University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Midterm 2

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due date: **Wednesday, 14 November 2018** at the beginning of class, or before that by email or canvas.
Please solve **at most 4 of the 6 exercises**!
Beware: **Collaboration is not allowed** on midterms!

Notations

Here is a list of notations that are used in this homework:

- We shall use the Iverson bracket notation as well as the notation $[n]$ for the set $\{1, 2, \ldots, n\}$ (when $n \in \mathbb{Z}$).
- If $n \in \mathbb{N}$, then S_n denotes the set of all permutations of $[n]$.
- If $n \in \mathbb{N}$ and $\sigma \in S_n$, then:
  - a *descent* of the permutation σ denotes an element $k \in [n-1]$ satisfying $\sigma(k) > \sigma(k+1)$.
  - the *descent set* $\operatorname{Des} \sigma$ of σ is defined as the set of all descents of σ.
  - the *descent number* $\operatorname{des} \sigma$ of σ is defined as the number of all descents of σ (that is, $\operatorname{des} \sigma = |\operatorname{Des} \sigma|$).
  - the *one-line notation* $\operatorname{OLN} \sigma$ of σ is defined as the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n))$. Often, this n-tuple is written with square brackets, i.e., as $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$.

- for each $i \in [n]$, we define $\ell_i(\sigma)$ to be the number of all $j \in \{i+1, i+2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$.
- we say that σ is *312-avoiding* if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(j) < \sigma(k) < \sigma(i)$.
- we say that σ is *321-avoiding* if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(k) < \sigma(j) < \sigma(i)$.

- For any $n \in \mathbb{N}$ and any $i \in [n-1]$, we let s_i denote the permutation in S_n that swaps i with $i+1$ while leaving all other elements of $[n]$ unchanged. (This assumes that n is determined by the context.)

- For any $n \in \mathbb{N}$ and any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\operatorname{cyc}_{i_1, i_2, \ldots, i_k}$ be the permutation in S_n that sends $i_1, i_2, \ldots, i_{k-1}, i_k$ to $i_2, i_3, \ldots, i_k, i_1$ (respectively) while leaving all the other elements of $[n]$ unchanged. (Again, this relies on n being clear from the context.)

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, the notation $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ denotes the number of all permutations $\sigma \in S_n$ having exactly k descents. This is called an *Eulerian number.*

- If X is a set, and if $\alpha : X \to X$ and $\beta : X \to X$ are two maps, then the composition $\alpha \circ \beta : X \to X$ is simply denoted by $\alpha\beta$, and is called the *product* of α and β. This notation is used for permutations, in particular.

- If X is a set, if $k \in \mathbb{N}$, and if $f : X \to X$ is any map, then the map $f^k : X \to X$ is defined by
$$f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}} = \underbrace{ff \cdots f}_{k \text{ times}}.$$
This map f^k is called the *k-th power of f* (or *k-th composition power of f*). These powers behave as one would expect as long as you have only one map $f : X \to X$ (meaning that $f^{a+b} = f^a f^b$ and $f^{ab} = (f^a)^b$ for any $a, b \in \mathbb{N}$); but be careful with several maps (e.g., two maps $f : X \to X$ and $g : X \to X$ don't always satisfy $(fg)^a = f^a g^a$). See [Grinbe16, Section 2.13.8] for details (where I write $f^{\circ k}$ instead of f^k).

- If X is a set, and if $f : X \to X$ is a map, then:
  - we say that f is an *involution* if and only if $f^2 = \mathrm{id}$. (Note that every involution is automatically a permutation.)
  - we say that f is *fixed-point-free* if each $x \in X$ satisfies $f(x) \neq x$ (that is, if f has no fixed points). (Note that the fixed-point-free permutations are precisely the derangements.)

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1 Exercise 1

1.1 Problem

Let n and k be positive integers.

For each $i \in \{0, 1, \ldots, n-1\}$ and $\tau \in S_{n-1}$, we let $\tau^{i}- \in S_n$ be the permutation such that

$$\operatorname{OLN}\left(\tau^{i}-\right) = \left(\tau\left(1\right), \tau\left(2\right), \ldots, \tau\left(i\right), n, \tau\left(i+1\right), \tau\left(i+2\right), \ldots, \tau\left(n-1\right)\right)$$

(that is, $\operatorname{OLN}\left(\tau^{i}-\right)$ is obtained from $\operatorname{OLN}\tau$ by inserting an n right after the i-th entry).

(a) Prove that each $i \in \{0, 1, \ldots, n-1\}$ and $\tau \in S_{n-1}$ satisfy

$$\begin{aligned}&\left[\operatorname{des}\left(\tau^{i}-\right) = k\right] \\ &= \left[\operatorname{des}\tau = k-1 \text{ and } \tau\left(i\right) < \tau\left(i+1\right)\right] + \left[\operatorname{des}\tau = k \text{ and } \tau\left(i\right) > \tau\left(i+1\right)\right],\end{aligned}$$

where we set $\tau\left(0\right) = 0$ and $\tau\left(n\right) = 0$.

(b) Prove that the map

$$\begin{aligned}\{0, 1, \ldots, n-1\} \times S_{n-1} &\to S_n, \\ \left(i, \tau\right) &\mapsto \tau^{i}-\end{aligned}$$

is a bijection.

(c) Prove that

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \left(k+1\right) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + \left(n-k\right) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle .$$

[**Hint:** You don't need to write more than a few sentences for parts **(a)** and **(b)**.]

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. For each $i \in [n]$, let

$$a_i = \operatorname{cyc}_{i', i'-1, \ldots, i} = s_{i'-1} s_{i'-2} \cdots s_i \in S_n, \qquad \text{where } i' = i + \ell_i\left(\sigma\right).$$

Prove that $\sigma = a_1 a_2 \cdots a_n$.

[**Hint:** Prove, "more generally", that if $j \in \{0, 1, \ldots, n\}$ is such that $1, 2, \ldots, j$ are fixed points of σ, then $\sigma = a_{j+1} a_{j+2} \cdots a_n$.]

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$.

(a) Prove that any $\sigma \in S_n$ and any $i \in [n]$ satisfy $\sigma(i) \leq i + \ell_i(\sigma)$.

(b) Prove that, for a given $\sigma \in S_n$, the following three statements are equivalent:

- A: We have $\sigma(i) \leq i + 1$ for all $i \in [n-1]$.
- B: The permutation σ is both 321-avoiding and 312-avoiding.
- C: We have $\ell_i(\sigma) \in \{0, 1\}$ for each $i \in [n]$. (In other words, the Lehmer code of σ consists only of 0's and 1's.)

(c) Assuming that $n \geq 1$, prove that the number of $\sigma \in S_n$ satisfying these three statements is 2^{n-1}.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n \geq 2$, and set $S = [n]$. Let $i \in [n-1]$. Prove that:

(a) The number of maps $f : S \to S$ with $f(i) = n$ and $f^n(S) = \{n\}$ is $2n^{n-3}$.

(b) Let $j \in [n-1]$ be such that $i \neq j$. The number of maps $f : S \to S$ with $f(i) = j$ and $f^n(S) = \{n\}$ is n^{n-3}.

[**Hint:** Substitute appropriate numbers for the variables in the Matrix-Tree Theorem.]

4.2 Solution

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5 Exercise 5

5.1 Problem

(a) For each $n \in \mathbb{N}$, prove that the number of fixed-point-free involutions $[n] \to [n]$ is

$$\begin{cases} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (n-1), & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(b) For each $n \in \mathbb{N}$, we let t_n be the number of all involutions in S_n. Prove that

$$t_n = \sum_{k=0}^{n} \binom{n}{2k} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)) \qquad \text{for each } n \in \mathbb{N}.$$

(c) For each $n \in \mathbb{N}$, prove that the number of maps $f : [n] \to [n]$ satisfying $f^3 = f$ is

$$\sum_{k=0}^{n} \binom{n}{k} k^{n-k} t_k.$$

5.2 Remark

The numbers in part **(a)** form the sequence A123023 in the OEIS. (And if you omit the terms for odd n, which are all zero, then you obtain sequence A001147, known as the *double factorials*.)

The numbers $t_0, t_1, t_2, \ldots$ in part **(b)** are sometimes called the *telephone numbers*, because an involution in S_n is a way how phone calls can be happening between n people $1, 2, \ldots, n$, assuming there are no conference calls. This is sequence A000085 in the OEIS.

Finally, the numbers in part **(c)** form sequence A060905.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let n be a positive integer, and let $p \in \{0, 1, \ldots, n\}$.

A permutation $\sigma \in S_n$ shall be called a *p-desarrangement* if it satisfies

$$\textbf{either } (\sigma = \mathrm{id} \text{ and } 2 \mid n) \textbf{ or } \sigma(1) \leq p \textbf{ or } (\sigma \neq \mathrm{id} \text{ and } 2 \mid \min(\mathrm{Des}\,\sigma)).$$

(The condition $2 \mid \min(\mathrm{Des}\,\sigma)$ means that the smallest descent of σ is even.[1] This is well-defined, since $\sigma \neq \mathrm{id}$ shows that σ has at least one descent. Further p-desarrangements are id when n is even, and all permutations starting with a number $\leq p$ (in one-line notation).)

Prove that the number of p-desarrangements in S_n is

$$\sum_{k=0}^{n-p} \binom{n-p}{k} \cdot (-1)^k (n-k)!.$$

[1] Here are all permutations $\sigma \neq \mathrm{id}$ in S_5 that satisfy this condition (written in one-line notation, with an

6.2 Remark

This number is exactly the number of p-derangements in S_n, as defined in Exercise 5 of midterm #1. This suggests the existence of a bijection between the p-desarrangements and the p-derangements. Such a thing has indeed been found in the case when $p = 0$. In this case, the 0-desarrangements are known as *desarrangements* (a pun on the name Désarmenien and the word "derangement"), whereas the 0-derangements are precisely the derangements. The desarrangements are just the permutations $\sigma \in S_n$ satisfying either $\sigma = \operatorname{id}$ or ($\sigma \neq \operatorname{id}$ and $2 \mid \min(\operatorname{Des} \sigma)$). One known bijection between the derangements and the desarrangements proceeds as follows:

- Let $\sigma \in S_n$ be a derangement. We want to define the corresponding desarrangement $F(\sigma)$.
- Compute the disjoint cycle decomposition of σ, and write it in such a way that each cycle contains its **largest** entry in its **second** position, and that the cycles are ordered in **increasing order of their largest entries**. That is, write

  $$\sigma = \operatorname{cyc}_{a_{1,1},a_{1,2},\ldots,a_{1,n_1}} \operatorname{cyc}_{a_{2,1},a_{2,2},\ldots,a_{2,n_2}} \cdots \operatorname{cyc}_{a_{k,1},a_{k,2},\ldots,a_{k,n_k}},$$

  where each of the numbers $1, 2, \ldots, n$ appears exactly once among the $a_{i,j}$, and where

  $$a_{i,2} \geq a_{i,j} \qquad \text{for all } i \text{ and } j, \qquad \text{and} \qquad a_{1,2} < a_{2,2} < \cdots < a_{k,2}.$$

- Now, let $F(\sigma)$ be the permutation whose one-line notation is

  $$(a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}, \ldots, a_{k,1}, a_{k,2}, \ldots, a_{k,n_k}).$$

For example, if $n = 7$ and $\sigma = [5, 3, 7, 6, 1, 4, 2]$ in one-line notation, then the appropriate representation of σ is $\sigma = \operatorname{cyc}_{1,5} \operatorname{cyc}_{4,6} \operatorname{cyc}_{3,7,2}$ and thus $F(\sigma) = [1, 5, 4, 6, 3, 7, 2]$ in one-line notation.

It is far from trivial to check that this is actually a well-defined bijection. I don't know if anything like that exists for $p \neq 0$. Feel free to explore. (But the simplest way to solve the exercise is not by bijection.)

6.3 Solution

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underline marking the position of the smallest descent):

$$\begin{array}{lllll}
[1,2,3,\underline{5},4], & [1,2,4,\underline{5},3], & [1,\underline{3},2,4,5], & [1,\underline{3},2,5,4], & [1,3,4,\underline{5},2], \\
[1,\underline{4},2,3,5], & [1,\underline{4},2,5,3], & [1,\underline{4},3,2,5], & [1,\underline{4},3,5,2], & [1,\underline{5},2,3,4], \\
[1,\underline{5},2,4,3], & [1,\underline{5},3,2,4], & [1,\underline{5},3,4,2], & [1,\underline{5},4,2,3], & [1,\underline{5},4,3,2], \\
[2,\underline{3},1,4,5], & [2,\underline{3},1,5,4], & [2,3,4,\underline{5},1], & [2,\underline{4},1,3,5], & [2,\underline{4},1,5,3], \\
[2,\underline{4},3,1,5], & [2,\underline{4},3,5,1], & [2,\underline{5},1,3,4], & [2,\underline{5},1,4,3], & [2,\underline{5},3,1,4], \\
[2,\underline{5},3,4,1], & [2,\underline{5},4,1,3], & [2,\underline{5},4,3,1], & [3,\underline{4},1,2,5], & [3,\underline{4},1,5,2], \\
[3,\underline{4},2,1,5], & [3,\underline{4},2,5,1], & [3,\underline{5},1,2,4], & [3,\underline{5},1,4,2], & [3,\underline{5},2,1,4], \\
[3,\underline{5},2,4,1], & [3,\underline{5},4,1,2], & [3,\underline{5},4,2,1], & [4,\underline{5},1,2,3], & [4,\underline{5},1,3,2], \\
 & [4,\underline{5},2,1,3], & [4,\underline{5},2,3,1], & [4,\underline{5},3,1,2], & [4,\underline{5},3,2,1].
\end{array}$$

References

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .