University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Midterm 2 (preliminary version)

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Notations

Here is a list of notations that are used in this homework:

- We shall use the Iverson bracket notation as well as the notation $[n]$ for the set $\{1, 2, \ldots, n\}$ (when $n \in \mathbb{Z}$).
- If $n \in \mathbb{N}$, then S_n denotes the set of all permutations of $[n]$.
- If $n \in \mathbb{N}$ and $\sigma \in S_n$, then:
  - a *descent* of the permutation σ denotes an element $k \in [n-1]$ satisfying $\sigma(k) > \sigma(k+1)$.
  - the *descent set* $\operatorname{Des} \sigma$ of σ is defined as the set of all descents of σ.
  - the *descent number* $\operatorname{des} \sigma$ of σ is defined as the number of all descents of σ (that is, $\operatorname{des} \sigma = |\operatorname{Des} \sigma|$).
  - the *one-line notation* $\operatorname{OLN} \sigma$ of σ is defined as the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n))$. Often, this n-tuple is written with square brackets, i.e., as $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$.
  - for each $i \in [n]$, we define $\ell_i(\sigma)$ to be the number of all $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$.

  - we say that σ is *312-avoiding* if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(j) < \sigma(k) < \sigma(i)$.
  - we say that σ is *321-avoiding* if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(k) < \sigma(j) < \sigma(i)$.

- For any $n \in \mathbb{N}$ and any $i \in [n-1]$, we let s_i denote the permutation in S_n that swaps i with $i+1$ while leaving all other elements of $[n]$ unchanged. (This assumes that n is determined by the context.)

- For any $n \in \mathbb{N}$ and any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\operatorname{cyc}_{i_1, i_2, \ldots, i_k}$ be the permutation in S_n that sends $i_1, i_2, \ldots, i_{k-1}, i_k$ to $i_2, i_3, \ldots, i_k, i_1$ (respectively) while leaving all the other elements of $[n]$ unchanged. (Again, this relies on n being clear from the context.)

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, the notation $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ denotes the number of all permutations $\sigma \in S_n$ having exactly k descents. This is called an *Eulerian number*.

- If X is a set, and if $\alpha : X \to X$ and $\beta : X \to X$ are two maps, then the composition $\alpha \circ \beta : X \to X$ is simply denoted by $\alpha\beta$, and is called the *product* of α and β. This notation is used for permutations, in particular.

- If X is a set, if $k \in \mathbb{N}$, and if $f : X \to X$ is any map, then the map $f^k : X \to X$ is defined by
$$f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}} = \underbrace{ff \cdots f}_{k \text{ times}}.$$
This map f^k is called the *k-th power of f* (or *k-th composition power of f*). These powers behave as one would expect as long as you have only one map $f : X \to X$ (meaning that $f^{a+b} = f^a f^b$ and $f^{ab} = (f^a)^b$ for any $a, b \in \mathbb{N}$); but be careful with several maps (e.g., two maps $f : X \to X$ and $g : X \to X$ don't always satisfy $(fg)^a = f^a g^a$). See [Grinbe16, Section 2.13.8] for details (where I write $f^{\circ k}$ instead of f^k).

- If X is a set, and if $f : X \to X$ is a map, then:
  - we say that f is an *involution* if and only if $f^2 = \mathrm{id}$. (Note that every involution is automatically a permutation.)
  - we say that f is *fixed-point-free* if each $x \in X$ satisfies $f(x) \neq x$ (that is, if f has no fixed points). (Note that the fixed-point-free permutations are precisely the derangements.)

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1 Exercise 1

1.1 Problem

Let n and k be positive integers.

For each $i \in \{0, 1, \ldots, n-1\}$ and $\tau \in S_{n-1}$, we let $\tau^i\text{–} \in S_n$ be the permutation such that

$$\operatorname{OLN}\left(\tau^i\text{–}\right) = \left(\tau(1), \tau(2), \ldots, \tau(i), n, \tau(i+1), \tau(i+2), \ldots, \tau(n-1)\right)$$

(that is, $\operatorname{OLN}\left(\tau^i\text{–}\right)$ is obtained from $\operatorname{OLN}\tau$ by inserting an n right after the i-th entry).

(a) Prove that each $i \in \{0, 1, \ldots, n-1\}$ and $\tau \in S_{n-1}$ satisfy

$$\begin{aligned} &\left[\operatorname{des}\left(\tau^i\text{–}\right) = k\right] \\ &= \left[\operatorname{des}\tau = k-1 \text{ and } \tau(i) < \tau(i+1)\right] + \left[\operatorname{des}\tau = k \text{ and } \tau(i) > \tau(i+1)\right], \end{aligned}$$

where we set $\tau(0) = 0$ and $\tau(n) = 0$.

(b) Prove that the map

$$\begin{aligned} \{0, 1, \ldots, n-1\} \times S_{n-1} &\to S_n, \\ (i, \tau) &\mapsto \tau^i\text{–} \end{aligned}$$

is a bijection.

(c) Prove that

$$\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle.$$

1.2 Solution sketch

(a) Let $i \in \{0, 1, \ldots, n-1\}$ and $\tau \in S_{n-1}$. Set $\tau(0) = 0$ and $\tau(n) = 0$.

Let us first recall some definitions: If $\sigma \in S_m$ is a permutation for some $m \in \mathbb{N}$, then the descents of σ are the numbers $g \in [m-1]$ satisfying $\sigma(g) > \sigma(g+1)$; in other words, the descents of σ are the positions at which an entry in $\operatorname{OLN}\sigma$ is followed by a smaller entry. Furthermore, $\operatorname{des}\sigma$ is the number of these descents.

Now, we have

$$\begin{aligned} \operatorname{OLN}\tau &= \left(\tau(1), \tau(2), \ldots, \tau(i), \tau(i+1), \tau(i+2), \ldots, \tau(n-1)\right) \qquad \text{and} \\ \operatorname{OLN}\left(\tau^i\text{–}\right) &= \left(\tau(1), \tau(2), \ldots, \tau(i), n, \tau(i+1), \tau(i+2), \ldots, \tau(n-1)\right) \end{aligned}$$

(by the definition of $\tau^i\text{–}$). Thus, the descents of $\tau^i\text{–}$ can be characterized as follows:

- Any number $g \in [i-1]$ is a descent of $\tau^i\text{–}$ if and only if it is a descent of τ (because the first i entries of $\operatorname{OLN}\left(\tau^i\text{–}\right)$ are precisely the first i entries of $\operatorname{OLN}\tau$).
- The number i is never a descent of $\tau^i\text{–}$ (since $\tau(i) > n$ never holds).
- If $i \neq n-1$, then the number $i+1$ is always a descent of $\tau^i\text{–}$ (since $n > \tau(i+1)$ always holds).
- Any number $g \in \{i+2, i+3, \ldots, n-1\}$ is a descent of $\tau^i\text{–}$ if and only if $g-1$ is a descent of τ (because the last $n-1-i$ entries of $\operatorname{OLN}\left(\tau^i\text{–}\right)$ are precisely the last $n-1-i$ entries of $\operatorname{OLN}\tau$).

Thus, when we go from τ to $\tau^i\text{–}$, the descents are "more or less" preserved in the sense that

- some of the descents (namely, those that are larger than i) get shifted by 1;

- the descent i is lost (if i was a descent of τ to begin with); and
- a descent $i+1$ is created if $i \neq n-1$.

Hence, the total number of descents decreases by 1 if i was a descent of τ, and furthermore increases by 1 if $i \neq n-1$. In other words,

$$\begin{aligned}
\operatorname{des}\left(\tau^{i}-\right) &= \operatorname{des}\tau - \underbrace{[i \text{ is a descent of } \tau]}_{=[i \in \operatorname{Des}\tau]} + [i \neq n-1] = \operatorname{des}\tau - [i \in \operatorname{Des}\tau] + [i \neq n-1] \\
&= \operatorname{des}\tau + [i \neq n-1] - [i \in \operatorname{Des}\tau]. \qquad (1)
\end{aligned}$$

But it is easy to see that

$$[i \neq n-1] - [i \in \operatorname{Des}\tau] = [\tau(i) < \tau(i+1)]. \qquad (2)$$

[*Proof of* (2)*:* In order to prove (2), we can distinguish between the following three cases:

- *Case 1:* We have $i \neq n-1$ and $\tau(i) < \tau(i+1)$.
- *Case 2:* We have $i \neq n-1$ and $\tau(i) > \tau(i+1)$.
- *Case 3:* We have $i = n-1$.

(No other cases can occur, because $\tau(i) \neq \tau(i+1)$.)

In Case 1, the equality (2) boils down to $1 - 0 = 1$ (since $\tau(i) < \tau(i+1)$ yields $i \notin \operatorname{Des}\tau$), which is true.

In Case 2, the equality (2) boils down to $1 - 1 = 0$ (since $\tau(i) > \tau(i+1)$ yields $i \in \operatorname{Des}\tau$), which is true.

In Case 3, the equality (2) boils down to $0 - 0 = 0$ (since $i = n-1$ yields $n = i+1$ and thus $\tau(i) > 0 = \tau\left(\underbrace{n}_{=i+1}\right) = \tau(i+1)$), which is true.

Thus, the equality (2) is proven in all three cases.]

Now, (1) becomes

$$\operatorname{des}\left(\tau^{i}-\right) = \operatorname{des}\tau + \underbrace{[i \neq n-1] - [i \in \operatorname{Des}\tau]}_{\substack{=[\tau(i)<\tau(i+1)] \\ (\text{by } (2))}} = \operatorname{des}\tau + [\tau(i) < \tau(i+1)].$$

Hence,

$$\begin{aligned}
&\left[\operatorname{des}\left(\tau^{i}-\right)=k\right] \\
&=\left[\operatorname{des}\tau+\left[\tau\left(i\right)<\tau\left(i+1\right)\right]=k\right] \\
&=\left[\tau\left(i\right)<\tau\left(i+1\right)\text{ and }\operatorname{des}\tau+\underbrace{\left[\tau\left(i\right)<\tau\left(i+1\right)\right]}_{\substack{=1\\ \text{(since }\tau\left(i\right)<\tau\left(i+1\right)\text{)}}}=k\right] \\
&\qquad+\left[\tau\left(i\right)>\tau\left(i+1\right)\text{ and }\operatorname{des}\tau+\underbrace{\left[\tau\left(i\right)<\tau\left(i+1\right)\right]}_{\substack{=0\\ \text{(since }\tau\left(i\right)>\tau\left(i+1\right)\text{)}}}=k\right] \\
&\qquad\left(\begin{array}{c}\text{since we always have either }\tau\left(i\right)<\tau\left(i+1\right)\text{ or }\tau\left(i\right)>\tau\left(i+1\right),\\ \text{but never both at once}\end{array}\right) \\
&=\underbrace{\left[\tau\left(i\right)<\tau\left(i+1\right)\text{ and }\operatorname{des}\tau+1=k\right]}_{\substack{=\left[\tau\left(i\right)<\tau\left(i+1\right)\text{ and }\operatorname{des}\tau=k-1\right]\\ =\left[\operatorname{des}\tau=k-1\text{ and }\tau\left(i\right)<\tau\left(i+1\right)\right]}}+\underbrace{\left[\tau\left(i\right)>\tau\left(i+1\right)\text{ and }\operatorname{des}\tau=k\right]}_{=\left[\operatorname{des}\tau=k\text{ and }\tau\left(i\right)>\tau\left(i+1\right)\right]} \\
&=\left[\operatorname{des}\tau=k-1\text{ and }\tau\left(i\right)<\tau\left(i+1\right)\right]+\left[\operatorname{des}\tau=k\text{ and }\tau\left(i\right)>\tau\left(i+1\right)\right].
\end{aligned}$$

This solves part **(a)** of the exercise.

(b) The map

$$\begin{aligned}
\left\{0,1,\ldots,n-1\right\}\times S_{n-1}&\to S_{n}, \\
\left(i,\tau\right)&\mapsto\tau^{i}-
\end{aligned}$$

is invertible. In fact, its inverse is the map that sends each $\sigma\in S_n$ to the pair $\left(i,\tau\right)\in\left\{0,1,\ldots,n-1\right\}\times S_{n-1}$, where $i=\sigma^{-1}\left(n\right)-1$ and where $\tau\in S_{n-1}$ is (uniquely) determined by

$$\operatorname{OLN}\tau=\left(\sigma\left(1\right),\sigma\left(2\right),\ldots,\sigma\left(i\right),\sigma\left(i+2\right),\sigma\left(i+3\right),\ldots,\sigma\left(n\right)\right).$$

(Proving this is straightforward.)

(c) We WLOG assume that $n\neq 1$ (since the proof in the case $n=1$ is straightforward). Thus, $n-1\neq 0$ and $n>1$.

The definition of $\left\langle\begin{matrix}n\\k\end{matrix}\right\rangle$ yields

$$\begin{aligned}
\left\langle\begin{matrix}n\\k\end{matrix}\right\rangle&=\left(\text{the number of all permutations }\sigma\in S_{n}\text{ having exactly }k\text{ descents}\right) \\
&=\left(\text{the number of all permutations }\sigma\in S_{n}\text{ such that }\operatorname{des}\sigma=k\right) \\
&=\left|\left\{\sigma\in S_{n}\ \mid\ \operatorname{des}\sigma=k\right\}\right| \\
&=\sum_{\sigma\in S_{n}}\left[\operatorname{des}\sigma=k\right] \qquad (3)
\end{aligned}$$

(since

$$\sum_{\sigma\in S_n}[\operatorname{des}\sigma=k]=\sum_{\substack{\sigma\in S_n;\\ \operatorname{des}\sigma=k}}\underbrace{[\operatorname{des}\sigma=k]}_{\substack{=1\\ (\text{since }\operatorname{des}\sigma=k)}}+\sum_{\substack{\sigma\in S_n;\\ \operatorname{des}\sigma\neq k}}\underbrace{[\operatorname{des}\sigma=k]}_{\substack{=0\\ (\text{since }\operatorname{des}\sigma\neq k)}}=\sum_{\substack{\sigma\in S_n;\\ \operatorname{des}\sigma=k}}1+\underbrace{\sum_{\substack{\sigma\in S_n;\\ \operatorname{des}\sigma\neq k}}0}_{=0}$$
$$=\sum_{\substack{\sigma\in S_n;\\ \operatorname{des}\sigma=k}}1=\left|\{\sigma\in S_n\ \mid\ \operatorname{des}\sigma=k\}\right|\cdot 1=\left|\{\sigma\in S_n\ \mid\ \operatorname{des}\sigma=k\}\right|$$

). The same argument (applied to $n-1$ instead of n) yields

$$\left\langle {n-1 \atop k} \right\rangle=\sum_{\sigma\in S_{n-1}}[\operatorname{des}\sigma=k]=\sum_{\tau\in S_{n-1}}[\operatorname{des}\tau=k]. \tag{4}$$

The same argument (applied to $k-1$ instead of k) yields

$$\left\langle {n-1 \atop k-1} \right\rangle=\sum_{\tau\in S_{n-1}}[\operatorname{des}\tau=k-1]. \tag{5}$$

Now, (3) becomes

$$\left\langle {n \atop k} \right\rangle=\sum_{\sigma\in S_n}[\operatorname{des}\sigma=k]=\underbrace{\sum_{(i,\tau)\in\{0,1,\ldots,n-1\}\times S_{n-1}}}_{=\sum_{\tau\in S_{n-1}}\sum_{i\in\{0,1,\ldots,n-1\}}}\underbrace{\left[\operatorname{des}\left(\tau^{i}-\right)=k\right]}_{\substack{=[\operatorname{des}\tau=k-1\text{ and }\tau(i)<\tau(i+1)]+[\operatorname{des}\tau=k\text{ and }\tau(i)>\tau(i+1)]\\ (\text{by part }\textbf{(a)}\text{ of the exercise})}}$$
$$\left(\begin{array}{c}\text{here, we have substituted }\tau^{i}-\text{ for }\sigma\text{ in the sum,}\\ \text{since the map }\{0,1,\ldots,n-1\}\times S_{n-1}\to S_n,\ (i,\tau)\mapsto\tau^{i}-\\ \text{is a bijection (by part }\textbf{(b)}\text{ of the exercise)}\end{array}\right)$$
$$=\sum_{\tau\in S_{n-1}}\sum_{i\in\{0,1,\ldots,n-1\}}\left([\operatorname{des}\tau=k-1\text{ and }\tau(i)<\tau(i+1)]+[\operatorname{des}\tau=k\text{ and }\tau(i)>\tau(i+1)]\right)$$
$$=\sum_{\tau\in S_{n-1}}\sum_{i\in\{0,1,\ldots,n-1\}}\underbrace{[\operatorname{des}\tau=k-1\text{ and }\tau(i)<\tau(i+1)]}_{=[\operatorname{des}\tau=k-1][\tau(i)<\tau(i+1)]}$$
$$+\sum_{\tau\in S_{n-1}}\sum_{i\in\{0,1,\ldots,n-1\}}\underbrace{[\operatorname{des}\tau=k\text{ and }\tau(i)>\tau(i+1)]}_{=[\operatorname{des}\tau=k-1][\tau(i)>\tau(i+1)]}$$
$$=\sum_{\tau\in S_{n-1}}[\operatorname{des}\tau=k-1]\sum_{i\in\{0,1,\ldots,n-1\}}[\tau(i)<\tau(i+1)]$$
$$+\sum_{\tau\in S_{n-1}}[\operatorname{des}\tau=k]\sum_{i\in\{0,1,\ldots,n-1\}}[\tau(i)>\tau(i+1)]. \tag{6}$$

But each $\tau \in S_{n-1}$ satisfies

$$\begin{aligned}
&\sum_{i\in\{0,1,\ldots,n-1\}} [\tau(i) < \tau(i+1)] \\
&= \underbrace{[\tau(0) < \tau(1)]}_{\substack{=1 \\ (\text{since } \tau(0)=0<\tau(1))}} + \sum_{i\in[n-2]} \underbrace{[\tau(i) < \tau(i+1)]}_{\substack{=[\text{not } \tau(i)\geq\tau(i+1)] \\ =[\text{not } \tau(i)>\tau(i+1)] \\ (\text{since } \tau(i)\neq\tau(i+1))}} + \underbrace{[\tau(n-1) < \tau(n)]}_{\substack{=0 \\ (\text{since } \tau(n-1)>0=\tau(n))}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addends for } i=0 \text{ and for } i=n-1 \\ \text{from the sum (and these are indeed two distinct addends,} \\ \text{since } n-1\neq 0)\end{array}\right) \\
&= 1 + \underbrace{\sum_{i\in[n-2]} [\text{not } \tau(i) > \tau(i+1)]}_{=|\{i\in[n-2] \ | \ \text{not } \tau(i)>\tau(i+1)\}|} \\
&= 1 + \left| \underbrace{\{i\in[n-2] \ | \ \text{not } \tau(i) > \tau(i+1)\}}_{\substack{=[n-2]\setminus\operatorname{Des}\tau \\ (\text{since } \tau\in S_{n-1} \text{ and thus } \operatorname{Des}\tau=\{i\in[n-2] \ | \ \tau(i)>\tau(i+1)\})}} \right| = 1 + \underbrace{|[n-2]\setminus\operatorname{Des}\tau|}_{\substack{=(n-2)-|\operatorname{Des}\tau| \\ (\text{since } \operatorname{Des}\tau\subseteq[n-2])}} \\
&= 1 + (n-2) - |\operatorname{Des}\tau| = n-1-\underbrace{|\operatorname{Des}\tau|}_{=\operatorname{des}\tau} = n-1-\operatorname{des}\tau
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i\in\{0,1,\ldots,n-1\}} [\tau(i) > \tau(i+1)] \\
&= \underbrace{[\tau(0) > \tau(1)]}_{\substack{=0 \\ (\text{since } \tau(0)=0<\tau(1))}} + \underbrace{\sum_{i\in[n-2]} [\tau(i) > \tau(i+1)]}_{=|\{i\in[n-2] \ | \ \tau(i)>\tau(i+1)\}|} + \underbrace{[\tau(n-1) > \tau(n)]}_{\substack{=1 \\ (\text{since } \tau(n-1)>0=\tau(n))}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addends for } i=0 \text{ and for } i=n-1 \\ \text{from the sum (and these are indeed two distinct addends,} \\ \text{since } n-1\neq 0)\end{array}\right) \\
&= 1 + \left| \underbrace{\{i\in[n-2] \ | \ \tau(i) > \tau(i+1)\}}_{\substack{=\operatorname{Des}\tau \\ (\text{since } \tau\in S_{n-1})}} \right| = 1 + \underbrace{|\operatorname{Des}\tau|}_{=\operatorname{des}\tau} = 1 + \operatorname{des}\tau = \operatorname{des}\tau + 1.
\end{aligned}$$

Hence, (6) becomes

$$\begin{aligned}
\left\langle {n \atop k} \right\rangle &= \sum_{\tau \in S_{n-1}} [\operatorname{des} \tau = k-1] \underbrace{\sum_{i \in \{0,1,\ldots,n-1\}} [\tau(i) < \tau(i+1)]}_{= n-1-\operatorname{des}\tau} \\
&\qquad + \sum_{\tau \in S_{n-1}} [\operatorname{des} \tau = k] \underbrace{\sum_{i \in \{0,1,\ldots,n-1\}} [\tau(i) > \tau(i+1)]}_{= \operatorname{des}\tau + 1} \\
&= \sum_{\tau \in S_{n-1}} \underbrace{[\operatorname{des} \tau = k-1] \cdot (n-1-\operatorname{des}\tau)}_{\substack{= [\operatorname{des}\tau = k-1] \cdot (n-1-(k-1)) \\ \text{(indeed, this equality clearly holds} \\ \text{when } \operatorname{des}\tau = k-1\text{; but when } \operatorname{des}\tau \neq k-1, \\ \text{it simply boils down to } 0=0)}} + \sum_{\tau \in S_{n-1}} \underbrace{[\operatorname{des} \tau = k] \cdot (\operatorname{des}\tau + 1)}_{\substack{= [\operatorname{des}\tau = k] \cdot (k+1) \\ \text{(indeed, this equality clearly holds} \\ \text{when } \operatorname{des}\tau = k\text{; but when } \operatorname{des}\tau \neq k, \\ \text{it simply boils down to } 0=0)}} \\
&= \underbrace{\sum_{\tau \in S_{n-1}} [\operatorname{des} \tau = k-1]}_{\substack{= \left\langle {n-1 \atop k-1} \right\rangle \\ \text{(by (5))}}} \cdot \underbrace{(n-1-(k-1))}_{= n-k} + \underbrace{\sum_{\tau \in S_{n-1}} [\operatorname{des} \tau = k]}_{\substack{= \left\langle {n-1 \atop k} \right\rangle \\ \text{(by (4))}}} \cdot (k+1) \\
&= \left\langle {n-1 \atop k-1} \right\rangle \cdot (n-k) + \left\langle {n-1 \atop k} \right\rangle \cdot (k+1) = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle .
\end{aligned}$$

This solves part **(c)** of the exercise.

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. For each $i \in [n]$, let

$$a_i = \operatorname{cyc}_{i', i'-1, \ldots, i} = s_{i'-1} s_{i'-2} \cdots s_i \in S_n, \qquad \text{where } i' = i + \ell_i(\sigma).$$

Prove that $\sigma = a_1 a_2 \cdots a_n$.

[**Hint:** Prove, "more generally", that if $j \in \{0, 1, \ldots, n\}$ is such that $1, 2, \ldots, j$ are fixed points of σ, then $\sigma = a_{j+1} a_{j+2} \cdots a_n$.]

2.2 Remark

This exercise shows a direct way of expressing every $\sigma \in S_n$ as a product of $\ell(\sigma)$ many simple transpositions (indeed, it represents σ as the product $a_1 a_2 \cdots a_n$, but we can then rewrite each a_i as $s_{i'-1} s_{i'-2} \cdots s_i$, which turns $a_1 a_2 \cdots a_n$ into a product of $\ell_1(\sigma) + \ell_2(\sigma) + \cdots + \ell_n(\sigma) = \ell(\sigma)$ many simple transpositions). This way is occasionally stated visually in terms of the Rothe diagram of σ (see, for example, `https://sumidiot.blogspot.com/2008/05/rothe-diagram.html` or [Kerber99, Corollary 11.3.5]).

The exercise also appears in [Grinbe16, Exercise 5.21 **(c)**]. The solution I give below follows the same strategy as the solution given in [Grinbe16], but differs in the execution.

2.3 Solution sketch

Forget that we fixed n and σ. First, we need the following definition:

Definition 2.1. Let T and Q be two subsets of $\mathbb{Z}$. Let $f : T \to Q$ be any map. Then, we say that the map f is *strictly increasing* if every two elements t_1 and t_2 of T satisfying $t_1 < t_2$ satisfy $f(t_1) < f(t_2)$. (Thus, if $T = \{i_1 < i_2 < \cdots < i_k\}$ is a finite set, then f is strictly increasing if and only if $f(i_1) < f(i_2) < \cdots < f(i_k)$.) It is easy to see that if f is strictly increasing, then for any two elements t_1 and t_2 of T, we have the logical equivalence

$$(t_1 > t_2) \iff (f(t_1) > f(t_2)).$$

Note that this notion of "strictly increasing" is defined for maps on arbitrary subsets of $\mathbb{Z}$, not just on intervals.

Example 2.2. Let $\pi \in S_5$ be the permutation given in one-line notation as $[2, 1, 4, 3, 5]$. Then, π itself is not strictly increasing (since, for example, $1 < 2$ but we don't have $\pi(1) < \pi(2)$). But its restriction $\pi \mid_{\{1,4,5\}} : \{1, 4, 5\} \to [5]$ is strictly increasing (since the images of $1, 4, 5$ under π are $2, 3, 5$, and these satisfy $2 < 3 < 5$).

Lemma 2.3. *Let $n \in \mathbb{N}$. Let $a, b \in [n]$ satisfy $a \leq b$. Let $\alpha = \operatorname{cyc}_{a,a+1,\ldots,b} \in S_n$. Let U be a subset of $[n]$ such that $b \notin U$. Then, the restriction $\alpha \mid_U : U \to [n]$ is strictly increasing.*

Proof of Lemma 2.3. We have $U \subseteq [n] \setminus \{b\}$ (since U is a subset of $[n]$ such that $b \notin U$). Thus, the restriction $\alpha \mid_U : U \to [n]$ is a restriction of the restriction $\alpha \mid_{[n]\setminus\{b\}} : [n] \setminus \{b\} \to [n]$ (because if X, Y and Z are three sets such that $X \subseteq Y \subseteq Z$, and if f is a map from Z, then $f \mid_X = (f \mid_Y) \mid_X$). Note that a restriction of a strictly increasing map to a subset is always strictly increasing.

But $\alpha = \operatorname{cyc}_{a,a+1,\ldots,b}$. Hence, the map α sends the elements

$$\begin{array}{cccccccccccccc} & 1, & 2, & \ldots, & a-1, & a, & a+1, & \ldots, & b-1, & b+1, & b+2, & \ldots, & n \\ \text{to} & 1, & 2, & \ldots, & a-1, & a+1, & a+2, & \ldots, & b, & b+1, & b+2, & \ldots, & n, \end{array}$$

respectively. In other words, the map α sends the elements $1, 2, \ldots, b-1, b+1, b+2, \ldots, n$ to $1, 2, \ldots, a-1, a+1, a+2, \ldots, n$ in this order. Thus, its restriction $\alpha \mid_{[n]\setminus\{b\}} : [n] \setminus \{b\} \to [n]$ is strictly increasing (since $1 < 2 < \cdots < a-1 < a+1 < a+2 < \cdots < n$). Hence, the restriction $\alpha \mid_U : U \to [n]$ is strictly increasing as well (since it is a restriction of the restriction $\alpha \mid_{[n]\setminus\{b\}} : [n] \setminus \{b\} \to [n]$). Hence, Lemma 2.3 is proven. $\square$

Lemma 2.4. *Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. Let $i \in [n]$. Then:*

(a) *We have $i + \ell_i(\sigma) \in [n]$ and $i + \ell_i(\sigma) \geq i$.*

(b) *We have $\ell_i(\sigma) = |[\sigma(i) - 1] \setminus \sigma([i-1])|$.*

(c) *Let $\alpha \in S_n$ be a further permutation such that the restriction $\alpha \mid_{\sigma(\{i,i+1,\ldots,n\})} : \sigma(\{i, i+1, \ldots, n\}) \to [n]$ is strictly increasing. Let $\tau = \alpha \circ \sigma$. Then, $\ell_i(\tau) = \ell_i(\sigma)$.*

Proof of Lemma 2.4. Recall that $\ell_i(\sigma)$ was defined as the number of all $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$. Thus, $\ell_i(\sigma)$ is $\leq$ to the number of **all** $j \in \{i+1, i+2, \ldots, n\}$. In other words, $\ell_i(\sigma)$ is $\leq$ to $n - i$ (since the number of all $j \in \{i+1, i+2, \ldots, n\}$ is $n - i$). In other words, $\ell_i(\sigma) \leq n - i$, so that $i + \ell_i(\sigma) \leq n$ and thus $i + \ell_i(\sigma) \in [n]$ (because $i + \underbrace{\ell_i(\sigma)}_{\geq 0} \geq i \geq 1$). Moreover, $i + \underbrace{\ell_i(\sigma)}_{\geq 0} \geq i$. This proves Lemma 2.4 **(a)**.

(b) This is [Grinbe16, Lemma 5.48 **(b)**]; see [Grinbe16, solution to Exercise 5.18] for a detailed proof. Here is a sketch: The definition of $\ell_i(\sigma)$ rewrites as

$$\ell_i(\sigma) = |\{j \in \{i+1, i+2, \ldots, n\} \mid \sigma(i) > \sigma(j)\}|.$$

But there is a bijection $\{j \in \{i+1, i+2, \ldots, n\} \mid \sigma(i) > \sigma(j)\} \to [\sigma(i) - 1] \setminus \sigma([i-1])$ (namely, the map sending each j to $\sigma(j)$); so we have

$$|\{j \in \{i+1, i+2, \ldots, n\} \mid \sigma(i) > \sigma(j)\}| = |[\sigma(i) - 1] \setminus \sigma([i-1])| .$$

Combining these two equations, we obtain Lemma 2.4 **(b)**.

(c) Recall that $\ell_i(\sigma)$ was defined as the number of all $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$. Likewise, $\ell_i(\tau)$ was defined as the number of all $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\tau(i) > \tau(j)$. Comparing these two definitions, we see that in order to prove $\ell_i(\tau) = \ell_i(\sigma)$, it suffices to prove that the $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$ are precisely the $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\tau(i) > \tau(j)$. In other words, it suffices to prove that for each $j \in \{i+1, i+2, \ldots, n\}$, we have the logical equivalence $(\sigma(i) > \sigma(j)) \iff (\tau(i) > \tau(j))$.

So let us prove this. Fix $j \in \{i+1, i+2, \ldots, n\}$. Then, both i and j belong to the set $\{i, i+1, \ldots, n\}$. Hence, both $\sigma(i)$ and $\sigma(j)$ belong to the set $\sigma(\{i, i+1, \ldots, n\})$. Therefore, we have the logical equivalence

$$(\sigma(i) > \sigma(j)) \iff \left(\left(\alpha \mid_{\sigma(\{i,i+1,\ldots,n\})}\right)(\sigma(i)) > \left(\alpha \mid_{\sigma(\{i,i+1,\ldots,n\})}\right)(\sigma(j))\right)$$

(since the map $\alpha \mid_{\sigma(\{i,i+1,\ldots,n\})} : \sigma(\{i, i+1, \ldots, n\}) \to [n]$ is strictly increasing).

Thus, we have the following chain of equivalences:

$$\begin{aligned} &(\sigma(i) > \sigma(j)) \\ \iff &\left(\underbrace{\left(\alpha \mid_{\sigma(\{i,i+1,\ldots,n\})}\right)(\sigma(i))}_{=\alpha(\sigma(i))=(\alpha\circ\sigma)(i)} > \underbrace{\left(\alpha \mid_{\sigma(\{i,i+1,\ldots,n\})}\right)(\sigma(j))}_{=\alpha(\sigma(j))=(\alpha\circ\sigma)(j)}\right) \\ \iff &((\alpha \circ \sigma)(i) > (\alpha \circ \sigma)(j)) \iff (\tau(i) > \tau(j)) \qquad (\text{since } \alpha \circ \sigma = \tau). \end{aligned}$$

Hence, we have the logical equivalence $(\sigma(i) > \sigma(j)) \iff (\tau(i) > \tau(j))$. This completes the proof of Lemma 2.4 **(c)**. □

Now, for each $i \in [n]$, the permutations $\operatorname{cyc}_{i',i'-1,\ldots,i}$ and $s_{i'-1}s_{i'-2}\cdots s_i$ appearing in the exercise are well-defined (because Lemma 2.4 **(a)** yields that $i' = i + \ell_i(\sigma)$ satisfies $i' \in [n]$ and $i' \geq i$). Moreover, these permutations are equal (for each $i \in [n]$ separately), because of the following fact:

Lemma 2.5. *Let $n \in \mathbb{N}$. Let i and i' be two elements of $[n]$ such that $i' \geq i$. Then, $\operatorname{cyc}_{i',i'-1,\ldots,i} = s_{i'-1}s_{i'-2}\cdots s_i$.*

Proof of Lemma 2.5. Proposition 4.3 **(a)** in the class notes from 2018-10-17 (applied to $k = i' - i + 1$ and $(i_1, i_2, \ldots, i_k) = (i', i'-1, \ldots, i)$) yields

$$\operatorname{cyc}_{i',i'-1,\ldots,i} = \underbrace{t_{i',i'-1}}_{=t_{i'-1,i'}=s_{i'-1}} \underbrace{t_{i'-1,i'-2}}_{=t_{i'-2,i'-1}=s_{i'-2}} \cdots \underbrace{t_{i+1,i}}_{=t_{i,i+1}=s_i} = s_{i'-1}s_{i'-2}\cdots s_i.$$

This proves Lemma 2.5. □

Thus, the definition of a_i given in the exercise makes sense.

Next, we recall the principle of *backwards induction* – i.e., the following induction principle (stated here in a form tailored to our specific situation):

Theorem 2.6. *Let $n \in \mathbb{N}$. For each $p \in \{0, 1, \ldots, n\}$, let $\mathcal{A}(p)$ be a logical statement.*

Assume the following:

Assumption 1: The statement $\mathcal{A}(n)$ holds.

Assumption 2: If $j \in [n]$ is such that $\mathcal{A}(j)$ holds, then $\mathcal{A}(j-1)$ also holds.

Then, $\mathcal{A}(p)$ holds for each $p \in \{0, 1, \ldots, n\}$.

It is easy to prove Theorem 2.6 by deriving it from the usual induction principle. (Namely, argue by induction on k that $\mathcal{A}(n-k)$ holds for each $k \in \{0, 1, \ldots, n\}$. Assumption 1 provides the induction base, while Assumption 2 provides the induction step.)

Now, let us solve the exercise. We shall follow the hint. Fix $n \in \mathbb{N}$. For each $p \in \{0, 1, \ldots, n\}$, let us define a statement $\mathcal{A}(p)$ as follows:

> *Statement $\mathcal{A}(p)$:* Let $\sigma \in S_n$ be such that $1, 2, \ldots, p$ are fixed points of σ.
>
> For each $i \in \{p+1, p+2, \ldots, n\}$, let $a_i = \operatorname{cyc}_{i', i'-1, \ldots, i}$, where $i' = i + \ell_i(\sigma)$. (This is well-defined, because Lemma 2.4 **(a)** shows that $i' \in [n]$ and $i' \geq i$.) Then, $\sigma = a_{p+1} a_{p+2} \cdots a_n$.

Our goal is to prove that $\mathcal{A}(p)$ holds for each $p \in \{0, 1, \ldots, n\}$. This is, as the hint says, "more general" than the exercise, because the claim of the exercise is precisely the statement $\mathcal{A}(0)$ (indeed, the requirement that $1, 2, \ldots, 0$ are fixed points of σ is vacuously true). But the words "more general" are in quotation marks because all of these statements $\mathcal{A}(p)$ can be easily derived from the exercise, once the latter is solved; they are thus mere stepping stones for our solution.

We take aim at proving that Assumptions 1 and 2 of Theorem 2.6 hold for these statements $\mathcal{A}(0), \mathcal{A}(1), \ldots, \mathcal{A}(n)$:

[*Proof of Assumption 1:* We must prove that if $\sigma \in S_n$ is such that $1, 2, \ldots, n$ are fixed points of σ, then $\sigma = a_{n+1} a_{n+2} \cdots a_n$. But this is clear: If $\sigma \in S_n$ is such that $1, 2, \ldots, n$ are fixed points of σ, then $\sigma = \operatorname{id} = (\text{empty product}) = a_{n+1} a_{n+2} \cdots a_n$. Thus, Statement $\mathcal{A}(n)$ holds. This concludes the proof of Assumption 1.]

[*Proof of Assumption 2:* Let $j \in [n]$ be such that $\mathcal{A}(j)$ holds. We must prove that $\mathcal{A}(j-1)$ also holds.

Let $\sigma \in S_n$ be such that $1, 2, \ldots, j-1$ are fixed points of σ. For each $i \in \{j, j+1, \ldots, n\}$, set $i' = i + \ell_i(\sigma)$ and $a_i = \operatorname{cyc}_{i', i'-1, \ldots, i}$. (This is well-defined, because Lemma 2.4 **(a)** shows that $i' \in [n]$ and $i' \geq i$.) We are going to prove that $\sigma = a_j a_{j+1} \cdots a_n$.

First, however, let us check that $\sigma(j) = j'$. Indeed, $\sigma([j-1]) = [j-1]$ (since $1, 2, \ldots, j-1$ are fixed points of σ). Moreover, $\sigma(j) \geq j$ (because otherwise, we would have $\sigma(j) < j$, so that $\sigma(j) \in [j-1] = \sigma([j-1])$, which would mean that $\sigma(j) = \sigma(k)$ for some $k \in [j-1]$; but this would contradict the injectivity of σ).

Now, Lemma 2.4 **(b)** (applied to j instead of i) yields

$$\ell_j(\sigma) = \left| [\sigma(j) - 1] \setminus \underbrace{\sigma([j-1])}_{=[j-1]} \right| = \left| \underbrace{[\sigma(j) - 1] \setminus [j-1]}_{=\{j, j+1, \ldots, \sigma(j)-1\}} \right| = |\{j, j+1, \ldots, \sigma(j) - 1\}|$$
$$= \sigma(j) - j \qquad (\text{since } \sigma(j) \geq j).$$

Hence, $j + \ell_j(\sigma) = \sigma(j)$. Now, the definition of j' yields $j' = j + \ell_j(\sigma) = \sigma(j)$. But the definition of a_j yields $a_j = \operatorname{cyc}_{j', j'-1, \ldots, j}$, whence $a_j(j) = j' = \sigma(j)$, so that $a_j^{-1}(\sigma(j)) = j$.

We have $a_j = \operatorname{cyc}_{j', j'-1, \ldots, j}$ (by the definition of a_j). Now, each $k \in [j-1]$ satisfies $k \leq j - 1 < j$ and thus $k \notin \{j', j'-1, \ldots, j\}$ (since all elements of $\{j', j'-1, \ldots, j\}$ are

$\geq j$) and therefore $\operatorname{cyc}_{j',j'-1,\ldots,j}(k) = k$ and thus $\underbrace{a_j}_{=\operatorname{cyc}_{j',j'-1,\ldots,j}}(k) = \operatorname{cyc}_{j',j'-1,\ldots,j}(k) = k$ and thus

$$a_j^{-1}(k) = k. \tag{7}$$

Now, let $\tau \in S_n$ be the permutation $a_j^{-1} \circ \sigma$. Then, $\tau(j) = \left(a_j^{-1} \circ \sigma\right)(j) = a_j^{-1}(\sigma(j)) = j$. In other words, j is a fixed point of τ.

Moreover, each $k \in [j-1]$ satisfies $\sigma(k) = k$ (since $1, 2, \ldots, j-1$ are fixed points of σ) and thus

$$\underbrace{\tau}_{=a_j^{-1}\circ\sigma}(k) = \left(a_j^{-1} \circ \sigma\right)(k) = a_j^{-1}\left(\underbrace{\sigma(k)}_{=k}\right) = a_j^{-1}(k) = k \qquad (\text{by } (7)).$$

In other words, $1, 2, \ldots, j-1$ are fixed points of τ. Since j is also a fixed point of τ, we thus conclude that $1, 2, \ldots, j$ are fixed points of τ.

Next, let $i \in \{j+1, j+2, \ldots, n\}$ be arbitrary. Thus, $i \geq j+1 > j$. Therefore, $j' \notin \sigma(\{i, i+1, \ldots, n\})$ [1]. Also, $j' = j + \underbrace{\ell_j(\sigma)}_{\geq 0} \geq j$, so that $j \leq j'$.

Define a permutation $\alpha \in S_n$ by $\alpha = a_j^{-1}$. Then, $\tau = \underbrace{a_j^{-1}}_{=\alpha} \circ \sigma = \alpha \circ \sigma$.

From $a_j = \operatorname{cyc}_{j',j'-1,\ldots,j}$, we obtain $a_j^{-1} = \left(\operatorname{cyc}_{j',j'-1,\ldots,j}\right)^{-1} = \operatorname{cyc}_{j,j+1,\ldots,j'}$. Thus, $\alpha = a_j^{-1} = \operatorname{cyc}_{j,j+1,\ldots,j'}$. Hence, Lemma 2.3 (applied to $a = j$, $b = j'$ and $U = \sigma(\{i, i+1, \ldots, n\})$) yields that the restriction $\alpha\mid_{\sigma(\{i,i+1,\ldots,n\})} : \sigma(\{i, i+1, \ldots, n\}) \to [n]$ is strictly increasing (since $j' \notin \sigma(\{i, i+1, \ldots, n\})$). Hence, Lemma 2.4 **(c)** yields $\ell_i(\tau) = \ell_i(\sigma)$.

Now, forget that we fixed i. We thus have shown that each $i \in \{j+1, j+2, \ldots, n\}$ satisfies

$$\ell_i(\tau) = \ell_i(\sigma). \tag{8}$$

Thus, each $i \in \{j+1, j+2, \ldots, n\}$ satisfies

$$i' = i + \underbrace{\ell_i(\sigma)}_{\substack{=\ell_i(\tau) \\ (\text{by } (8))}} = i + \ell_i(\tau). \tag{9}$$

Now let us see where we stand: The permutation $\tau \in S_n$ has the property that $1, 2, \ldots, j$ are fixed points of τ. For each $i \in \{j+1, j+2, \ldots, n\}$, we have $a_i = \operatorname{cyc}_{i',i'-1,\ldots,i}$, where $i' = i + \ell_i(\tau)$ (by (9)). Hence, we can apply Statement $\mathcal{A}(j)$ (which we have assumed to hold) to τ instead of σ. We thus conclude that $\tau = a_{j+1}a_{j+2}\cdots a_n$. From $\tau = a_j^{-1} \circ \sigma = a_j^{-1}\sigma$, we obtain

$$\sigma = a_j \underbrace{\tau}_{=a_{j+1}a_{j+2}\cdots a_n} = a_j\left(a_{j+1}a_{j+2}\cdots a_n\right) = a_j a_{j+1} \cdots a_n.$$

Now, forget that we fixed σ. We thus have shown that if $\sigma \in S_n$ is such that $1, 2, \ldots, j-1$ are fixed points of σ, and if we set

$$a_i = \operatorname{cyc}_{i',i'-1,\ldots,i} \ (\text{where } i' = i + \ell_i(\sigma)) \qquad \text{for each } i \in \{j, j+1, \ldots, n\},$$

[1] *Proof.* Assume the contrary. Thus, $j' \in \sigma(\{i, i+1, \ldots, n\})$. In other words, there exists some $k \in \{i, i+1, \ldots, n\}$ such that $j' = \sigma(k)$. Consider this k. Comparing $j' = \sigma(k)$ with $j' = \sigma(j)$, we obtain $\sigma(k) = \sigma(j)$, and thus $k = j$ (since σ is injective). From $k \in \{i, i+1, \ldots, n\}$, we obtain $k \geq i > j$ and therefore $k \neq j$. This contradicts $k = j$. This contradiction shows that our assumption was false, qed.

then $\sigma = a_j a_{j+1} \cdots a_n$. But this is precisely the statement $\mathcal{A}(j-1)$. So we have shown that $\mathcal{A}(j-1)$ holds. This proves Assumption 2.]

We have now verified that both Assumptions 1 and 2 of Theorem 2.6 hold. Hence, Theorem 2.6 shows that $\mathcal{A}(p)$ holds for each $p \in \{0, 1, \ldots, n\}$. Thus, in particular, $\mathcal{A}(0)$ holds. But $\mathcal{A}(0)$ is precisely the claim of the exercise (since every permutation $\sigma \in S_n$ has the property that $1, 2, \ldots, 0$ are fixed points of σ). Thus, the exercise is solved.

2.4 Remark

Let w_0 be the permutation of $[n]$ that sends each $k \in [n]$ to $n+1-k$. Applying the exercise to $\sigma = w_0$, we obtain

$$\begin{aligned} w_0 &= \operatorname{cyc}_{n,n-1,\ldots,1} \operatorname{cyc}_{n,n-1,\ldots,2} \cdots \operatorname{cyc}_{n,n-1,n-2} \operatorname{cyc}_{n,n-1} \operatorname{cyc}_n \\ &= (s_{n-1}s_{n-2}\cdots s_1)(s_{n-1}s_{n-2}\cdots s_2)\cdots(s_{n-1}s_{n-2}s_{n-3})(s_{n-1}s_{n-2})(s_{n-1}) \end{aligned} \tag{10}$$

(because setting $\sigma = w_0$, we get $\ell_i(\sigma) = n - i$ and thus $a_i = \operatorname{cyc}_{n,n-1,\ldots,i} = s_{n-1}s_{n-2}\cdots s_i$ for each $i \in [n]$).

On the other hand, Part of Proposition 4.3 **(h)** in the class notes from 2018-10-17 is the claim that

$$\begin{aligned} w_0 &= \operatorname{cyc}_1 \operatorname{cyc}_{2,1} \operatorname{cyc}_{3,2,1} \cdots \operatorname{cyc}_{n,n-1,\ldots,1} \\ &= s_1(s_2s_1)(s_3s_2s_1)\cdots(s_{n-1}s_{n-2}\cdots s_1). \end{aligned} \tag{11}$$

Do you see why the two equalities (10) and (11) are equivalent? (*Hint:* We have $w_0 = w_0^{-1}$ and also $w_0 s_i w_0^{-1} = s_{n-i}$ for each $i \in [n-1]$.)

---

3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$.

(a) Prove that any $\sigma \in S_n$ and any $i \in [n]$ satisfy $\sigma(i) \leq i + \ell_i(\sigma)$.

(b) Prove that, for a given $\sigma \in S_n$, the following three statements are equivalent:

- A: We have $\sigma(i) \leq i + 1$ for all $i \in [n-1]$.
- B: The permutation σ is both 321-avoiding and 312-avoiding.
- C: We have $\ell_i(\sigma) \in \{0, 1\}$ for each $i \in [n]$. (In other words, the Lehmer code of σ consists only of 0's and 1's.)

(c) Assuming that $n \geq 1$, prove that the number of $\sigma \in S_n$ satisfying these three statements is 2^{n-1}.

3.2 Remark

Part **(a)** of this exercise is exactly [Grinbe16, Lemma 5.48 **(c)**].

3.3 Solution sketch

(a) Let $\sigma \in S_n$ and $i \in [n]$. Lemma 2.4 **(b)** yields

$$\ell_i(\sigma) = |[\sigma(i)-1] \setminus \sigma([i-1])| \geq |[\sigma(i)-1]| - \underbrace{|\sigma([i-1])|}_{\substack{=|[i-1]| \\ \text{(since the map } \sigma \text{ is injective)}}}$$
$$\left(\text{since } |A \setminus B| \geq |A| - |B| \text{ for any two finite sets } A \text{ and } B\right)$$
$$= \underbrace{|[\sigma(i)-1]|}_{=\sigma(i)-1} - \underbrace{|[i-1]|}_{=i-1} = (\sigma(i)-1) - (i-1) = \sigma(i) - i.$$

In other words, $\sigma(i) \leq i + \ell_i(\sigma)$. This solves part **(a)** of the exercise.

(b) We shall prove the three implications $A \Longrightarrow B$, $B \Longrightarrow C$ and $C \Longrightarrow A$:

Proof of the implication $A \Longrightarrow B$: Assume that statement A holds. Thus, $\sigma(i) \leq i+1$ for all $i \in [n-1]$. This inequality clearly also holds for $i = n$ (since $\sigma(n) \leq n \leq n+1$); thus, it holds for all $i \in [n]$. In other words, we have

$$\sigma(i) \leq i+1 \qquad \text{for all } i \in [n]. \tag{12}$$

Let $i, j, k \in [n]$ be three elements satisfying $i < j < k$ and $\sigma(j) < \sigma(k) < \sigma(i)$. We shall derive a contradiction.

Indeed, every $p \in [i]$ satisfies

$$\sigma(p) \leq p+1 \qquad \text{(by (12), applied to } p \text{ instead of } i\text{)}$$
$$\leq i+1 \qquad \text{(since } p \leq i\text{)}$$

and thus $\sigma(p) \in [i+1]$. In other words, $\sigma(1), \sigma(2), \ldots, \sigma(i)$ are i elements of the set $[i+1]$.

Also, $\sigma(j) < \sigma(i) \leq i+1$ (by (12)); hence, $\sigma(j)$ is an element of the set $[i+1]$ as well. Similarly, $\sigma(k)$ is an element of the set $[i+1]$ as well.

The $i+2$ elements $\underbrace{1, 2, \ldots, i}_{\text{the elements of } [i]}, j, k$ are distinct (since $i < j < k$). Hence, their images under σ are distinct as well (since σ is injective). In other words, the $i+2$ elements $\sigma(1), \sigma(2), \ldots, \sigma(i), \sigma(j), \sigma(k)$ are distinct. But we know that these $i+2$ distinct elements must belong to the $(i+1)$-element set $[i+1]$ (since we have shown that $\sigma(1), \sigma(2), \ldots, \sigma(i)$ are i elements of the set $[i+1]$, that $\sigma(j)$ is an element of the set $[i+1]$, and that $\sigma(k)$ is an element of the set $[i+1]$). This, of course, contradicts the Pigeonhole Principle (as $i+2 > i+1$).

Now, forget that we fixed i, j, k. We thus have derived a contradiction for each three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(j) < \sigma(k) < \sigma(i)$. Hence, there exist no such three elements i, j, k. In other words, σ is 312-avoiding (by the definition of "312-avoiding"). An analogous argument shows that σ is 321-avoiding. Hence, the permutation σ is both 321-avoiding and 312-avoiding. In other words, statement B holds. This proves the implication $A \Longrightarrow B$.

Proof of the implication $B \Longrightarrow C$: Assume that statement B holds. Thus, the permutation σ is both 321-avoiding and 312-avoiding.

Now, let $i \in [n]$. We shall show that $\ell_i(\sigma) \in \{0, 1\}$.

Indeed, assume the contrary. Thus, $\ell_i(\sigma) \notin \{0, 1\}$. Hence, $\ell_i(\sigma) \geq 2$ (since $\ell_i(\sigma)$ is a nonnegative integer). In other words, there exist at least two $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$ (since $\ell_i(\sigma)$ is the number of all such j's). Fix two distinct such j;

denote them by j_1 and j_2. Thus, j_1 and j_2 are two distinct $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$.

Hence, $j_1 \in \{i+1, i+2, \ldots, n\}$; therefore, $j_1 \geq i+1 > i$ and $j_1 \in [n]$. Similarly, $j_2 > i$ and $j_2 \in [n]$. Also, $\sigma(i) > \sigma(j_1)$ (by the definition of j_1) and $\sigma(i) > \sigma(j_2)$ (similarly).

We WLOG assume that $j_1 \leq j_2$ (since otherwise, we can simply swap j_1 with j_2). Hence, $j_1 < j_2$ (since j_1 and j_2 are distinct), so that $i < j_1 < j_2$. Moreover, σ is injective, and thus $\sigma(j_1) \neq \sigma(j_2)$ (since j_1 and j_2 are distinct).

But if we had $\sigma(j_1) < \sigma(j_2)$, then σ would not be 312-avoiding (since the three elements $i, j_1, j_2 \in [n]$ would satisfy $i < j_1 < j_2$ and $\sigma(j_1) < \sigma(j_2) < \sigma(i)$, which would make them the exact kind of three elements i, j, k that the definition of "312-avoiding" disallows). Hence, we cannot have $\sigma(j_1) < \sigma(j_2)$.

If we had $\sigma(j_1) > \sigma(j_2)$, then σ would not be 321-avoiding (since the three elements $i, j_1, j_2 \in [n]$ would satisfy $i < j_1 < j_2$ and $\sigma(j_2) < \sigma(j_1) < \sigma(i)$, which would make them the exact kind of three elements i, j, k that the definition of "321-avoiding" disallows). Hence, we cannot have $\sigma(j_1) > \sigma(j_2)$.

Therefore, neither $\sigma(j_1) < \sigma(j_2)$ nor $\sigma(j_1) > \sigma(j_2)$ is possible. Hence, we must have $\sigma(j_1) = \sigma(j_2)$. This contradicts $\sigma(j_1) \neq \sigma(j_2)$.

This contradiction shows that our assumption was false. Hence, $\ell_i(\sigma) \in \{0, 1\}$ is proven.

Now, forget that we fixed i. We thus have shown that we have $\ell_i(\sigma) \in \{0, 1\}$ for each $i \in [n]$. In other words, statement C holds. This proves the implication $B \Longrightarrow C$.

Proof of the implication $C \Longrightarrow A$: Assume that statement C holds. Thus, we have $\ell_i(\sigma) \in \{0, 1\}$ for each $i \in [n]$. In other words,

$$\ell_i(\sigma) \leq 1 \qquad \text{for each } i \in [n]. \tag{13}$$

Hence, for each $i \in [n-1]$, we have

$$\sigma(i) \leq i + \underbrace{\ell_i(\sigma)}_{\substack{\leq 1 \\ \text{(by (13))}}} \qquad \text{(by part (a) of the exercise)}$$
$$\leq i + 1.$$

In other words, statement A holds. This proves the implication $C \Longrightarrow A$.

We have now proven the three implications $A \Longrightarrow B$, $B \Longrightarrow C$ and $C \Longrightarrow A$. Combining them, we conclude that the three statements A, B and C are equivalent. Thus, part **(b)** of the exercise is solved.

(c) Assume that $n \geq 1$.

Whenever m is an integer, we shall use the notation $[m]_0$ for the set $\{0, 1, \ldots, m\}$. (This is an empty set when $m < 0$.)

Let H denote the set $[n-1]_0 \times [n-2]_0 \times \cdots \times [n-n]_0$.

Let H_1 denote the subset $\underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n-1 \text{ times}} \times \{0\}$ of H. Thus, $|H_1| = 2^{n-1}$.

Define the map $L : S_n \to H$ by

$$(L(\sigma) = (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) \qquad \text{for each } \sigma \in S_n).$$

It is known that this map L is well-defined and is a bijection (see, e.g., [Grinbe16, Theorem 5.52]).[2] Hence,

$$(\text{the number of } \sigma \in S_n \text{ satisfying } L(\sigma) \in H_1) = |H_1| = 2^{n-1}.$$

[2]This map L is known as the *Lehmer code* (or, rather, $L(\sigma)$ is known as the Lehmer code of the permutation $\sigma \in S_n$).

Note that each $\sigma \in S_n$ satisfies

$$\ell_n(\sigma) = 0 \tag{14}$$

(since $\ell_n(\sigma)$ is defined as the number of all $j \in \{n+1, n+2, \ldots, n\}$ satisfying $\sigma(n) > \sigma(j)$; but there are clearly no such j).

Hence, for each $\sigma \in S_n$, we have the following logical equivalence:

$$(\ell_i(\sigma) \in \{0,1\} \text{ for all } i \in [n]) \iff (L(\sigma) \in H_1). \tag{15}$$

[*Proof of* (15)*:* Let $\sigma \in S_n$. We must prove the equivalence (15). We shall prove its "$\Longrightarrow$" and "$\Longleftarrow$" directions separately:

$\Longrightarrow$: Assume that $\ell_i(\sigma) \in \{0,1\}$ for all $i \in [n]$. We must prove that $L(\sigma) \in H_1$.

We have assumed that $\ell_i(\sigma) \in \{0,1\}$ for all $i \in [n]$. Hence, $\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_{n-1}(\sigma) \in \{0,1\}$. Furthermore, (14) yields $\ell_n(\sigma) = 0 \in \{0\}$. Now, the definition of L yields

$$\begin{aligned} L(\sigma) &= (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) \in \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n-1 \text{ times}} \times \{0\} \\ &\qquad (\text{since } \ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_{n-1}(\sigma) \in \{0,1\} \text{ and } \ell_n(\sigma) \in \{0\}) \\ &= H_1 \qquad (\text{by the definition of } H_1). \end{aligned}$$

Now, forget that we assumed that $\ell_i(\sigma) \in \{0,1\}$ for all $i \in [n]$. We thus have shown that if $\ell_i(\sigma) \in \{0,1\}$ for all $i \in [n]$, then $L(\sigma) \in H_1$. In other words, we have proven the "$\Longrightarrow$" direction of the equivalence (15).

$\Longleftarrow$: Assume that $L(\sigma) \in H_1$. We must prove that $\ell_i(\sigma) \in \{0,1\}$ for all $i \in [n]$.

We have assumed that $L(\sigma) \in H_1$. But the definition of L yields $L(\sigma) = (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))$. Hence,

$$\begin{aligned} (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) = L(\sigma) \in H_1 &= \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n-1 \text{ times}} \times \underbrace{\{0\}}_{\subseteq \{0,1\}} \\ &\qquad (\text{by the definition of } H_1) \\ &\subseteq \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n-1 \text{ times}} \times \{0,1\} \\ &= \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n \text{ times}} = \{0,1\}^n. \end{aligned}$$

In other words, $\ell_i(\sigma) \in \{0,1\}$ for all $i \in [n]$.

Now, forget that we assumed that $L(\sigma) \in H_1$. We thus have shown that if $L(\sigma) \in H_1$, then $\ell_i(\sigma) \in \{0,1\}$ for all $i \in [n]$. In other words, we have proven the "$\Longleftarrow$" direction of the equivalence (15).

We now have proven both the "$\Longrightarrow$" and the "$\Longleftarrow$" directions of the equivalence (15). Hence, this equivalence is proven.]

Now, consider the statements A, B and C from part **(b)** of the exercise. These three statements are equivalent (by part **(b)**). Hence,

$$\begin{aligned} &(\text{the number of } \sigma \in S_n \text{ satisfying statements } A, B \text{ and } C) \\ &= (\text{the number of } \sigma \in S_n \text{ satisfying statement } C) \\ &= \left(\text{the number of } \sigma \in S_n \text{ satisfying } \underbrace{\ell_i(\sigma) \in \{0,1\} \text{ for all } i \in [n]}_{\substack{\iff (L(\sigma) \in H_1) \\ (\text{by } (15))}} \right) \\ &= (\text{the number of } \sigma \in S_n \text{ satisfying } L(\sigma) \in H_1) = 2^{n-1}. \end{aligned}$$

This solves part **(c)** of the exercise.

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4 Exercise 4

4.1 Problem

Let $n \geq 2$, and set $S = [n]$. Let $i \in [n-1]$. Prove that:

(a) The number of maps $f : S \to S$ with $f(i) = n$ and $f^n(S) = \{n\}$ is $2n^{n-3}$.

(b) Let $j \in [n-1]$ be such that $i \neq j$. The number of maps $f : S \to S$ with $f(i) = j$ and $f^n(S) = \{n\}$ is n^{n-3}.

[**Hint:** Substitute appropriate numbers for the variables in the Matrix-Tree Theorem.]

4.2 Solution sketch

Forget that we fixed i.

Before we solve anything, let us agree on a notation: If i and j are two elements of $[n]$, then we define a set $A_{i,j}$ by

$$A_{i,j} = \{f : S \to S \mid f(i) = j \text{ and } f^n(S) = \{n\}\}.$$

Thus, $|A_{i,j}|$ is the number of maps $f : S \to S$ with $f(i) = j$ and $f^n(S) = \{n\}$. Hence, part **(a)** of the exercise is equivalent to saying that $|A_{i,n}| = 2n^{n-3}$ for any $i \in [n-1]$, whereas part **(b)** of the exercise is equivalent to saying that $|A_{i,j}| = n^{n-3}$ for any $i \in [n-1]$ and $j \in [n-1]$ satisfying $i \neq j$. It is in these forms that we shall solve the exercise.

(a) *First solution to part **(a)**:* Here is the matrix-tree theorem, as stated in class (Theorem 5.4 in the class notes from 2018-11-05):[3]

Theorem 4.1. *Let $n \geq 1$. Let $S = [n]$. For any distinct $i, j \in [n]$, let $a_{i,j}$ be a number (or an indeterminate).*

For each $i \in [n]$, we let $b_i = a_{i,1} + a_{i,2} + \cdots + \widehat{a_{i,i}} + \cdots + a_{i,n}$. Here, the hat over the "$a_{i,i}$" means that the addend $a_{i,i}$ should not be included in the sum (so that the sum is $a_{i,1} + a_{i,2} + \cdots + a_{i,i-1} + a_{i,i+1} + a_{i,i+2} + \cdots + a_{i,n}$).

Let L be the $(n-1) \times (n-1)$-matrix whose (i,j)-th entry is $\begin{cases} b_i, & \text{if } i = j; \\ -a_{i,j}, & \text{if } i \neq j \end{cases}$ for all $i, j \in [n-1]$.

Then,

$$\sum_{\substack{f:S\to S;\\ f^n(S)=\{n\}}} \prod_{i\in[n-1]} a_{i,f(i)} = \det L.$$

Now, recall that our goal is to solve part **(a)** of the exercise; in other words, our goal is to prove that $|A_{i,n}| = 2n^{n-3}$ for any $i \in [n-1]$ (because part **(a)** is equivalent to this). First, we notice that the specific value of the number i is irrelevant: If i_1 and i_2 are two

[3]Keep in mind that we have not fixed i, so we can use this letter for various other needs.

elements of $[n-1]$, then $|A_{i_1,n}| = |A_{i_2,n}|$ [4]. Applying this to $i_1 = i$ and $i_2 = n-1$, we conclude that

$$|A_{i,n}| = |A_{n-1,n}| \qquad \text{for each } i \in [n-1]. \tag{16}$$

Hence, in order to prove that $|A_{i,n}| = 2n^{n-3}$ for any $i \in [n-1]$, it suffices to prove that $|A_{n-1,n}| = 2n^{n-3}$. This is what we shall prove in the following.

We WLOG assume that $n \geq 3$, since otherwise the result is easy to check by hand.

For any distinct $i, j \in [n]$, let $a_{i,j} = [i \neq n-1 \text{ or } j = n]$. Thus, $a_{i,j} = 1$ whenever $i \neq n-1$, whereas $a_{n-1,j} = [j = n]$. Now, define b_i and L as in Theorem 4.1. Then, each $i \in [n-1]$ satisfies

$$b_i = a_{i,1} + a_{i,2} + \cdots + \widehat{a_{i,i}} + \cdots + a_{i,n} = \sum_{\substack{j \in [n];\\ j \neq i}} \underbrace{a_{i,j}}_{=[i \neq n-1 \text{ or } j=n]}$$
$$= \sum_{\substack{j \in [n];\\ j \neq i}} [i \neq n-1 \text{ or } j = n] = \begin{cases} n-1, & \text{if } i \neq n-1; \\ 1, & \text{if } i = n-1 \end{cases}$$

(because if $i \neq n-1$, then all $n-1$ addends of the sum $\sum_{\substack{j \in [n];\\ j \neq i}} [i \neq n-1 \text{ or } j = n]$ equal 1, whereas otherwise one of these addends equals 1 whereas all others equal 0). Hence, the $(n-1) \times (n-1)$-matrix L has the following form:[5]

$$L = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & n-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{(n-1)\times(n-1)}$$

(viz.: its diagonal entries are $\underbrace{n-1, n-1, \ldots, n-1}_{n-2 \text{ times}}, 1$; its last row is $\underbrace{0, 0, \ldots, 0}_{n-2 \text{ times}}, 1$; and all

[4] *Proof.* Let i_1 and i_2 be two elements of $[n-1]$. We must prove that $|A_{i_1,n}| = |A_{i_2,n}|$. If $i_1 = i_2$, then this is obvious. Thus, WLOG assume that $i_1 \neq i_2$. Hence, the transposition $t_{i_1,i_2} \in S_n$ is well-defined. Clearly, n is a fixed point of t_{i_1,i_2}, and t_{i_1,i_2} is an involution. Now, it is easy to see that the map

$$A_{i_1,n} \to A_{i_2,n}, \qquad f \mapsto t_{i_1,i_2} \circ f \circ t_{i_1,i_2}$$

is well-defined (after all, it simply interchanges the roles of i_1 and i_2, so that it sends a map f satisfying $f(i_1) = j$ to a map f satisfying $f(i_2) = j$ without disrupting the "$f^n(S) = \{n\}$" behavior) and is a bijection (its inverse map is defined in the same way). Thus, $|A_{i_1,n}| = |A_{i_2,n}|$, qed.

[5] We are using a slightly nonstandard notation here: We are putting the size of a matrix as a subscript on the bottom right of the matrix. Thus, for example, "$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{7\times 7}$" will mean the 7×7-matrix whose all entries equal 1. This prevents ambiguities (for example, without the subscript, "$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$" would be ambiguous, because the size of the matrix would not be clear).

its other entries equal -1). Hence,

$$\det L = \det \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}_{(n-2)\times(n-2)}$$

$$\left(\begin{array}{c}\text{here, we expanded the determinant along the last row,} \\ \text{which gave us only one cofactor because the last row} \\ \text{of } L \text{ has only one nonzero entry}\end{array}\right)$$

$$= \det \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -n & n & 0 & \cdots & 0 \\ -n & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -n & 0 & 0 & \cdots & n \end{pmatrix}_{(n-2)\times(n-2)}$$

$$\left(\begin{array}{c}\text{here, we have subtracted the first row of the matrix} \\ \text{from each of the other rows}\end{array}\right)$$

$$= n^{n-3} \det \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-2)\times(n-2)}$$

$$\left(\begin{array}{c}\text{here, we have factored out an } n \text{ from each row of the} \\ \text{matrix except of the first row}\end{array}\right)$$

$$= n^{n-3} \underbrace{\det \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-2)\times(n-2)}}_{\substack{=2\cdot 1\cdot 1\cdot\cdots\cdot 1 \\ \text{(since the determinant of a lower-triangular} \\ \text{matrix equals the product of its diagonal entries)}}}$$

$$\left(\begin{array}{c}\text{here, we have added each row of the matrix except of} \\ \text{the first row to the first row}\end{array}\right)$$

$$= n^{n-3} \cdot (2 \cdot 1 \cdot 1 \cdot \cdots \cdot 1) = 2n^{n-3}.$$

Thus, Theorem 4.1 yields

$$\sum_{\substack{f:S\to S; \\ f^n(S)=\{n\}}} \prod_{i\in[n-1]} a_{i,f(i)} = \det L = 2n^{n-3}.$$

Comparing this with

$$\sum_{\substack{f:S\to S;\\ f^n(S)=\{n\}}} \prod_{i\in[n-1]} \underbrace{a_{i,f(i)}}_{\substack{=[i\neq n-1 \text{ or } f(i)=n] \\ \text{(by the definition of } a_{i,f(i)})}}$$

$$= \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\}}} \underbrace{\prod_{i\in[n-1]} [i\neq n-1 \text{ or } f(i)=n]}_{\substack{=\left(\prod_{i\in[n-2]}[i\neq n-1 \text{ or } f(i)=n]\right)\cdot[n-1\neq n-1 \text{ or } f(n-1)=n] \\ \text{(here, we have split off the factor for } i=n-1 \text{ from the product)}}}$$

$$= \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\}}} \left(\prod_{i\in[n-2]} \underbrace{[i\neq n-1 \text{ or } f(i)=n]}_{\substack{=1\\ \text{(since } i\neq n-1)}}\right) \cdot \underbrace{[n-1\neq n-1 \text{ or } f(n-1)=n]}_{\substack{=[f(n-1)=n] \\ \text{(since } n-1\neq n-1 \text{ does not hold)}}}$$

$$= \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\}}} \underbrace{\left(\prod_{i\in[n-2]} 1\right)}_{=1} \cdot [f(n-1)=n] = \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\}}} [f(n-1)=n]$$

$$= \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\};\\ f(n-1)=n}} \underbrace{[f(n-1)=n]}_{\substack{=1\\ \text{(since } f(n-1)=n)}} + \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\};\\ f(n-1)\neq n}} \underbrace{[f(n-1)=n]}_{\substack{=0\\ \text{(since } f(n-1)\neq n)}}$$

$$= \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\};\\ f(n-1)=n}} 1 + \underbrace{\sum_{\substack{f:S\to S;\\ f^n(S)=\{n\};\\ f(n-1)\neq n}} 0}_{=0} = \sum_{\substack{f:S\to S;\\ f^n(S)=\{n\};\\ f(n-1)=n}} 1$$

$$= |\{f:S\to S \mid f^n(S)=\{n\} \text{ and } f(n-1)=n\}|\cdot 1$$

$$= |\{f:S\to S \mid f^n(S)=\{n\} \text{ and } f(n-1)=n\}|$$

$$= \left|\underbrace{\{f:S\to S \mid f(n-1)=n \text{ and } f^n(S)=\{n\}\}}_{\substack{=A_{n-1,n}\\ \text{(by the definition of } A_{n-1,n})}}\right| = |A_{n-1,n}|,$$

we obtain $|A_{n-1,n}| = 2n^{n-3}$. As we have explained, this solves part **(a)** of the exercise.

*Second solution to part **(a)**:* I shall **not** follow the hint this time. Instead, let me recall a fact proven in class (during the proof of Theorem 5.2 in the class notes from 2018-11-05):

Proposition 4.2. *Let $n\geq 1$, and set $S=[n]$. For each subset T of S, we set*

$$\Phi(T) = \{f:S\to S \mid f^n(S)\subseteq T \text{ and } T\subseteq \operatorname{Fix} f\}, \tag{17}$$

where $\operatorname{Fix} f := \{x\in S \mid f(x)=x\}$.

Then, there are integers $g_0, g_1, \ldots, g_n$ such that

$$g_i = |\Phi(T)| \qquad \text{for every } i\text{-element subset } T \text{ of } S, \tag{18}$$

and these integers are given by

$$g_k = kn^{n-k-1} \qquad \text{for each } k\in\{0,1,\ldots,n\}. \tag{19}$$

Consider the integers $g_0, g_1, \ldots, g_n$ from Proposition 4.2. Applying (19) to $k = 2$, we obtain $g_2 = 2n^{n-2-1} = 2n^{n-3}$.

Also, let us introduce another notation: If $f : S \to S$ is any map, and if a and b are any two elements of S, then

$$(f \text{ but } a \mapsto b)$$

shall denote the map from S to S that sends each $s \in S$ to $\begin{cases} f(s), & \text{if } s \neq a; \\ b, & \text{if } s = a. \end{cases}$ In other words, the map $(f \text{ but } a \mapsto b)$ differs from f only in that it sends a to b.

Now, fix $i \in [n-1]$. Let T be the subset $\{i, n\}$ of S. Then, T is a 2-element subset (since $i \neq n$ (because $i \in [n-1]$)); thus, (18) (applied to $i = 2$) yields $g_2 = |\Phi(T)|$. Hence, $|\Phi(T)| = g_2 = 2n^{n-3}$. The definition of $\Phi(T)$ yields

$$\begin{aligned} \Phi(T) &= \{f : S \to S \mid f^n(S) \subseteq T \text{ and } T \subseteq \operatorname{Fix} f\} \\ &= \{f : S \to S \mid f^n(S) \subseteq \{i, n\} \text{ and } \{i, n\} \subseteq \operatorname{Fix} f\} \qquad (\text{since } T = \{i, n\}) \\ &= \{f : S \to S \mid f^n(S) \subseteq \{i, n\} \text{ and } f(i) = i \text{ and } f(n) = n\}. \end{aligned}$$

The definition of $A_{i,n}$ yields

$$A_{i,n} = \{f : S \to S \mid f(i) = n \text{ and } f^n(S) = \{n\}\}. \tag{20}$$

It is easy to see that the map

$$\begin{aligned} \Phi(T) &\to A_{i,n}, \\ f &\mapsto (f \text{ but } i \mapsto n) \end{aligned}$$

is well-defined[6]. It is also easy to see that the map

$$\begin{aligned} A_{i,n} &\to \Phi(T), \\ f &\mapsto (f \text{ but } i \mapsto i) \end{aligned}$$

is well-defined. Furthermore, these two maps are mutually inverse (since each $f \in \Phi(T)$ sends i to i, while each $f \in A_{i,n}$ sends i to n). Thus, these maps are bijections. Hence, we have found a bijection from $A_{i,n}$ to $\Phi(T)$. Thus, $|A_{i,n}| = |\Phi(T)| = 2n^{n-3}$. In view of (20), this rewrites as

$$|\{f : S \to S \mid f(i) = n \text{ and } f^n(S) = \{n\}\}| = 2n^{n-3}.$$

In other words, the number of maps $f : S \to S$ with $f(i) = n$ and $f^n(S) = \{n\}$ is $2n^{n-3}$. This solves part **(a)** of the exercise again.

(b) While it is certainly possible to solve part **(b)** using the Matrix-Tree Theorem (as we did with part **(a)**), it is also more complicated than what we did above. Fortunately, there is a simpler approach:

[6]To prove this, you need to show that every $f \in \Phi(T)$ satisfies $(f \text{ but } i \mapsto n)(i) = n$ and $(f \text{ but } i \mapsto n)^n(S) = \{n\}$. The first of these two equalities is obvious. The second can be argued (roughly speaking) as follows: Since $f \in \Phi(T)$, we have $f^n(S) \subseteq \{i, n\}$; hence, each element of S will eventually reach either i or n when we apply f to it many times. Thus, if we keep applying $(f \text{ but } i \mapsto n)$ to it instead of f, it will eventually reach n (because it will move in the same way as if we apply f to it, unless and until it reaches i; but at that point, $(f \text{ but } i \mapsto n)$ will send it directly to n). Hence, $(f \text{ but } i \mapsto n)^n(S) = \{n\}$, qed.

As we know, part **(b)** of the exercise is equivalent to saying that $|A_{i,j}| = n^{n-3}$ for any $i \in [n-1]$ and $j \in [n-1]$ satisfying $i \neq j$. Thus, it suffices to prove that $|A_{i,j}| = n^{n-3}$ for any $i \in [n-1]$ and $j \in [n-1]$ satisfying $i \neq j$.

Fix $i \in [n-1]$. We must thus show that $|A_{i,j}| = n^{n-3}$ for any $j \in [n-1]$ satisfying $i \neq j$.

We notice that the specific value of the number j is irrelevant (as long as $i \neq j$ holds): If j_1 and j_2 are two elements of $[n-1]$ such that $i \neq j_1$ and $i \neq j_2$, then

$$|A_{i,j_1}| = |A_{i,j_2}| \tag{21}$$

[7]. Furthermore, $A_{i,i} = \varnothing$ (indeed, any $f \in A_{i,i}$ would have to satisfy both $f(i) = i$ and $f^n(S) = \{n\}$; but these two equalities contradict each other[8]), and thus $|A_{i,i}| = 0$.

Now, let $j \in [n-1]$ be such that $i \neq j$. We must prove that $|A_{i,j}| = n^{n-3}$. Note that i and j are two distinct elements of the set $[n-1]$ (since $i \neq j$); thus, this set $[n-1]$ has at least two elements. In other words, $|[n-1]| \geq 2$. Thus, $n - 1 = |[n-1]| \geq 2$, so that $n \geq 3$. Hence, $n - 2 \neq 0$.

We have $i \in [n-1]$. Hence, we can split off the addend for $u = i$ from the sum $\sum_{u \in [n-1]} |A_{i,u}|$. We thus obtain

$$\sum_{u\in[n-1]} |A_{i,u}| = \underbrace{|A_{i,i}|}_{=0} + \sum_{\substack{u\in[n-1];\\ u\neq i}} \underbrace{|A_{i,u}|}_{\substack{=|A_{i,j}| \\ \text{(by (21), applied to } j_1=u \\ \text{and } j_2=j)}} = \sum_{\substack{u\in[n-1];\\ u\neq i}} |A_{i,j}| = (n-2)\,|A_{i,j}| \tag{22}$$

(since the number of $u \in [n-1]$ satisfying $u \neq i$ is $n-2$). But each $u \in [n]$ satisfies

$$A_{i,u} = \{f : S \to S \ \mid\ f(i) = u \text{ and } f^n(S) = \{n\}\}$$

(by the definition of $A_{i,u}$). Thus, the n sets $A_{i,1}, A_{i,2}, \ldots, A_{i,n}$ are disjoint, and their union is $\{f : S \to S \ \mid\ f^n(S) = \{n\}\}$. Hence,

$$\begin{aligned} |A_{i,1}| + |A_{i,2}| + \cdots + |A_{i,n}| &= |\{f : S \to S \ \mid\ f^n(S) = \{n\}\}| \\ &= (\text{the number of maps } f : S \to S \text{ satisfying } f^n(S) = \{n\}) \\ &= n^{n-2} \end{aligned}$$

[7] *Proof.* Let j_1 and j_2 be two elements of $[n-1]$ satisfying $i \neq j_1$ and $i \neq j_2$. We must prove that $|A_{i,j_1}| = |A_{i,j_2}|$. If $j_1 = j_2$, then this is obvious. Thus, WLOG assume that $j_1 \neq j_2$. Hence, the transposition $t_{j_1,j_2} \in S_n$ is well-defined. Clearly, both i and n are fixed points of t_{j_1,j_2} (since $i \neq j_1$ and $i \neq j_2$ and $j_1, j_2 \in [n-1]$), and t_{j_1,j_2} is an involution. Now, it is easy to see that the map

$$A_{i,j_1} \to A_{i,j_2}, \qquad f \mapsto t_{j_1,j_2} \circ f \circ t_{j_1,j_2}$$

is well-defined (after all, it simply interchanges the roles of j_1 and j_2, so that it sends a map f satisfying $f(i) = j_1$ to a map f satisfying $f(i) = j_2$ without disrupting the "$f^n(S) = \{n\}$" behavior) and is a bijection (its inverse map is defined in the same way). Thus, $|A_{i,j_1}| = |A_{i,j_2}|$, qed.

[8] Indeed, the equality $f(i) = i$ leads to $f^k(i) = i$ for all $k \in \mathbb{N}$; thus, $f^n(i) = i \notin \{n\}$, which contradicts $f^n\left(\underbrace{i}_{\in S}\right) \in f^n(S) = \{n\}$.

(by Theorem 5.2 in the class notes from 2018-11-05). Hence,

$$n^{n-2} = |A_{i,1}| + |A_{i,2}| + \cdots + |A_{i,n}| = \sum_{u\in[n]} |A_{i,u}| = \underbrace{\sum_{u\in[n-1]} |A_{i,u}|}_{\substack{=(n-2)|A_{i,j}| \\ \text{(by (22))}}} + \underbrace{|A_{i,n}|}_{\substack{=2n^{n-3} \\ \text{(by part (a)} \\ \text{of the exercise)}}}$$

$$\text{(here, we have split off the addend for } u = n \text{ from the sum)}$$

$$= (n-2)|A_{i,j}| + 2n^{n-3}.$$

We can solve this equation for $|A_{i,j}|$ (since $n - 2 \neq 0$), and obtain $|A_{i,j}| = \dfrac{n^{n-2} - 2n^{n-3}}{n-2} = \dfrac{(n-2)\,n^{n-3}}{n-2} = n^{n-3}$. This is exactly what we wanted to show. Thus, part **(b)** of the exercise is solved.

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5 Exercise 5

5.1 Problem

(a) For each $n \in \mathbb{N}$, prove that the number of fixed-point-free involutions $[n] \to [n]$ is

$$\begin{cases} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (n-1), & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(b) For each $n \in \mathbb{N}$, we let t_n be the number of all involutions in S_n. Prove that

$$t_n = \sum_{k=0}^{n} \binom{n}{2k} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)) \qquad \text{for each } n \in \mathbb{N}.$$

(c) For each $n \in \mathbb{N}$, prove that the number of maps $f : [n] \to [n]$ satisfying $f^3 = f$ is

$$\sum_{k=0}^{n} \binom{n}{k} k^{n-k} t_k.$$

5.2 Remark

The numbers in part **(a)** form the sequence A123023 in the OEIS. (And if you omit the terms for odd n, which are all zero, then you obtain sequence A001147, known as the *double factorials*.)

The numbers $t_0, t_1, t_2, \ldots$ in part **(b)** are sometimes called the *telephone numbers*, because an involution in S_n is a way how phone calls can be happening between n people $1, 2, \ldots, n$, assuming there are no conference calls. This is sequence A000085 in the OEIS.

Finally, the numbers in part **(c)** form sequence A060905.

5.3 Solution sketch

(a) *First solution to part **(a)** (sketched):* The fixed-point-free involutions $[n] \to [n]$ are precisely the derangements in S_n that are also involutions. But the latter derangements have been counted in UMN Spring 2018 Math 4707 notes from 2018-05-02 (pages 9–20).

*Second solution to part **(a)** (sketched):* We WLOG assume that $n \geq 2$, since the cases when $n < 2$ are trivial to check.

The fixed-point-free involutions $[n] \to [n]$ are precisely the permutations $\sigma \in S_n$ whose cycles are all 2-cycles (because any cycle of length > 2 would prevent σ from being an involution, whereas any cycle of length < 2 would prevent σ from being fixed-point-free). Thus, they are precisely the permutations $\sigma \in S_n$ that have exactly $n/2$ many 2-cycles and no cycles of any other length[9]. Thus, if n is odd, then there exist no such involutions (since $n/2$ is not an integer in this case), i.e., their number is 0. For the same reason, if n is even, we have

$$\begin{aligned}
&(\text{the number of fixed-point-free involutions } [n] \to [n]) \\
&= (\text{the number of permutations } \sigma \in S_n \text{ that have exactly } n/2 \text{ many 2-cycles} \\
&\qquad \text{and no cycles of any other length}) \\
&= \frac{n!}{0!\,(n/2)!0!0!\cdots 0!1^0 2^{n/2} 3^0 4^0 \cdots n^0} \qquad \left(\begin{array}{c}\text{by the exercise on pages 234–236} \\ \text{of the 2018-10-24 notes}\end{array}\right) \\
&= \frac{n!}{(n/2)!2^{n/2}} \qquad \left(\text{since } 0! = 1 \text{ and } k^0 = 1 \text{ for each } k \in \mathbb{Z}\right) \\
&= \frac{1}{2^{n/2}} \cdot \frac{1}{(n/2)!} \cdot \underbrace{n!}_{\substack{=1\cdot 2\cdot\cdots\cdot n \\ =(1\cdot 3\cdot 5\cdot\cdots\cdot(n-1))\cdot(2\cdot 4\cdot 6\cdot\cdots\cdot n) \\ (\text{since } n \text{ is even})}} \\
&= \frac{1}{2^{n/2}} \cdot \frac{1}{(n/2)!} \cdot (1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)) \cdot \underbrace{(2\cdot 4\cdot 6\cdot\cdots\cdot n)}_{=2^{n/2}\cdot(1\cdot 2\cdot 3\cdot\cdots\cdot(n/2))} \\
&= \frac{1}{2^{n/2}} \cdot \frac{1}{(n/2)!} \cdot (1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)) \cdot 2^{n/2} \cdot (1\cdot 2\cdot 3\cdot\cdots\cdot(n/2)) \\
&= \frac{1}{(n/2)!} \cdot (1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)) \cdot \underbrace{(1\cdot 2\cdot 3\cdot\cdots\cdot(n/2))}_{=(n/2)!} = 1\cdot 3\cdot 5\cdot\cdots\cdot(n-1).
\end{aligned}$$

Combining the claims in the previous two sentences, we conclude that part **(a)** of the exercise is solved.

(b) Let $n \in \mathbb{N}$. To construct an involution $\sigma \in S_n$, we can use the following algorithm:

- First choose the number m of fixed points of σ; this must be a number in $\{0, 1, \ldots, n\}$.
- Next, choose the set P of all fixed points of σ; this must be an m-element subset of $[n]$. There are $\dbinom{n}{m}$ ways of choosing this subset P.
- Note that $|P| = m$ and thus $|[n] \setminus P| = n - m$.
- Now, the values $\sigma(p)$ for all $p \in P$ are already determined (indeed, we must have $\sigma(p) = p$ for all $p \in P$, since P should be the set of all fixed points of σ), and we

[9] Indeed, if the cycles of σ are all 2-cycles, then there must be exactly $n/2$ of these 2-cycles (since each of the n elements of $[n]$ must be caught up in exactly 1 of these 2-cycles, but each 2-cycle catches exactly two elements of $[n]$).

have $\sigma(P) = P$. Thus, it remains to choose the values $\sigma(q)$ for $q \in [n] \setminus P$. Since σ should be a permutation, we must have $\sigma([n] \setminus P) = [n] \setminus \underbrace{\sigma(P)}_{=P} = [n] \setminus P$; therefore, these values $\sigma(q)$ must belong to $[n] \setminus P$. Thus, the restriction of σ to $[n] \setminus P$ should be a permutation of $[n] \setminus P$. This permutation should be an involution (since σ should be an involution) and should be fixed-point-free (since we want P to be the set of all fixed points of σ, and thus any fixed points of σ must lie in P). Thus, the restriction of σ to $[n] \setminus P$ should be a fixed-point-free involution of $[n] \setminus P$. Hence, the number of ways of choosing this restriction (i.e., choosing the values $\sigma(q)$ for $q \in [n] \setminus P$) is

$$\begin{aligned}
&(\text{the number of all fixed-point-free involutions of } [n] \setminus P) \\
&= (\text{the number of all fixed-point-free involutions of } [n-m]) \\
&\qquad \left(\begin{array}{c}\text{since there is a bijection between } [n] \setminus P \text{ and } [n-m] \\ (\text{because } |[n] \setminus P| = n-m)\end{array}\right) \\
&= \begin{cases} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot ((n-m)-1), & \text{if } n-m \text{ is even;} \\ 0, & \text{if } n-m \text{ is odd} \end{cases} \\
&\qquad (\text{by part } \textbf{(a)} \text{ of this exercise, applied to } n-m \text{ instead of } n).
\end{aligned}$$

Hence, the total number of involutions $\sigma \in S_n$ is

$$\begin{aligned}
&\sum_{m \in \{0,1,\ldots,n\}} \binom{n}{m} \begin{cases} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot ((n-m)-1), & \text{if } n-m \text{ is even;} \\ 0, & \text{if } n-m \text{ is odd} \end{cases} \\
&= \sum_{\substack{m \in \{0,1,\ldots,n\};\\ n-m \text{ is even}}} \binom{n}{m} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot ((n-m)-1)) \\
&\qquad \left(\begin{array}{c}\text{here, we have removed the addends for which } n-m \text{ is odd,} \\ \text{since these addends are } 0\end{array}\right) \\
&= \sum_{\substack{m \in \{0,1,\ldots,n\};\\ m \text{ is even}}} \underbrace{\binom{n}{n-m}}_{=\binom{n}{m}} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (m-1)) \\
&\qquad (\text{here, we have substituted } m \text{ for } n-m \text{ in the sum}) \\
&= \sum_{\substack{m \in \{0,1,\ldots,n\};\\ m \text{ is even}}} \binom{n}{m} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (m-1)) \\
&= \sum_{\substack{m \in \{0,1,\ldots,2n\};\\ m \text{ is even}}} \binom{n}{m} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (m-1)) \\
&\qquad \left(\begin{array}{c}\text{here, we have extended the sum by loosening} \\ \text{the ``} m \in \{0,1,\ldots,n\} \text{'' condition to ``} m \in \{0,1,\ldots,2n\} \text{'';} \\ \text{this did not affect the value of the sum, since} \\ \text{all addends with } m > n \text{ are } 0\end{array}\right) \\
&= \sum_{k=0}^{n} \binom{n}{2k} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)) \\
&\qquad (\text{here, we have substituted } 2k \text{ for } m \text{ in the sum}).
\end{aligned}$$

In other words, $t_n = \sum_{k=0}^{n} \binom{n}{2k} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1))$ (since t_n is the number of involutions $\sigma \in S_n$). This solves part **(b)** of the exercise.

(c) It is easy to see that a map $f : [n] \to [n]$ satisfies $f^3 = f$ if and only if its restriction $f\mid_{f([n])}$ (to its own image) is an involution. Thus, any map $f : [n] \to [n]$ satisfying $f^3 = f$ can be constructed as follows:

- Choose the size k of its image $f([n])$; this is an integer in $\{0, 1, \ldots, n\}$.
- Then choose this image $f([n])$ as a k-element subset of $[n]$; there are $\binom{n}{k}$ choices for this subset.
- Then choose the restriction $f\mid_{f([n])}$ as an involution of $f([n])$; there are t_k choices for this involution[10].
- Finally, choose the values of f on the $n-k$ elements of $[n] \setminus f([n])$. These values must belong to $f([n])$, so we have k^{n-k} choices here (because $f([n])$ is a k-element set).

Hence, the total number of maps $f : [n] \to [n]$ satisfying $f^3 = f$ is

$$\sum_{k=0}^{n} \binom{n}{k} t_k k^{n-k} = \sum_{k=0}^{n} \binom{n}{k} k^{n-k} t_k.$$

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6 Exercise 6

6.1 Problem

Let n be a positive integer, and let $p \in \{0, 1, \ldots, n\}$.

A permutation $\sigma \in S_n$ shall be called a *p-desarrangement* if it satisfies

$$\textbf{either } (\sigma = \operatorname{id} \text{ and } 2 \mid n) \textbf{ or } \sigma(1) \leq p \textbf{ or } (\sigma \neq \operatorname{id} \text{ and } 2 \mid \min(\operatorname{Des}\sigma)).$$

(The condition $2 \mid \min(\operatorname{Des}\sigma)$ means that the smallest descent of σ is even.[11] This is well-defined, since $\sigma \neq \operatorname{id}$ shows that σ has at least one descent. Further p-desarrangements are

[10] Indeed, $f([n])$ is a k-element set, and thus there exists a bijection between $f([n])$ and $[k]$. Hence, the number of involutions of $f([n])$ equals the number of involutions of $[k]$. But the latter number is precisely t_k (by the definition of t_k). Hence, the number of involutions of $f([n])$ is t_k.

[11] Here are all permutations $\sigma \neq \operatorname{id}$ in S_5 that satisfy this condition (written in one-line notation, with an underline marking the position of the smallest descent):

$[1,2,3,\underline{5},4]$,	$[1,2,4,\underline{5},3]$,	$[1,\underline{3},2,4,5]$,	$[1,\underline{3},2,5,4]$,	$[1,3,4,\underline{5},2]$,
$[1,\underline{4},2,3,5]$,	$[1,\underline{4},2,5,3]$,	$[1,\underline{4},3,2,5]$,	$[1,\underline{4},3,5,2]$,	$[1,\underline{5},2,3,4]$,
$[1,\underline{5},2,4,3]$,	$[1,\underline{5},3,2,4]$,	$[1,\underline{5},3,4,2]$,	$[1,\underline{5},4,2,3]$,	$[1,\underline{5},4,3,2]$,
$[2,\underline{3},1,4,5]$,	$[2,\underline{3},1,5,4]$,	$[2,3,4,\underline{5},1]$,	$[2,\underline{4},1,3,5]$,	$[2,\underline{4},1,5,3]$,
$[2,\underline{4},3,1,5]$,	$[2,\underline{4},3,5,1]$,	$[2,\underline{5},1,3,4]$,	$[2,\underline{5},1,4,3]$,	$[2,\underline{5},3,1,4]$,
$[2,\underline{5},3,4,1]$,	$[2,\underline{5},4,1,3]$,	$[2,\underline{5},4,3,1]$,	$[3,\underline{4},1,2,5]$,	$[3,\underline{4},1,5,2]$,
$[3,\underline{4},2,1,5]$,	$[3,\underline{4},2,5,1]$,	$[3,\underline{5},1,2,4]$,	$[3,\underline{5},1,4,2]$,	$[3,\underline{5},2,1,4]$,
$[3,\underline{5},2,4,1]$,	$[3,\underline{5},4,1,2]$,	$[3,\underline{5},4,2,1]$,	$[4,\underline{5},1,2,3]$,	$[4,\underline{5},1,3,2]$,
	$[4,\underline{5},2,1,3]$,	$[4,\underline{5},2,3,1]$,	$[4,\underline{5},3,1,2]$,	$[4,\underline{5},3,2,1]$.

id when n is even, and all permutations starting with a number $\leq p$ (in one-line notation).)

Prove that the number of p-desarrangements in S_n is

$$\sum_{k=0}^{n-p} \binom{n-p}{k} \cdot (-1)^k (n-k)!.$$

6.2 Remark

This number is exactly the number of p-derangements in S_n, as defined in Exercise 5 of midterm #1. This suggests the existence of a bijection between the p-desarrangements and the p-derangements. Such a thing has indeed been found in the case when $p = 0$. In this case, the 0-desarrangements are known as *desarrangements* (a pun on the name Désarmenien and the word "derangement"), whereas the 0-derangements are precisely the derangements. The desarrangements are just the permutations $\sigma \in S_n$ satisfying either $\sigma = \operatorname{id}$ or ($\sigma \neq \operatorname{id}$ and $2 \mid \min(\operatorname{Des}\sigma)$). One known bijection between the derangements and the desarrangements proceeds as follows:

- Let $\sigma \in S_n$ be a derangement. We want to define the corresponding desarrangement $F(\sigma)$.
- Compute the disjoint cycle decomposition of σ, and write it in such a way that each cycle contains its **largest** entry in its **second** position, and that the cycles are ordered in **increasing order of their largest entries**. That is, write

  $$\sigma = \operatorname{cyc}_{a_{1,1},a_{1,2},\ldots,a_{1,n_1}} \operatorname{cyc}_{a_{2,1},a_{2,2},\ldots,a_{2,n_2}} \cdots \operatorname{cyc}_{a_{k,1},a_{k,2},\ldots,a_{k,n_k}},$$

  where each of the numbers $1, 2, \ldots, n$ appears exactly once among the $a_{i,j}$, and where

  $$a_{i,2} \geq a_{i,j} \qquad \text{for all } i \text{ and } j, \qquad \text{and} \qquad a_{1,2} < a_{2,2} < \cdots < a_{k,2}.$$

- Now, let $F(\sigma)$ be the permutation whose one-line notation is

  $$(a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}, \ldots, a_{k,1}, a_{k,2}, \ldots, a_{k,n_k}).$$

For example, if $n = 7$ and $\sigma = [5, 3, 7, 6, 1, 4, 2]$ in one-line notation, then the appropriate representation of σ is $\sigma = \operatorname{cyc}_{1,5} \operatorname{cyc}_{4,6} \operatorname{cyc}_{3,7,2}$ and thus $F(\sigma) = [1, 5, 4, 6, 3, 7, 2]$ in one-line notation.

It is far from trivial to check that this is actually a well-defined bijection. I don't know if anything like that exists for $p \neq 0$. Feel free to explore. (But the simplest way to solve the exercise is not by bijection.)

6.3 Solution sketch

Let us introduce some notations first. If $\sigma \in S_n$ is any permutation, then we set $\sigma(k) = 0$ for all $k > n$.

For each $k \in \mathbb{N}$, we let

$$B_k = \{\sigma \in S_n \mid p < \sigma(1) < \sigma(2) < \cdots < \sigma(k)\}.$$

Note that the chain of inequalities $p < \sigma(1) < \sigma(2) < \cdots < \sigma(k)$ is vacuously true when $k = 0$; thus, B_0 is simply the set S_n of all permutations $\sigma \in S_n$. In other words, every

$\sigma \in S_n$ satisfies $\sigma \in B_0$. Also, for any $k > n$, we have $B_k = \varnothing$, because no $\sigma \in S_n$ satisfies $p < \sigma(1) < \sigma(2) < \cdots < \sigma(k)$ (indeed, the last member of this chain of inequalities is $\sigma(k) = 0$, which is not larger than p). Thus, the sequence $(|B_0|, |B_1|, |B_2|, \ldots)$ is finitely supported.

For each $k \in \{0, 1, \ldots, n\}$, we have

$$|B_k| = \binom{n-p}{k} (n-k)!. \qquad (23)$$

[*Proof of* (23)*:* Let $k \in \{0, 1, \ldots, n\}$. In order to construct a permutation $\sigma \in S_n$ satisfying $p < \sigma(1) < \sigma(2) < \cdots < \sigma(k)$, we can proceed as follows: First choose the first k values $\sigma(1), \sigma(2), \ldots, \sigma(k)$ in $\binom{n-p}{k}$ many ways (since we only need to choose their set as a k-element subset of the $(n-p)$-element set $\{p+1, p+2, \ldots, n\}$; their order is then uniquely determined), and then choose the remaining $n-k$ values $\sigma(k+1), \sigma(k+2), \ldots, \sigma(n)$ in $(n-k)!$ many ways. Thus, the total number of such permutations is $\binom{n-p}{k}(n-k)!$. In other words, $|B_k| = \binom{n-p}{k}(n-k)!$ (since the set of all such permutations is B_k). This proves (23).]

It is also clear that $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$.

Now, it is easy to see that a permutation $\sigma \in S_n$ is a p-desarrangement if and only if it satisfies

- **either** $p \geq \sigma(1)$,
- **or** $p < \sigma(1) < \sigma(2) \geq \sigma(3)$,
- **or** $p < \sigma(1) < \sigma(2) < \sigma(3) < \sigma(4) \geq \sigma(5)$,
- **or** $p < \sigma(1) < \sigma(2) < \sigma(3) < \sigma(4) < \sigma(5) < \sigma(6) \geq \sigma(7)$,
- etc.

In other words, a permutation $\sigma \in S_n$ is a p-desarrangement if and only if it satisfies

- **either** (not $p < \sigma(1)$),
- **or** ($p < \sigma(1) < \sigma(2)$ but not $p < \sigma(1) < \sigma(2) < \sigma(3)$),
- **or** ($p < \sigma(1) < \sigma(2) < \sigma(3) < \sigma(4)$ but not $p < \sigma(1) < \sigma(2) < \cdots < \sigma(5)$),
- **or** ($p < \sigma(1) < \sigma(2) < \cdots < \sigma(6)$ but not $p < \sigma(1) < \sigma(2) < \cdots < \sigma(7)$),
- etc.

In other words, a permutation $\sigma \in S_n$ is a p-desarrangement if and only if it satisfies

- **either** (not $\sigma \in B_1$),
- **or** ($\sigma \in B_2$ but not $\sigma \in B_3$),
- **or** ($\sigma \in B_4$ but not $\sigma \in B_5$),
- **or** ($\sigma \in B_6$ but not $\sigma \in B_7$),

- etc.

(since for any given $k \in \mathbb{N}$, the condition "$p < \sigma(1) < \sigma(2) < \cdots < \sigma(k)$" is equivalent to the condition "$\sigma \in B_k$").

Moreover, all these possibilities are mutually exclusive (since $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$, and therefore no $\sigma \in S_n$ can fail to satisfy $\sigma \in B_{2i+1}$ for some i and yet satisfy $\sigma \in B_{2j}$ for some larger j). Hence,

$$\begin{aligned}
&(\text{the number of } p\text{-desarrangements}) \\
&= \underbrace{(\text{the number of } \sigma \in S_n \text{ such that } (\text{not } \sigma \in B_1))}_{\substack{=(\text{the number of } \sigma\in S_n \text{ such that } (\sigma\in B_0 \text{ but not } \sigma\in B_1)) \\ (\text{because every } \sigma\in S_n \text{ satisfies } \sigma\in B_0)}} \\
&\quad + (\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_2 \text{ but not } \sigma \in B_3)) \\
&\quad + (\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_4 \text{ but not } \sigma \in B_5)) \\
&\quad + (\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_6 \text{ but not } \sigma \in B_7)) \\
&\quad + \cdots \\
&= (\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_0 \text{ but not } \sigma \in B_1)) \\
&\quad + (\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_2 \text{ but not } \sigma \in B_3)) \\
&\quad + (\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_4 \text{ but not } \sigma \in B_5)) \\
&\quad + (\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_6 \text{ but not } \sigma \in B_7)) \\
&\quad + \cdots \\
&= \sum_{i\geq 0} \underbrace{(\text{the number of } \sigma \in S_n \text{ such that } (\sigma \in B_{2i} \text{ but not } \sigma \in B_{2i+1}))}_{\substack{=|B_{2i}\setminus B_{2i+1}|=|B_{2i}|-|B_{2i+1}| \\ (\text{since } B_{2i+1}\subseteq B_{2i} \ (\text{because } B_0\supseteq B_1\supseteq B_2\supseteq\cdots))}} \\
&= \sum_{i\geq 0} (|B_{2i}| - |B_{2i+1}|) = \sum_{k\geq 0} (-1)^k |B_k| \\
&\qquad \left(\begin{array}{c}\text{because every finitely supported sequence } (a_0, a_1, a_2, \ldots) \text{ of numbers} \\ \text{satisfies } \sum_{i\geq 0} (a_{2i} - a_{2i+1}) = \sum_{k\geq 0} (-1)^k a_k\end{array}\right) \\
&= \sum_{k=0}^{n} (-1)^k \underbrace{|B_k|}_{\substack{=\binom{n-p}{k}(n-k)! \\ (\text{by (23)})}} + \sum_{k>n} (-1)^k \underbrace{|B_k|}_{\substack{=0 \\ (\text{since } B_k=\varnothing \text{ for } k>n)}} = \sum_{k=0}^{n} (-1)^k \binom{n-p}{k} (n-k)! \\
&= \sum_{k=0}^{n-p} (-1)^k \binom{n-p}{k} (n-k)! + \sum_{k=n-p+1}^{n} (-1)^k \underbrace{\binom{n-p}{k}}_{\substack{=0 \\ (\text{since } k>n-p)}} (n-k)! \\
&= \sum_{k=0}^{n-p} \binom{n-p}{k} \cdot (-1)^k (n-k)!.
\end{aligned}$$

This solves the exercise.

References

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.

http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10 .

[Kerber99] Adalbert Kerber, *Applied Finite Group Actions*, 2nd edition, Springer 1999.