{d\geq 1}\sum_{\substack{k\in\{1,2,\ldots,n\};\\ d\mid k}}}_{\substack{=\sum_{k\in\{1,2,\ldots,n\}}\sum_{\substack{d\geq 1;\\ d\mid k}} \\ \text{(this interchange of sums was legitimate, since} \\ \text{there are only finitely many pairs } (d,k) \text{ such that} \\ d \text{ is a positive integer and } k\in\{1,2,\ldots,n\} \text{ satisfies } d\mid k)}} a(d)\,p(n-k)$$

$$= \underbrace{\sum_{k\in\{1,2,\ldots,n\}}}_{=\sum_{k=1}^{n}} \underbrace{\sum_{\substack{d\geq 1;\\ d\mid k}}}_{=\sum_{d\mid k}} a(d)\,p(n-k) = \sum_{k=1}^{n}\sum_{d\mid k} a(d)\,p(n-k)$$

$$= \sum_{k=1}^{n}\left(\sum_{d\mid k} a(d)\right) p(n-k).$$

This solves part **(c)** of the exercise.

(a) If k is any positive integer, then

$$\begin{aligned}\sigma(k) &= (\text{the sum of all positive divisors of } k) \qquad (\text{by the definition of } \sigma)\\ &= \sum_{d\mid k} d \qquad (31)\end{aligned}$$

(since the summation sign "$\sum_{d\mid k}$" means "sum over all positive divisors d of k").

Let $a : \{1, 2, 3, \ldots\} \to \mathbb{Q}$ be the map that sends each $i \in \{1, 2, 3, \ldots\}$ to i. Thus, $a(i) = i$ for each $i \in \{1, 2, 3, \ldots\}$.

Let $n \in \mathbb{N}$. Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n satisfies

$$\begin{aligned}a(\lambda) &= a(\lambda_1) + a(\lambda_2) + \cdots + a(\lambda_k) \qquad (\text{by the definition of } a(\lambda))\\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_k \qquad (\text{since } a(i) = i \text{ for each } i \in \{1, 2, 3, \ldots\})\\ &= n \qquad (\text{since } (\lambda_1, \lambda_2, \ldots, \lambda_k) \text{ is a partition of } n).\end{aligned}$$

Hence,

$$\sum_{\lambda\vdash n} \underbrace{a(\lambda)}_{=n} = \sum_{\lambda\vdash n} n = \underbrace{(\text{the number of all partitions of } n)}_{\substack{=p(n)\\ (\text{by the definition of } p(n))}} \cdot n = p(n) \cdot n = np(n).$$

Thus,

$$\begin{aligned}np(n) &= \sum_{\lambda\vdash n} a(\lambda) = \sum_{k=1}^{n} \left(\sum_{d\mid k} \underbrace{a(d)}_{\substack{=d\\ (\text{since } a(i)=i\\ \text{for each } i\in\{1,2,3,\ldots\})}} \right) p(n-k) \qquad (\text{by part } \textbf{(c)} \text{ of the exercise})\\ &= \sum_{k=1}^{n} \underbrace{\left(\sum_{d\mid k} d \right)}_{\substack{=\sigma(k)\\ (\text{by (31)})}} p(n-k) = \sum_{k=1}^{n} \sigma(k)\, p(n-k).\end{aligned}$$

This solves part **(a)** of the exercise.

(b) If k is any positive integer, then

$$\begin{aligned}\partial(k) &= (\text{the number of positive divisors of } k) \qquad (\text{by the definition of } \partial)\\ &= \sum_{d\mid k} 1 \qquad (32)\end{aligned}$$

[14].

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[14]since

$$\begin{aligned}\sum_{d\mid k} 1 = \sum_{\substack{d \text{ is a positive}\\ \text{divisor of } k}} 1 &= (\text{the number of positive divisors of } k) \cdot 1\\ &= (\text{the number of positive divisors of } k)\end{aligned}$$

Let $a : \{1, 2, 3, \ldots\} \to \mathbb{Q}$ be the map that sends each $i \in \{1, 2, 3, \ldots\}$ to 1. Thus, $a(i) = 1$ for each $i \in \{1, 2, 3, \ldots\}$.

Let $n \in \mathbb{N}$. If λ is a partition of n into k parts (for some $k \in \mathbb{N}$), then

$$a(\lambda) = k \tag{33}$$

[15].

But every $k \in \mathbb{N}$ satisfies

$$p_k(n) = (\text{the number of all partitions of } n \text{ into } k \text{ parts}) \tag{34}$$

(by the definition of $p_k(n)$). Hence,

$$p_k(n) = 0 \qquad \text{for each integer } k > n \tag{35}$$

[16].

Now, each partition λ of n is a partition of n into k parts for a **unique** $k \in \mathbb{N}$. Hence,

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[15] *Proof of* (33)*:* Let $k \in \mathbb{N}$. Let λ be a partition of n into k parts. Thus, we can write λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Hence,

$$\begin{aligned} a(\lambda) &= a(\lambda_1) + a(\lambda_2) + \cdots + a(\lambda_k) && (\text{by the definition of } a(\lambda)) \\ &= \underbrace{1 + 1 + \cdots + 1}_{k \text{ times}} && (\text{since } a(i) = 1 \text{ for each } i \in \{1, 2, 3, \ldots\}) \\ &= k. \end{aligned}$$

This proves (33).

[16] *Proof of* (35)*:* Let $k > n$ be an integer. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n into k parts. Since λ is a partition of n, we have $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, so that

$$\begin{aligned} n = \lambda_1 + \lambda_2 + \cdots + \lambda_k &\geq \underbrace{1 + 1 + \cdots + 1}_{k \text{ times}} \qquad (\text{since } \lambda_i \geq 1 \text{ for each } i \in [k]) \\ &= k > n. \end{aligned}$$

This is clearly a contradiction.

Forget that we fixed λ. We thus have obtained a contradiction for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n into k parts. Thus, there exists no such partition. In other words, (the number of all partitions of n into k parts) $= 0$. Thus,

$$p_k(n) = (\text{the number of all partitions of } n \text{ into } k \text{ parts}) = 0.$$

This proves (35).

the summation sign "$\sum\limits_{\lambda \vdash n}$" can be replaced by "$\sum\limits_{k \in \mathbb{N}} \sum\limits_{\substack{\lambda \text{ is a partition} \\ \text{of } n \text{ into } k \text{ parts}}}$". We thus find

$$\begin{aligned}
\sum_{\lambda \vdash n} a(\lambda) &= \sum_{k \in \mathbb{N}} \sum_{\substack{\lambda \text{ is a partition} \\ \text{of } n \text{ into } k \text{ parts}}} \underbrace{a(\lambda)}_{\substack{=k \\ (\text{by (33)})}} = \sum_{k \in \mathbb{N}} \underbrace{\sum_{\substack{\lambda \text{ is a partition} \\ \text{of } n \text{ into } k \text{ parts}}} k}_{=(\text{the number of all partitions of } n \text{ into } k \text{ parts}) \cdot k} \\
&= \sum_{k \in \mathbb{N}} \underbrace{(\text{the number of all partitions of } n \text{ into } k \text{ parts})}_{\substack{=p_k(n) \\ (\text{by (34)})}} \cdot k = \sum_{k \in \mathbb{N}} p_k(n) \cdot k \\
&= \sum_{k \in \mathbb{N}} k p_k(n) = \sum_{k=0}^{n} k p_k(n) + \sum_{k \geq n+1} k \underbrace{p_k(n)}_{\substack{=0 \\ (\text{by (35), since } k \geq n+1 > n)}} \\
&= \sum_{k=0}^{n} k p_k(n) + \underbrace{\sum_{k \geq n+1} k0}_{=0} = \sum_{k=0}^{n} k p_k(n).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=0}^{n} k p_k(n) &= \sum_{\lambda \vdash n} a(\lambda) \\
&= \sum_{k=1}^{n} \left(\sum_{d \mid k} \underbrace{a(d)}_{\substack{=1 \\ (\text{since } a(i)=1 \\ \text{for each } i \in \{1,2,3,\ldots\})}} \right) p(n-k) \qquad (\text{by part } \textbf{(c)} \text{ of the exercise}) \\
&= \sum_{k=1}^{n} \underbrace{\left(\sum_{d \mid k} 1 \right)}_{\substack{=\partial(k) \\ (\text{by (32)})}} p(n-k) = \sum_{k=1}^{n} \partial(k) p(n-k).
\end{aligned}$$

This solves part **(b)** of the exercise.

6.3 Remark

Parts **(a)** and **(b)** of this exercise reveal the connection between partitions and divisors. Results in a similar vein can be found in [BiFoFo95].

Part **(a)** of the exercise has also been discussed in `https://mathoverflow.net/questions/127000/` .

Part **(b)** of the exercise also appears in `https://math.stackexchange.com/questions/2864176` (in a slightly modified version[17]), where a proof using generating functions is given.

[17] Note that if n is a positive integer, then $p_0(n) = 0$, and thus $\sum\limits_{k=0}^{n} k p_k(n)$ can be rewritten as $\sum\limits_{k=1}^{n} k p_k(n)$.

Also, the sum $\sum\limits_{k=1}^{n} \partial(k) p(n-k)$ can be rewritten as $\sum\limits_{\substack{r \geq 1;\ s \geq 1; \\ rs \leq n}} p(n - rs)$.

7 Exercise 7

7.1 Problem

Let n be an even positive integer.

(a) For each $\sigma \in S_n$, prove that there exist two distinct elements i and j of $[n]$ such that $\sigma(i) - i \equiv \sigma(j) - j \mod n$.

(b) If n people are seated around a table at lunch, and the same n people are seated around the same table at dinner, then prove that you can find two distinct people which have the same distance at the lunch as they have at the dinner. (The *distance* between two people means the number of persons sitting between them, counted along the shorter arc, plus 1. For instance, two neighbors will have distance 1.)

(c) Prove that both **(a)** and **(b)** are false if $n = 5$.

(d) What about $n = 7$?

[**Hint:** **(a)** If there are no such i and j, what can you say about the remainders of the n numbers $\sigma(1) - 1, \sigma(2) - 2, \ldots, \sigma(n) - n$ modulo n, and what can you say about $\sum_{i=1}^{n} (\sigma(i) - i)$?]

7.2 Remark

Part **(b)** of this exercise is [AndFen04, Example 0.3].

7.3 Solution sketch

(a) Let $\sigma \in S_n$. Thus, σ is a bijection from $[n]$ to $[n]$. Hence, we can substitute i for $\sigma(i)$ in the sum $\sum_{i \in [n]} \sigma(i)$. We thus obtain

$$\sum_{i \in [n]} \sigma(i) = \sum_{i \in [n]} i. \tag{36}$$

For each integer z, we let $z\%n$ denote the remainder of z when divided by n.

Assume (for the sake of contradiction) that the n numbers

$$(\sigma(1) - 1)\%n, \qquad (\sigma(2) - 2)\%n, \qquad \ldots, \qquad (\sigma(n) - n)\%n \tag{37}$$

are distinct. Thus, the map

$$[n] \to \{0, 1, \ldots, n-1\}, \qquad i \mapsto (\sigma(i) - i)\%n$$

[18] is injective. Therefore, by the Pigeonhole Principle for Injections, this map must also be bijective (since it is an injective map between two finite sets of the same size). In other

[18] This map is well-defined, because for each $i \in [n]$, we have $(\sigma(i) - i)\%n \in \{0, 1, \ldots, n-1\}$ (since $(\sigma(i) - i)\%n$ is a remainder of a division by n).

words, it is a bijection. Hence, we can substitute $(\sigma(i) - i) \% n$ for j in the sum $\sum\limits_{j \in \{0,1,\ldots,n-1\}} j$. We thus find

$$\sum_{j \in \{0,1,\ldots,n-1\}} j = \sum_{i \in [n]} \underbrace{((\sigma(i) - i) \% n)}_{\substack{\equiv \sigma(i) - i \bmod n \\ \text{(because } z\%n \equiv z \bmod n \\ \text{for each integer } z)}} \equiv \sum_{i \in [n]} (\sigma(i) - i) = \sum_{i \in [n]} \sigma(i) - \sum_{i \in [n]} i = 0 \bmod n$$

(by (36)). In view of

$$\sum_{j \in \{0,1,\ldots,n-1\}} j = 0 + 1 + \cdots + (n-1) = \frac{(n-1)((n-1)+1)}{2} \qquad \text{(by Little Gauss)}$$
$$= \frac{(n-1)n}{2},$$

this rewrites as $\frac{(n-1)n}{2} \equiv 0 \bmod n$. In other words, $n \mid \frac{(n-1)n}{2}$. In other words, $\frac{(n-1)n}{2} / n$ is an integer.

But $\frac{(n-1)n}{2} / n = \frac{n-1}{2}$ is not an integer, because $n - 1$ is odd (since n is even). This contradicts the fact that $\frac{(n-1)n}{2} / n$ is an integer.

This contradiction proves that our assumption was false. Hence, the n numbers listed in (37) are **not** all distinct. In other words, there exist two distinct elements i and j of $[n]$ such that $(\sigma(i) - i) \% n = (\sigma(j) - j) \% n$. These i and j must then satisfy

$$\begin{aligned} \sigma(i) - i &\equiv (\sigma(i) - i) \% n && \text{(since } z \equiv z\%n \bmod n \text{ for each integer } z) \\ &= (\sigma(j) - j) \% n \\ &\equiv \sigma(j) - j \bmod n && \text{(since } z\%n \equiv z \bmod n \text{ for each integer } z). \end{aligned}$$

Thus, we have shown that there exist two distinct elements i and j of $[n]$ such that $\sigma(i) - i \equiv \sigma(j) - j \mod n$. This solves part **(a)** of the exercise.

(b) The following solution is taken from [AndFen04, Example 0.3].

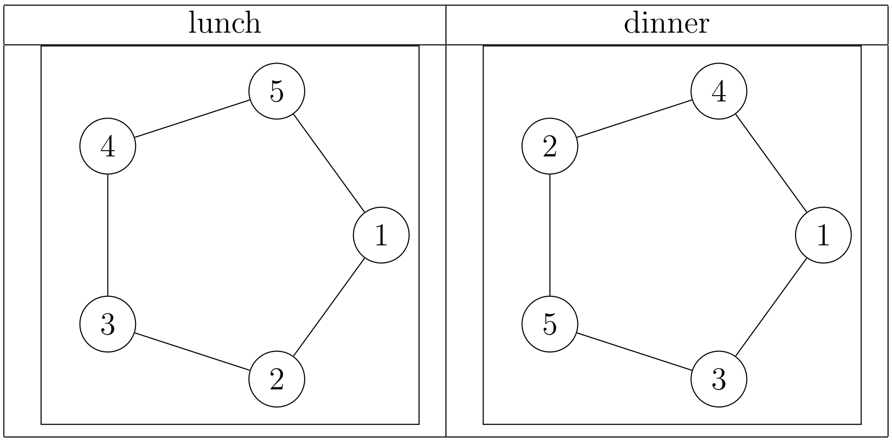
Label the n people by $1, 2, \ldots, n$ in the order in which they are seated at lunch (going around the table in clockwise order, starting somewhere – it does not matter where exactly). Then, at dinner, the n people at the table are $\sigma(1), \sigma(2), \ldots, \sigma(n)$ for some permutation $\sigma \in S_n$ (again we are listing them by going around the table in clockwise order, starting somewhere). Consider this σ. Part **(a)** of this exercise shows that there exist two distinct elements i and j of $[n]$ such that $\sigma(i) - i \equiv \sigma(j) - j \mod n$. Consider these i and j.

We have $\sigma(i) - i \equiv \sigma(j) - j \mod n$. In other words, $j - i \equiv \sigma(j) - \sigma(i) \mod n$. Thus, the integers $j - i$ and $\sigma(j) - \sigma(i)$ leave the same remainder when divided by n. In other words, $(j - i) \% n = (\sigma(j) - \sigma(i)) \% n$.

Therefore, the distance between the two people $\sigma(i)$ and $\sigma(j)$ at dinner is equal to the distance between these two people at lunch[19]. So these two people have the same distance at the lunch as they have at the dinner. This solves part **(b)** of the exercise.

[19] Indeed, their distance at dinner is $\min\{(j-i)\%n,\ n-(j-i)\%n\}$, whereas their distance at lunch is $\min\{(\sigma(j)-\sigma(i))\%n,\ n-(\sigma(j)-\sigma(i))\%n\}$. These two numbers are equal, since $(j-i)\%n = (\sigma(j)-\sigma(i))\%n$.

(c) Since part **(b)** follows from part **(a)** (as we have just seen), it suffices to prove that part **(b)** is false for $n = 5$. This is not hard: Let the seating arrangement at lunch be $1, 2, 3, 4, 5$, and let the seating arrangement at dinner be $1, 3, 5, 2, 4$. This means that they are seated as follows:



It is easy to see that there are no two people that have the same distance at the lunch as they have at the dinner: Indeed,

- any two people having distance 1 at the lunch have distance 2 at the dinner;
- any two people having distance 2 at the lunch have distance 1 at the dinner.

So part **(b)** of the problem fails for $n = 5$. Thus, part **(a)** also fails for $n = 5$.

(d) Again, both parts **(a)** and **(b)** are false for $n = 7$. The proof of this is similar to part **(c)**: Again, it suffices to prove that part **(b)** is false for $n = 7$. This is not hard: Let the seating arrangement at lunch be $1, 2, 3, 4, 5, 6, 7$, and let the seating arrangement at dinner be $1, 3, 5, 7, 2, 4, 6$. It is easy to see that there are no two people that have the same distance at the lunch as they have at the dinner: Indeed,

- any two people having distance 1 at the lunch have distance 3 at the dinner;
- any two people having distance 2 at the lunch have distance 1 at the dinner;
- any two people having distance 3 at the lunch have distance 2 at the dinner.

So part **(b)** of the problem fails for $n = 7$. Thus, part **(a)** also fails for $n = 7$.

7.4 Remark

Part **(b)** of the exercise can be restated as follows: For each $\sigma \in S_n$, prove that there exist two distinct elements i and j of $[n]$ such that

$$\sigma(i) - \sigma(j) \equiv i - j \mod n \qquad \textbf{or} \qquad \sigma(i) - \sigma(j) \equiv j - i \mod n.$$

This shows once again that it is a weaker statement than part **(a)**.

Parts **(c)** and **(d)** can be generalized: If n is any odd number **not divisible by** 3, then parts **(a)** and **(b)** of the exercise are false. To prove this, one merely needs to generalize the construction we did to solve parts **(c)** and **(d)** (that is: let the seating arrangement at lunch be $1, 2, \ldots, n$, and let the seating arrangement at dinner be $\underbrace{1, 3, 5, \ldots, n}_{\text{all odd numbers}}, \underbrace{2, 4, 6, \ldots, n-2}_{\text{all even numbers}}$).

Part **(a)** fails for **all** odd numbers n. Indeed, just take σ to be the permutation whose one-line notation is $\left(\underbrace{1,3,5,\ldots,n}_{\text{all odd numbers}},\underbrace{2,4,6,\ldots,n-2}_{\text{all even numbers}}\right)$, and check that the claim of **(a)** does not hold for this σ.

More interestingly, part **(b)** of the exercise holds not only when n is even, but also when n is divisible by 3. This is supposedly a result of Pólya from 1918, and appears with proof in Kløve's article [Klove77]. See `https://math.stackexchange.com/a/281468/` for an outline of this proof.

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