University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Midterm 3

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due date: **Wednesday, 12 December 2018** at the beginning of class, or before that by email or canvas.
Please solve **at most 3 of the 7 exercises**!
Beware: **Collaboration is not allowed** on midterms!

Notations

Here is a list of notations that are used in this homework:

- As usual, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$ of all nonnegative integers.
- We shall use the Iverson bracket notation as well as the notation $[n]$ for the set $\{1, 2, \ldots, n\}$ (when $n \in \mathbb{Z}$).
- If $n \in \mathbb{N}$, then S_n denotes the set of all permutations of $[n]$.
- A *point* shall mean an element of $\mathbb{Z}^2$, that is, a pair of integers. We depict these points as lattice points on the Cartesian plane, and add and subtract them as vectors. Recall the notion of a *lattice path*, defined in §6.1 (class notes from 2018-11-12) and (equivalently) in UMN Spring 2018 Math 4707 Midterm 1. (Lattice paths have up-steps and right-steps.) We abbreviate "lattice path" as "*LP*".
- A *formal power series* (short *FPS*) shall always mean a formal power series in the indeterminate x with rational coefficients (as defined in class).

  If f is an FPS and if $n \in \mathbb{N}$, then $[x^n] f$ shall denote the coefficient of x^n in f.

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Let $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$.

(a) Prove that

$$\sum_{k=0}^{n} \binom{n}{k} (x+k)^k (y-k)^{n-k} = \sum_{t=0}^{n} \frac{n!}{t!} (x+y)^t .$$

(b) Prove that

$$\sum_{k=0}^{n} \binom{n}{k} x (x+k)^{k-1} (y-k)^{n-k} = (x+y)^n .$$

(Here, the "$x (x+k)^{k-1}$" expression should be understood as 1 when $k=0$; this gives it meaning even if $x=0$.)

[**Hint:** **(a)** Expand $(x+k)^k$ and $(y-k)^{n-k}$ by the binomial theorem, then try using [Grinbe18, Exercise 2].

(b) Rewrite $x (x+k)^{k-1}$ as $(x+k)^k - k (x+k)^{k-1}$, thus splitting the left hand side into two sums. Apply part **(a)** to both of them.]

1.2 Solution

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2 Exercise 2

2.1 Problem

Let n be a positive integer.

(a) Let A be the $n \times n$-matrix $([i \neq j])_{i,j \in [n]}$. (This is the $n \times n$-matrix whose diagonal entries are 0 while all its other entries are 1. For example, for $n = 3$, it is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.)

Prove that $\det A = (-1)^{n-1} (n-1)$.

(b) Prove that

$$\sum_{\substack{\sigma \in S_n \text{ is a} \\ \text{derangement}}} (-1)^{\sigma} = (-1)^{n-1} (n-1) .$$

(c) Let $b_1, b_2, \ldots, b_n$ be any n numbers. Let B be the $n \times n$-matrix $\left([i \neq j] \prod_{h \in [n] \setminus \{i,j\}} b_h \right)_{i,j \in [n]}$.

(For example, for $n = 4$, we have

$$B = \begin{pmatrix} 0 & b_3 b_4 & b_2 b_4 & b_2 b_3 \\ b_3 b_4 & 0 & b_1 b_4 & b_1 b_3 \\ b_2 b_4 & b_1 b_4 & 0 & b_1 b_2 \\ b_2 b_3 & b_1 b_3 & b_1 b_2 & 0 \end{pmatrix} .)$$

Prove that

$$\det B = (-1)^{n-1} (n-1) \prod_{h \in [n]} b_h^{n-2}.$$

(Here, the "$(-1)^{n-1} (n-1) \prod_{h \in [n]} b_h^{n-2}$" expression should be understood as 0 if $n = 1$, even if $\prod_{h \in [n]} b_h^{n-2}$ may be undefined in this case when some of the b_h are 0.)

[**Hint:** **(a)** If you need a reminder on the basic properties of determinants, see, e.g., [Grinbe16, Exercises 6.7 and 6.8].

(c) If you divide by some b_h in your proof, make sure to argue why this is legitimate, or separately treat the case when some of the b_h are 0. (There is a combinatorial proof that does not require any division.)]

2.2 Solution

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3 Exercise 3

3.1 Problem

Let n be a positive integer. If $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ and $k \in [n]$, then

- we say that k is a *1-position* of $\mathbf{i}$ if $i_k = 1$;
- we say that k is a *10-position* of $\mathbf{i}$ if $k < n$, $i_k = 1$ and $i_{k+1} = 0$;
- we say that k is a *cyclic 10-position* of $\mathbf{i}$ if $i_k = 1$ and $i_{k+1} = 0$, where i_{n+1} is understood to be i_1.

(The first two of these concepts have already been defined in Homework set #2 Exercise 5. The concept of a "cyclic 10-position" differs from that of a "10-position" only in that we consider the n-tuple to "wrap around".)

Let $k \in \mathbb{N}$ and $a \in \{0, 1, \ldots, n-1\}$. Prove the following:

(a) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly a 1-positions and exactly k 10-positions is $\binom{a}{k} \binom{n-a}{k}$.

(b) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly a 1-positions and exactly k cyclic 10-positions is $\dfrac{n}{n-a}\dbinom{a-1}{a-k}\dbinom{n-a}{k}$ (this expression should be interpreted as $[k=0]$ when $a=n$).

(c) The number of n-tuples $\mathbf{i} \in \{0,1\}^n$ starting with a 0 and having exactly a 1-positions and exactly k cyclic 10-positions is $\dbinom{a-1}{a-k}\dbinom{n-a}{k}$.

3.2 Remark

1. You can rewrite the "$\dbinom{a}{k}$" in part **(a)** as "$\dbinom{a}{a-k}$" in order to make the similarity to the other two parts more glaring. Likewise, you could rewrite the "$\dbinom{a-1}{a-k}$" in parts **(b)** and **(c)** as $\dbinom{a-1}{k-1}$ when $a>0$, but not in the border case when $a=0$.

2. Sanity check: By summing over all a, we conclude from part **(a)** that the number of n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly k 10-positions is

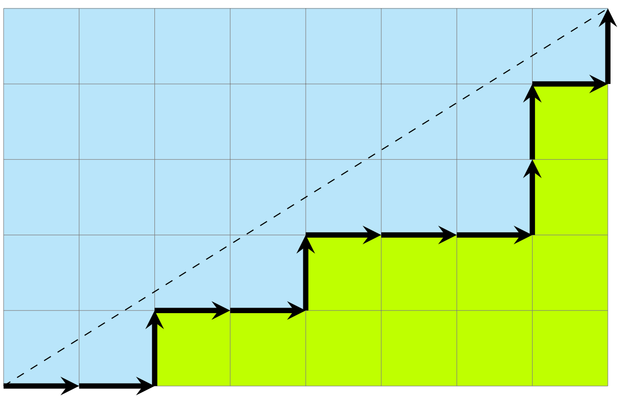
$$\sum_{a=0}^{n} \binom{a}{k}\binom{n-a}{k} = \binom{n+1}{2k+1}$$

(by a simple application of Proposition 2.21 in the class from 2018-09-26). This is exactly the result of Homework set #2 Exercise 5.

3. Part **(b)** has an application to counting (a,b)-legal paths in the sense of §6.4 from class.

Indeed, let a and b be two coprime positive integers. We say that an LP $\mathbf{v}$ is (a,b)-*legal* if each $(x,y) \in \mathbf{v}$ satisfies $ax \geq by$. Proposition 6.7 in the class from 2018-11-14 shows that the number of (a,b)-legal paths from $(0,0)$ to (b,a) is $\dfrac{1}{a+b}\dbinom{a+b}{a}$. (This is a so-called *rational Catalan number*[1].)

Now, let us define a *left turn* of an LP $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ to be an $i \in [n-1]$ such that the i-th step of $\mathbf{v}$ is a right-step (i.e., we have $v_i - v_{i-1} = (1,0)$) but the $(i+1)$-st step of $\mathbf{v}$ is an up-step (i.e., we have $v_{i+1} - v_i = (0,1)$). For instance, the LP from $(0,0)$ to $(8,5)$ depicted in



[1] The word "rational" refers to the fact that the line $ax = by$ has a rational (not integer in general) slope; the rational Catalan number is still an integer.

is $(5, 8)$-legal and has the left turns 2, 5, 9 and 12.

Now, given $k \in \mathbb{N}$, we claim that the number of (a, b)-legal LPs from $(0, 0)$ to (b, a) having exactly k left turns is $\frac{1}{b}\binom{a-1}{a-k}\binom{b}{k}$ (this is a so-called *rational Narayana number*). Indeed, set $n = a + b$; then, every LP from $(0, 0)$ to (b, a) can be encoded as an n-tuple $\mathbf{i} \in \{0, 1\}^n$ with a 1-positions (by encoding each right-step as a 0 and each up-step as a 1). If a given LP is encoded by an n-tuple $(i_1, i_2, \ldots, i_n)$, then its shift (defined as in the proof of Proposition 6.7 in the class from 2018-11-14) is encoded by the n-tuple $(i_2, i_3, \ldots, i_n, i_1)$. As we know, each cycle of S (again, see the proof of Proposition 6.7 in the class from 2018-11-14 for the definition of S) has size $a + b$ and contains exactly one (a, b)-legal LP; this (a, b)-legal LP clearly starts with a right-step and ends with an up-step; hence it is easy to see that its left turns are in bijection with the cyclic 10-positions of the corresponding n-tuple[2]. It is now easy to conclude from part **(b)** that the number of (a, b)-legal LPs from $(0, 0)$ to (b, a) having exactly k left turns is $\frac{1}{n-a}\binom{a-1}{a-k}\binom{n-a}{k} = \frac{1}{b}\binom{a-1}{a-k}\binom{b}{k}$.

4. It may be easiest to solve the problem starting with part **(c)**.

3.3 Solution

[...]

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4 Exercise 4

4.1 Problem

An *integer formal power series* (short *IFPS*) shall mean a formal power series whose coefficients all are integers. For example, $1 - 2x + 3x^2 - 4x^3 \pm \cdots$ is an IFPS, while $1 - \frac{1}{2}x$ is not.

If m is an integer, and if a and b are two IFPSs, then we say that $a \equiv b \mod m$ if and only if there exists an IFPS c such that $a - b = mc$. (This is completely analogous to the definition of congruence modulo m for integers.) The following facts hold:

(A1) Two IFPSs a and b and an integer m satisfy $a \equiv b \mod m$ if and only if each $n \in \mathbb{N}$ satisfies $[x^n]\, a \equiv [x^n]\, b \mod m$ (that is, each coefficient of a is congruent to the corresponding coefficient of b modulo m).

(A2) Each integer m and each IFPS a satisfy $a \equiv a \mod m$.

(A3) If m is an integer, and if a and b are two IFPSs satisfying $a \equiv b \mod m$, then $b \equiv a \mod m$.

(A4) If m is an integer, and if a, b and c are three IFPSs satisfying $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$.

[2] Indeed, if we define "cyclic 01-positions" in the obvious way, then the left turns of our (a, b)-legal LP are exactly the cyclic 01-positions of the corresponding n-tuple. But the latter are in bijection with the cyclic 10-positions, because the cyclic 01-positions and the cyclic 10-positions alternate.

(A5) If m is an integer, and if a, b, c and d are four IFPSs satisfying $a \equiv c \mod m$ and $b \equiv d \mod m$, then

$$a + b \equiv c + d \mod m, \qquad a - b \equiv c - d \mod m, \qquad \text{and } ab \equiv cd \mod m.$$

(A6) If m is an integer, and if a and b are two IFPSs with constant terms ± 1 (that is, $[x^0]\, a = \pm 1$ and $[x^0]\, b = \pm 1$) satisfying $a \equiv b \mod m$, then

$$a^{-1} \equiv b^{-1} \mod m.$$

(A7) If m is an integer and n is a nonnegative integer, and if a and b are two IFPSs satisfying $a \equiv b \mod m$, then

$$a^n \equiv b^n \mod m.$$

Furthermore, if a and b have constant terms ± 1, then this also holds for negative n.

(You can use all these seven facts without proof, but if you are curious: Fact (A1) is essentially obvious; facts (A2)–(A5) are proven just as for integers. Fact (A6) follows by observing that $a^{-1} - b^{-1} = -a^{-1}b^{-1}(a - b)$, since the assumption on the constant terms forces a^{-1} and b^{-1} to be well-defined IFPSs. Finally, fact (A7) is proven by forwards induction for $n \geq 0$ and then by backwards induction for $n < 0$.)

Now, let p be a prime.

(a) Prove that $(1 + x)^p \equiv 1 + x^p \mod p$.

(b) Prove *Lucas's congruence*: Any $a, b \in \mathbb{Z}$ and $c, d \in \{0, 1, \ldots, p - 1\}$ satisfy

$$\binom{ap + c}{bp + d} \equiv \binom{a}{b}\binom{c}{d} \mod p.$$

(c) Prove that if $m \in \mathbb{N}$, and if a and b are two IFPSs satisfying $a \equiv b \mod m$, then $a^m \equiv b^m \mod m^2$.

(d) Prove that any $a, b \in \mathbb{Z}$ satisfy

$$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^2.$$

[**Hint: (a)** What do you remember about $\dbinom{p}{k}$?

(b) $(1 + x)^{ap+c} = ((1 + x)^p)^a (1 + x)^c$. It is helpful to define $[x^n]\, f = 0$ for any negative n and any FPS f.

(c) Write a as $b + mc$ for some IFPS c. The same congruence holds for integers.]

4.2 Solution

[...]

5 Exercise 5

5.1 Problem

For each $n \in \mathbb{N}$, we define two FPSs P_n and S_n by

$$P_n = \prod_{s=1}^{n} (1 - x^s) \qquad \text{and} \qquad S_n = \sum_{s=-n}^{n} (-1)^s x^{s(3s+1)/2}.$$

(For example,

$$P_4 = (1-x)\left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right) = 1 - x - x^2 + 2x^5 - x^8 - x^9 + x^{10}$$

and

$$S_4 = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26}. \ \ )$$

(a) Show that S_n is well-defined, i.e., that all of the exponents $s(3s+1)/2$ are nonnegative integers (even when $s < 0$).

(b) Set

$$F_n = \sum_{s=0}^{n} (-1)^s \frac{P_n}{P_s} x^{sn+s(s+1)/2} \qquad \text{for each } n \in \mathbb{N}.$$

Show that $F_n = S_n$ for each $n \in \mathbb{N}$.

[**Hint:** **(b)** Induction on n. In the induction step, use $P_n = P_{n-1} - x^n P_{n-1}$ to split the sum defining F_n into two subsums after first splitting off the $s = n$ addend. This leads to $F_n - F_{n-1} = S_n - S_{n-1}$.]

5.2 Remark

If we take the limit $n \to \infty$ in the claim of part **(b)** (see [Loehr11, §7.5] for the meaning of "limit" here), then we quickly obtain

$$\prod_{s=1}^{\infty} (1 - x^s) = \sum_{s=-\infty}^{\infty} (-1)^s x^{s(3s+1)/2}.$$

(Indeed, each of the addends $(-1)^s \dfrac{P_n}{P_s} x^{sn+s(s+1)/2}$ for $s > 0$ tends to 0 when $n \to \infty$.) This is Euler's pentagonal number theorem again.

5.3 Solution

[...]

6 Exercise 6

6.1 Problem

Recall that we are using the following notations:

- A *partition* means a weakly decreasing sequence of positive integers. (Thus, a partition is the same as a partition of some $n \in \mathbb{N}$.)
- Given any $n \in \mathbb{Z}$, we let $p(n)$ denote the number of all partitions of n. (This is 0 when $n < 0$.)
- Given any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we let $p_k(n)$ denote the number of all partitions of n into k parts.

Prove the following:

(a) Any $n \in \mathbb{N}$ satisfies

$$np(n) = \sum_{k=1}^{n} \sigma(k) p(n-k),$$

where $\sigma(k)$ denotes the sum of all positive divisors of k.

(b) Any $n \in \mathbb{N}$ satisfies

$$\sum_{k=0}^{n} k p_k(n) = \sum_{k=1}^{n} \partial(k) p(n-k),$$

where $\partial(k)$ denotes the number of positive divisors of k.

(c) Let $a : \{1, 2, 3, \ldots\} \to \mathbb{Q}$ be any map. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we let $a(\lambda)$ denote the sum $a(\lambda_1) + a(\lambda_2) + \cdots + a(\lambda_k)$. Then, any $n \in \mathbb{N}$ satisfies

$$\sum_{\lambda \vdash n} a(\lambda) = \sum_{k=1}^{n} \left(\sum_{d \mid k} a(d) \right) p(n-k).$$

Here, the summation sign "$\sum_{\lambda \vdash n}$" means "sum over all partitions λ of n", whereas the summation sign "$\sum_{d \mid k}$" means "sum over all positive divisors d of k".

[**Hint:** Parts **(a)** and **(b)** are particular cases of **(c)**. Particular cases are not always easier to prove than generalizations.]

6.2 Solution

[...]

7 Exercise 7

7.1 Problem

Let n be an even positive integer.

(a) For each $\sigma \in S_n$, prove that there exist two distinct elements i and j of $[n]$ such that $\sigma(i) - i \equiv \sigma(j) - j \mod n$.

(b) If n people are seated around a table at lunch, and the same n people are seated around the same table at dinner, then prove that you can find two distinct people which have the same distance at the lunch as they have at the dinner. (The *distance* between two people means the number of persons sitting between them, counted along the shorter arc, plus 1. For instance, two neighbors will have distance 1.)

(c) Prove that both **(a)** and **(b)** are false if $n = 5$.

(d) What about $n = 7$?

[**Hint:** **(a)** If there are no such i and j, what can you say about the remainders of the n numbers $\sigma(1) - 1, \sigma(2) - 2, \ldots, \sigma(n) - n$ modulo n, and what can you say about $\sum_{i=1}^{n} (\sigma(i) - i)$?]

7.2 Remark

What about $n = 9$? What about other odd n ? (Talking about part **(b)** in particular; the answer for part **(a)** is simple.)

7.3 Solution

[...]

References

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

[Grinbe18] Darij Grinberg, *Solutions to UMN Spring 2018 Math 4707 homework set #3*.
`http://www.cip.ifi.lmu.de/~grinberg/t/18s/hw3s.pdf`

[Loehr11] Nicholas A. Loehr, *Bijective Combinatorics*, Chapman & Hall/CRC 2011.