
Math 5705: Enumerative Combinatorics, Fall 2018: Midterm 1 (preliminary version)

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December 14, 2019

1 EXERCISE 1

1.1 PROBLEM

Let A and B be two finite sets, and let $f : A \rightarrow B$ be a map.

- (a) Prove that the number of maps $g : B \rightarrow A$ satisfying $f \circ g \circ f = f$ is

$$|A|^{|B \setminus f(A)|} \prod_{b \in f(A)} |f^{-1}(b)|.$$

(Here and in the following, $f(A)$ denotes the set $\{f(a) \mid a \in A\}$, whereas $f^{-1}(b)$ denotes the set $\{a \in A \mid f(a) = b\}$.)

- (b) Prove that the number of maps $g : B \rightarrow A$ satisfying $f \circ g \circ f = f$ and $g \circ f \circ g = g$ is

$$|f(A)|^{|B \setminus f(A)|} \prod_{b \in f(A)} |f^{-1}(b)|.$$

[Hint: For part (a), observe that

$$|A|^{|B \setminus f(A)|} \prod_{b \in f(A)} |f^{-1}(b)| = \prod_{b \in B} \begin{cases} |f^{-1}(b)|, & \text{if } b \in f(A); \\ |A|, & \text{if } b \notin f(A). \end{cases}$$

What does this suggest about the construction of such maps g ?

1.2 REMARK

The maps g in part **(a)** are called “generalized inverses” of f . The maps g in part **(b)** are called “reflexive generalized inverses” of f . Note that one consequence of part **(b)** is that there is always at least one reflexive generalized inverse of f (unless A is empty).

One can similarly define generalized inverses for linear maps between vector spaces; the resulting notion is much more well-known and has books devoted to it (see the Wikipedia for an overview).

1.3 SOLUTION

(a) Our solution will rely on the following claim:

Claim 1: Let $g : B \rightarrow A$ be a map. Then, $f \circ g \circ f = f$ if and only if we have

$$(g(b) \in f^{-1}(b) \text{ for each } b \in f(A)).$$

[*Proof of Claim 1:* Claim 1 is an “if and only if” statement. We shall prove it by first proving the “ \implies ” part (i.e., the “only if” part), and then proving the “ \impliedby ” part (i.e., the “if” part).

\implies : Assume that $f \circ g \circ f = f$. We must prove that $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$. Let $b \in f(A)$. Thus, there exists some $x \in A$ such that $b = f(x)$. Consider this x . Now,

$$f\left(g\left(\underbrace{b}_{=f(x)}\right)\right) = f(g(f(x))) = \underbrace{(f \circ g \circ f)}_{=f}(x) = f(x) = b.$$

Hence, $g(b) \in \{a \in A \mid f(a) = b\} = f^{-1}(b)$.

Now, forget that we fixed b . We thus have proven that $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$. This proves the “ \implies ” part of Claim 1.

\impliedby : Assume that $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$. We must prove that $f \circ g \circ f = f$. Let $x \in A$. Then, $f(x) \in f(A)$.

But we have assumed that $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$. Applying this to $b = f(x)$, we obtain $g(f(x)) \in f^{-1}(f(x)) = \{a \in A \mid f(a) = f(x)\}$ (by the definition of $f^{-1}(f(x))$). In other words, $g(f(x))$ is an $a \in A$ satisfying $f(a) = f(x)$. Hence, $g(f(x))$ satisfies $f(g(f(x))) = f(x)$. Thus, $(f \circ g \circ f)(x) = f(g(f(x))) = f(x)$.

Forget that we fixed x . We thus have shown that $(f \circ g \circ f)(x) = f(x)$ for each $x \in A$. In other words, $f \circ g \circ f = f$. This proves the “ \impliedby ” part of Claim 1.

Thus, Claim 1 is proven.]

Claim 1 shows that a map $g : B \rightarrow A$ satisfies $f \circ g \circ f = f$ if and only if we have

$$(g(b) \in f^{-1}(b) \text{ for each } b \in f(A)). \quad (1)$$

Hence, in order to construct a map $g : B \rightarrow A$ satisfying $f \circ g \circ f = f$, we can proceed as follows:

- For each $b \in f(A)$, choose the value $g(b)$ to be one of the elements of the set $f^{-1}(b)$. (Indeed, $g(b)$ must belong to $f^{-1}(b)$, because our g should satisfy (1).) Note that we have $|f^{-1}(b)|$ many choices for each $b \in f(A)$.
- For each $b \in B \setminus f(A)$, choose the value $g(b)$ arbitrarily (among all $|A|$ elements of A). Note that we have $|A|$ many choices for each $b \in B \setminus f(A)$.

Thus, there are $\left(\prod_{b \in f(A)} |f^{-1}(b)| \right) \cdot |A|^{|B \setminus f(A)|}$ many ways to perform this construction. Hence, the number of maps $g : B \rightarrow A$ satisfying $f \circ g \circ f = f$ is

$$\left(\prod_{b \in f(A)} |f^{-1}(b)| \right) \cdot |A|^{|B \setminus f(A)|} = |A|^{|B \setminus f(A)|} \prod_{b \in f(A)} |f^{-1}(b)|.$$

This solves part (a) of the exercise.

(b) We need the following claim:

Claim 2: Let $g : B \rightarrow A$ be a map. Then, $(f \circ g \circ f = f$ and $g \circ f \circ g = g)$ if and only if the two statements

$$(g(b) \in f^{-1}(b) \text{ for each } b \in f(A)) \quad (2)$$

and

$$(g(b) \in g(f(A)) \text{ for each } b \in B \setminus f(A)) \quad (3)$$

hold.

[*Proof of Claim 2:* Claim 2 is an “if and only if” statement. We shall prove it by first proving the “ \implies ” part (i.e., the “only if” part), and then proving the “ \impliedby ” part (i.e., the “if” part).

\implies : Assume that $(f \circ g \circ f = f$ and $g \circ f \circ g = g)$. We must prove that the two statements (2) and (3) hold.

Claim 1 shows that $f \circ g \circ f = f$ if and only if we have $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$. Hence, we have $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$ (since $f \circ g \circ f = f$). In other words, the statement (2) holds.

We have $g \circ f \circ g = g$, thus $g = g \circ f \circ g$. For each $b \in B \setminus f(A)$, we have

$$\underbrace{g}_{=g \circ f \circ g}(b) = (g \circ f \circ g)(b) = g\left(f\left(\underbrace{g(b)}_{\in A}\right)\right) \in g(f(A)).$$

Hence, the statement (3) holds. Thus, we have shown that the two statements (2) and (3) hold. This proves the “ \implies ” part of Claim 1.

\impliedby : Assume that the two statements (2) and (3) hold.

We must prove that $(f \circ g \circ f = f$ and $g \circ f \circ g = g)$.

Claim 1 shows that $f \circ g \circ f = f$ if and only if we have $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$. Thus, $f \circ g \circ f = f$ (because we have $(g(b) \in f^{-1}(b) \text{ for each } b \in f(A))$ (since the statement (2) holds)).

Let $b \in B$. Then, $g(b) \in g(f(A))$ ¹. In other words, there exists some $a \in A$ such that $g(b) = g(f(a))$. Consider this a . Now,

$$\begin{aligned} (g \circ f \circ g)(b) &= g\left(f\left(\underbrace{g(b)}_{=g(f(a))}\right)\right) = g\left(\underbrace{f(g(f(a)))}_{=(f \circ g \circ f)(a)}\right) = g\left(\underbrace{(f \circ g \circ f)(a)}_{=f}\right) \\ &= g(f(a)) = g(b). \end{aligned}$$

¹*Proof.* If $b \in f(A)$, then this is obvious. Hence, for the rest of this proof, we WLOG assume that we don't have $b \in f(A)$. Thus, $b \in B \setminus f(A)$ (since $b \in B$ but not $b \in f(A)$). Hence, (3) shows that $g(b) \in g(f(A))$, qed.

Forget that we fixed b . We thus have shown that $(g \circ f \circ g)(b) = g(b)$ for each $x \in B$. In other words, $g \circ f \circ g = g$.

Altogether, we thus have proven that $(f \circ g \circ f = f$ and $g \circ f \circ g = g)$. This proves the “ \Leftarrow ” part of Claim 2.

Thus, Claim 2 is proven.]

Claim 2 shows that a map $g : B \rightarrow A$ satisfies $f \circ g \circ f = f$ and $g \circ f \circ g = g$ if and only if the two statements (2) and (3) hold. Hence, in order to construct a map $g : B \rightarrow A$ satisfying $f \circ g \circ f = f$ and $g \circ f \circ g = g$, we can proceed as follows:

- For each $b \in f(A)$, choose the value $g(b)$ to be one of the elements of the set $f^{-1}(b)$. (Indeed, $g(b)$ must belong to $f^{-1}(b)$, because our g should satisfy (2).) Note that we have $|f^{-1}(b)|$ many choices for each $b \in f(A)$.
- At this point, we have already set the values $g(b)$ for all $b \in f(A)$. Thus, the set $g(f(A))$ is already uniquely determined. Moreover, this set $g(f(A))$ has $|f(A)|$ elements².
- For each $b \in B \setminus f(A)$, choose the value $g(b)$ to be one of the $|f(A)|$ elements of this set $g(f(A))$ ³. (Indeed, $g(b)$ must belong to $g(f(A))$, because our g should satisfy (3).) Note that we have $|f(A)|$ many choices for each $b \in B \setminus f(A)$.

Thus, there are $\left(\prod_{b \in f(A)} |f^{-1}(b)| \right) \cdot |f(A)|^{|B \setminus f(A)|}$ many ways to perform this construction.

Hence, the number of maps $g : B \rightarrow A$ satisfying $f \circ g \circ f = f$ and $g \circ f \circ g = g$ is

$$\left(\prod_{b \in f(A)} |f^{-1}(b)| \right) \cdot |f(A)|^{|B \setminus f(A)|} = |f(A)|^{|B \setminus f(A)|} \prod_{b \in f(A)} |f^{-1}(b)|.$$

This solves part **(b)** of the exercise.

²*Proof.* Let b_1 and b_2 be two elements of $f(A)$ such that $g(b_1) = g(b_2)$. We shall show that $b_1 = b_2$.

Indeed, $g(b_1)$ is an element of the set $f^{-1}(b_1)$ (because of how $g(b_1)$ was chosen). In other words, $g(b_1) \in f^{-1}(b_1) = \{a \in A \mid f(a) = b_1\}$. Hence, $g(b_1)$ is an element of A and satisfies $f(g(b_1)) = b_1$.

Similarly, $g(b_2)$ is an element of A and satisfies $f(g(b_2)) = b_2$. Now, $b_1 = f\left(\underbrace{g(b_1)}_{=g(b_2)}\right) = f(g(b_2)) = b_2$.

Now, forget that we fixed b_1 and b_2 . We thus have proven that if b_1 and b_2 are two elements of $f(A)$ such that $g(b_1) = g(b_2)$, then $b_1 = b_2$. In other words, the elements $g(b)$ for all $b \in f(A)$ are distinct. Hence, the number of these elements $g(b)$ is $|f(A)|$. In other words,

$$|\{g(b) \mid b \in f(A)\}| = |f(A)|.$$

In view of $g(f(A)) = \{g(b) \mid b \in f(A)\}$, this rewrites as $|g(f(A))| = |f(A)|$. In other words, the set $g(f(A))$ has $|f(A)|$ elements.

³This is a meaningful instruction, because the set $g(f(A))$ is already determined and has $|f(A)|$ elements.

2 EXERCISE 2

2.1 PROBLEM

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Prove that

$$(n+m) \sum_{j=0}^m (-1)^j \frac{\binom{m}{j}}{\binom{n+j}{j}} = n.$$

[Hint: The fraction on the left hand side has too many j 's. Try to simplify it to get the number of j 's down to just 1 (not counting the exponent in $(-1)^j$).]

2.2 SOLUTION

Forget that we fixed n and m . We shall use the following identity:

Lemma 2.1. *Let $n \in \mathbb{N}$ be positive. Let $m \in \mathbb{N}$. Then,*

$$\sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}. \quad (4)$$

Lemma 2.1 is precisely the claim of Exercise 4 on homework set #2.

We shall also use the following classical formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for any } n \in \mathbb{N} \text{ and } k \in \mathbb{N} \text{ satisfying } n \geq k. \quad (5)$$

Now, let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Thus, $n \geq 0$ and $m \geq 0$, so that $n + \underbrace{m}_{\geq 0} \geq n \geq 0$. Hence,

(5) (applied to $n+m$ and n instead of n and k) yields

$$\binom{n+m}{n} = \frac{(n+m)!}{n!((n+m)-n)!} = \frac{(n+m)!}{n!m!}. \quad (6)$$

We shall first prove the following two claims:

Claim 1: We have

$$\frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{\binom{n+m}{m-j}}{\binom{n+m}{n}}$$

for each $j \in \{0, 1, \dots, m\}$.

[*Proof of Claim 1:* Let $j \in \{0, 1, \dots, m\}$. Thus, $0 \leq j \leq m$. Hence, $m \geq j \geq 0$. Thus, (5) (applied to m and j instead of n and k) yields

$$\binom{m}{j} = \frac{m!}{j!(m-j)!}. \quad (7)$$

Furthermore, $\underbrace{n}_{\geq 0} + m \geq m \geq m - j$ (since $j \geq 0$) and $m - j \in \mathbb{N}$ (since $m \geq j$). Hence, (5) (applied to $n + m$ and $m - j$ instead of n and k) yields

$$\binom{n+m}{m-j} = \frac{(n+m)!}{(m-j)!((n+m)-(m-j))!} = \frac{(n+m)!}{(m-j)!(n+j)!}.$$

Dividing this equality by the equality (6), we obtain

$$\frac{\binom{n+m}{m-j}}{\binom{n+m}{n}} = \frac{\left(\frac{(n+m)!}{(m-j)!(n+j)!}\right)}{\left(\frac{(n+m)!}{n!m!}\right)} = \frac{n!m!}{(m-j)!(n+j)!}. \quad (8)$$

Also, $\underbrace{n}_{\geq 0} + j \geq j$ and $j \in \{0, 1, \dots, m\} \subseteq \mathbb{N}$. Hence, (5) (applied to $n + j$ and j instead of n and k) yields

$$\binom{n+j}{j} = \frac{(n+j)!}{j!((n+j)-j)!} = \frac{(n+j)!}{j!n!}. \quad (9)$$

Dividing the equality (7) by the equality (9), we obtain

$$\frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{\left(\frac{m!}{j!(m-j)!}\right)}{\left(\frac{(n+j)!}{j!n!}\right)} = \frac{n!m!}{(m-j)!(n+j)!} = \frac{\binom{n+m}{m-j}}{\binom{n+m}{n}}$$

(by (8)). This proves Claim 1.]

Claim 2: We have

$$(n+m) \binom{n+m-1}{m} = n \binom{n+m}{n}.$$

[*Proof of Claim 2:* We are in one of the following two cases:

Case 1: We have $n = 0$.

Case 2: We have $n \neq 0$.

Let us first consider Case 1. In this case, we have $n = 0$. Hence,

$$\begin{aligned} (n+m) \binom{n+m-1}{m} &= \underbrace{(0+m)}_{=m} \binom{(0+m-1)}{m} \\ &= \binom{m-1}{m} = \frac{(m-1)((m-1)-1) \cdots ((m-1)-m+1)}{m!} \\ &\quad \text{(by the definition of } \binom{m-1}{m} \text{)} \\ &= m \cdot \frac{(m-1)((m-1)-1) \cdots ((m-1)-m+1)}{m!} \\ &= \frac{1}{m!} \cdot m \cdot \underbrace{(m-1)((m-1)-1) \cdots ((m-1)-m+1)}_{\substack{=(m-1)(m-2) \cdots (m-m) \\ =(m-1)(m-2) \cdots 0}} \\ &= \frac{1}{m!} \cdot m \cdot \underbrace{(m-1)(m-2) \cdots 0}_{\substack{=m(m-1) \cdots 0 \\ =(m(m-1) \cdots 1) \cdot 0 \\ =0}} = 0. \end{aligned}$$

Comparing this with $\underbrace{n}_{=0} \binom{n+m}{n} = 0$, we obtain $(n+m) \binom{n+m-1}{m} = n \binom{n+m}{n}$. Hence, Claim 2 is proven in Case 1.

Let us now consider Case 2. In this case, we have $n \neq 0$. Hence, $n \geq 1$ (since $n \in \mathbb{N}$). Thus, $\underbrace{n}_{\geq 1} + m - 1 \geq 1 + m - 1 = m \geq 0$, so that $n + m - 1 \in \mathbb{N}$. Hence, (5) (applied to $n + m - 1$ and m instead of n and k) yields

$$\binom{n+m-1}{m} = \frac{(n+m-1)!}{m!((n+m-1)-m)!} = \frac{(n+m-1)!}{m!(n-1)!}.$$

Multiplying both sides of this equality by $n + m$, we obtain

$$(n+m) \binom{n+m-1}{m} = (n+m) \cdot \frac{(n+m-1)!}{m!(n-1)!}. \quad (10)$$

But $n! = n(n-1)!$ (since $n \geq 1$) and $(n+m)! = (n+m)(n+m-1)!$ (since $n + \underbrace{m}_{\geq 0} \geq n \geq 1$). The equality (6) becomes

$$\begin{aligned} \binom{n+m}{n} &= \frac{(n+m)!}{n!m!} = \frac{(n+m)(n+m-1)!}{n(n-1)!m!} \\ &\quad (\text{since } n! = n(n-1)! \text{ and } (n+m)! = (n+m)(n+m-1)!). \end{aligned}$$

Multiplying both sides of this equality by n , we find

$$n \binom{n+m}{n} = n \cdot \frac{(n+m)(n+m-1)!}{n(n-1)!m!} = (n+m) \cdot \frac{(n+m-1)!}{m!(n-1)!}.$$

Comparing this with (10), we obtain $(n+m) \binom{n+m-1}{m} = n \binom{n+m}{n}$. Thus, Claim 2 is proven in Case 2.

We now have proven Claim 2 in both Cases 1 and 2. Hence, Claim 2 always holds.]

Let us now solve the actual exercise. We need to prove the identity

$$(n+m) \sum_{j=0}^m (-1)^j \frac{\binom{m}{j}}{\binom{n+j}{j}} = n.$$

If $n+m = 0$, then this identity clearly holds (because both sides of this identity are 0 in this case⁴). Hence, for the rest of this proof, we WLOG assume that we don't have $n+m = 0$. Hence, $n+m \neq 0$. Thus, $n+m \geq 1$ (since $n+m \in \mathbb{N}$), so that $n+m-1 \in \mathbb{N}$. The integer $n+m$ is positive (since $n+m \geq 1 > 0$).

⁴Proof. The left hand side is clearly 0, because it contains the factor $n+m = 0$. The right hand side is also 0, because from $n+m = 0$, we obtain $0 = n+m \geq n$, thus $n \leq 0$ and therefore $n = 0$ (since $n \in \mathbb{N}$). Thus, both sides are 0.

Now,

$$\begin{aligned}
 (n+m) \sum_{j=0}^m (-1)^j \frac{\binom{m}{j}}{\binom{n+j}{j}} &= (n+m) \sum_{j=0}^m (-1)^j \frac{\binom{n+m}{m-j}}{\binom{n+m}{n}} \\
 &= \frac{\binom{n+m}{m-j}}{\binom{n+m}{n}} \\
 &\quad \text{(by Claim 1)} \\
 &= \frac{n+m}{\binom{n+m}{n}} \sum_{j=0}^m (-1)^j \binom{n+m}{m-j}.
 \end{aligned}$$

In view of

$$\begin{aligned}
 \sum_{j=0}^m (-1)^j \binom{n+m}{m-j} &= \sum_{k=0}^m \underbrace{(-1)^{m-k}}_{=(-1)^{k-m} \text{ (since } m-k \equiv k-m \pmod{2})} \underbrace{\binom{n+m}{m-(m-k)}}_{=\binom{n+m}{k}} \\
 &\quad \text{(here, we have substituted } m-k \text{ for } j \text{ in the sum)} \\
 &= \sum_{k=0}^m \underbrace{(-1)^{k-m}}_{=(-1)^k (-1)^{-m}} \binom{n+m}{k} = (-1)^{-m} \underbrace{\sum_{k=0}^m (-1)^k \binom{n+m}{k}}_{=(-1)^m \binom{n+m-1}{m} \text{ (by Lemma 2.1, applied to } n+m \text{ instead of } n)} \\
 &= \underbrace{(-1)^{-m} (-1)^m}_{=1} \binom{n+m-1}{m} = \binom{n+m-1}{m},
 \end{aligned}$$

this becomes

$$\begin{aligned}
 (n+m) \sum_{j=0}^m (-1)^j \frac{\binom{m}{j}}{\binom{n+j}{j}} &= \frac{n+m}{\binom{n+m}{n}} \underbrace{\sum_{j=0}^m (-1)^j \binom{n+m}{m-j}}_{=\binom{n+m-1}{m}} = \frac{n+m}{\binom{n+m}{n}} \binom{n+m-1}{m} \\
 &= \frac{1}{\binom{n+m}{n}} \underbrace{(n+m) \binom{n+m-1}{m}}_{=n \binom{n+m}{n} \text{ (by Claim 2)}} = \frac{1}{\binom{n+m}{n}} n \binom{n+m}{n} \\
 &= n.
 \end{aligned}$$

This solves the exercise.

3 EXERCISE 3

3.1 PROBLEM

Let n be a positive integer. Let a_1, a_2, \dots, a_n be n integers. Let $F : \mathbb{Z} \rightarrow \mathbb{R}$ be any function. Prove that

$$F(\max\{a_1, a_2, \dots, a_n\}) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} F(\min\{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}).$$

[Hint: This generalizes Exercise 5 on Spring 2018 Math 4707 homework set #2. Will some of the solutions given there still apply to this generalization?]

3.2 SOLUTION

The exercise generalizes Exercise 5 on Spring 2018 Math 4707 homework set #2. Out of the three solutions given for the latter exercise, two can easily be modified to solve the exercise at hand: namely, the second and the third solutions. The modified versions of these two solutions will constitute the First and the Second solutions below.

3.2.1 FIRST SOLUTION

We will rely on the following lemma:

Lemma 3.1. *Let S be a finite set. Let g be an element of S . For any nonempty subset I of S , let b_I be a real number. Assume that for every subset K of S satisfying $g \notin K$ and $K \neq \emptyset$, we have*

$$b_{K \cup \{g\}} = b_K. \quad (11)$$

Then,

$$\sum_{\substack{I \subseteq S; \\ I \neq \emptyset}} (-1)^{|I|-1} b_I = b_{\{g\}}. \quad (12)$$

This lemma appears (with proof) in the Second solution to Exercise 5 on Spring 2018 Math 4707 homework set #2; we refer to the latter source for its proof.

Let us now solve the exercise at hand.

We have

$$\begin{aligned}
& \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} F(\min \{a_i \mid i \in I\}) \\
&= \sum_{k=1}^n \sum_{\substack{I \subseteq [n]; \\ |I|=k}} (-1)^{|I|-1} F(\min \{a_i \mid i \in I\}) \\
&\quad \text{(because the nonempty subsets } I \text{ of } [n] \text{ are precisely the subsets } I \text{ of } [n] \text{ such that } |I| \in \{1, 2, \dots, n\}) \\
&= \sum_{k=1}^n \sum_{\substack{I \subseteq [n]; \\ |I|=k}} \underbrace{(-1)^{|I|-1}}_{=(-1)^{k-1} \text{ (since } |I|=k)} F(\min \{a_i \mid i \in I\}) \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{\substack{I \subseteq [n]; \\ |I|=k}} F(\min \{a_i \mid i \in I\}) \\
&= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} F(\min \{a_i \mid i \in \{i_1, i_2, \dots, i_k\}\}) \\
&\quad \text{(here, we have substituted } \{i_1, i_2, \dots, i_k\} \text{ for } I \text{ in the sum, since each } k\text{-element subset } I \text{ of } [n] \text{ can be uniquely written in the form } \{i_1, i_2, \dots, i_k\} \text{ for some } k\text{-tuple } (i_1, i_2, \dots, i_k) \text{ of integers with } 1 \leq i_1 < i_2 < \dots < i_k \leq n) \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} F\left(\min \{a_i \mid i \in \{i_1, i_2, \dots, i_k\}\}\right) \\
&\quad \quad \quad = \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} F(\min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}).
\end{aligned}$$

Hence, it suffices to prove the equality

$$F(\max \{a_1, a_2, \dots, a_n\}) = \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} F(\min \{a_i \mid i \in I\}) \quad (13)$$

(because then, it will follow that

$$\begin{aligned}
F(\max \{a_1, a_2, \dots, a_n\}) &= \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} F(\min \{a_i \mid i \in I\}) \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} F(\min \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}),
\end{aligned}$$

and this will solve the exercise).

The set $\{a_1, a_2, \dots, a_n\}$ is nonempty (since n is positive) and finite. Thus, the set $\{a_1, a_2, \dots, a_n\}$ is a nonempty finite set of integers, and therefore has a maximum. In other words, there exists some $g \in [n]$ such that $a_g = \max \{a_1, a_2, \dots, a_n\}$. Consider such a g . (There may be several choices for g , but we choose one.)

For every subset K of $[n]$ satisfying $g \notin K$ and $K \neq \emptyset$, we have

$$\min \{a_i \mid i \in K \cup \{g\}\} = \min \{a_i \mid i \in K\}. \quad (14)$$

(This equality, too, appears in the Second solution to Exercise 5 on Spring 2018 Math 4707 homework set #2; thus, again, we refer to the latter source for its proof.)

Thus, Lemma 3.1 (applied to $S = [n]$ and $b_I = F(\min \{a_i \mid i \in I\})$) yields that

$$\begin{aligned} \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} F(\min \{a_i \mid i \in I\}) &= F\left(\min \underbrace{\{a_i \mid i \in \{g\}\}}_{=\{a_g\}}\right) = F\left(\underbrace{\min \{a_g\}}_{=a_g}\right) \\ &= F(a_g) = F(\max \{a_1, a_2, \dots, a_n\}) \end{aligned}$$

(since $a_g = \max \{a_1, a_2, \dots, a_n\}$). This proves (13). Thus, the exercise is solved (because we know that proving (13) is sufficient).

3.2.2 SECOND SOLUTION

Just as in the First solution above, we observe that it suffices to prove the equality (13). So let us prove this equality.

This equality clearly does not change when the numbers a_1, a_2, \dots, a_n are permuted (because when this happens, the addends $(-1)^{|I|-1} F(\min \{a_i \mid i \in I\})$ on the right hand side get permuted as well, while the left hand side $F(\max \{a_1, a_2, \dots, a_n\})$ is preserved). Hence, we WLOG assume that $a_1 \leq a_2 \leq \dots \leq a_n$ (because we can always achieve this by permuting the numbers a_1, a_2, \dots, a_n : this is called sorting). Hence, each nonempty subset I of $[n]$ satisfies

$$\min \{a_i \mid i \in I\} = a_{\min I}. \quad (15)$$

(For example, $\min \{a_3, a_5, a_6\} = a_3 = a_{\min \{3,5,6\}}$.)

We shall also use the following lemma:

Lemma 3.2. *Let $n \in \mathbb{N}$. Let $k \in [n]$. Then,*

$$\sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1} = [k = n].$$

Lemma 3.2 appears in the Third solution to Exercise 5 on Spring 2018 Math 4707 homework set #2; thus, we don't repeat its proof here.

Now,

$$\begin{aligned}
& \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} F \left(\underbrace{\min \{a_i \mid i \in I\}}_{\substack{= a_{\min I} \\ (\text{by (15)})}} \right) \\
&= \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} F(a_{\min I}) = \sum_{\substack{k \in [n] \\ = \sum_{k=1}^n}} \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1} F \left(\underbrace{a_{\min I}}_{\substack{= a_k \\ (\text{since } \min I = k)}} \right) \\
&\quad (\text{here, we have split the sum according to the value of } \min I) \\
&= \sum_{k=1}^n \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1} F(a_k) = \sum_{k=1}^n F(a_k) \underbrace{\sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset; \\ \min I = k}} (-1)^{|I|-1}}_{\substack{= [k=n] \\ (\text{by Lemma 3.2})}} \\
&= \sum_{k=1}^n F(a_k) [k=n] = \sum_{k=1}^{n-1} F(a_k) \underbrace{[k=n]}_{\substack{=0 \\ (\text{since we don't have } k=n \\ (\text{because } k \leq n-1 < n))}} + F(a_n) \underbrace{[n=n]}_{=1} \\
&= \underbrace{\sum_{k=1}^{n-1} F(a_k) 0}_{=0} + F(a_n) = F(a_n) = F(\max \{a_1, a_2, \dots, a_n\})
\end{aligned}$$

(since $a_n = \max \{a_1, a_2, \dots, a_n\}$ (because $a_1 \leq a_2 \leq \dots \leq a_n$)). This proves (13). Thus, the exercise is solved again.

4 EXERCISE 4

4.1 PROBLEM

Recall once again the *Fibonacci sequence* (f_0, f_1, f_2, \dots) , which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2. \quad (16)$$

Now, let us define f_n for **negative** integers n as well, by “applying (16) backwards”: This means that we set $f_{n-2} = f_n - f_{n-1}$ for all integers $n \leq 1$. This allows us to recursively compute $f_{-1}, f_{-2}, f_{-3}, \dots$ (in this order). For example,

$$\begin{aligned}
f_{-1} &= f_1 - f_0 = 1 - 0 = 1; \\
f_{-2} &= f_0 - f_{-1} = 0 - 1 = -1; \\
f_{-3} &= f_{-1} - f_{-2} = 1 - (-1) = 2,
\end{aligned}$$

etc.

- (a) Prove that $f_{-n} = (-1)^{n-1} f_n$ for each $n \in \mathbb{Z}$.
- (b) Prove that $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.
- (c) Prove that $7f_n = f_{n-4} + f_{n+4}$ for all $n \in \mathbb{Z}$.

[**Hint:** This is **not** an exercise about the combinatorial interpretations (domino tilings, lacunar subsets, etc.) of Fibonacci numbers. Make sure that your proofs cover all integers, not just elements of \mathbb{N} .]

4.2 SOLUTION

Our definition of the Fibonacci numbers f_n for negative integers n shows that the equality (16) holds not only for all $n \geq 2$, but also for all $n \in \mathbb{Z}$. In other words, we have

$$f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \in \mathbb{Z}. \quad (17)$$

- (a) We shall first prove a slightly less general claim (requiring $n \in \mathbb{N}$ instead of $n \in \mathbb{Z}$):

Claim 1: We have $f_{-n} = (-1)^{n-1} f_n$ for each $n \in \mathbb{N}$.

[*Proof of Claim 1:* We shall prove Claim 1 by strong induction on n :

Induction step: Let $N \in \mathbb{N}$. Assume that Claim 1 holds for all $n < N$. We must now prove that Claim 1 holds for $n = N$.

We have assumed that Claim 1 holds for all $n < N$. In other words, we have

$$f_{-n} = (-1)^{n-1} f_n \quad \text{for all } n < N. \quad (18)$$

Now, we must prove that Claim 1 holds for $n = N$. In other words, we must prove that $f_{-N} = (-1)^{N-1} f_N$. If $N \leq 1$, then this is easy to verify⁵. Hence, for the rest of this proof, we WLOG assume that we don't have $N \leq 1$. Hence, $N > 1$, so that $N \geq 2$ (since $N \in \mathbb{N}$). Hence, both $N - 1$ and $N - 2$ belong to \mathbb{N} . Hence, we obtain $f_{-(N-1)} = (-1)^{(N-1)-1} f_{N-1}$ (by applying (18) to $n = N - 1$) and $f_{-(N-2)} = (-1)^{(N-2)-1} f_{N-2}$ (by applying (18) to $n = N - 2$).

Now, (17) (applied to $n = N$) yields

$$f_N = f_{N-1} + f_{N-2} = f_{N-2} + f_{N-1}. \quad (19)$$

But (17) (applied to $n = -N + 2$) yields

$$f_{-N+2} = f_{(-N+2)-1} + f_{(-N+2)-2} = f_{-N+1} + f_{-N}.$$

⁵*Proof.* If $N \leq 1$, then N is either 0 or 1 (since $N \in \mathbb{N}$). But in each of these two cases, we can easily verify that $f_{-N} = (-1)^{N-1} f_N$:

- If $N = 0$, then $f_{-N} = f_{-0} = f_0 = 0$ and $f_N = f_0 = 0$ and $(-1)^{N-1} = (-1)^{0-1} = -1$. Thus, $f_{-N} = (-1)^{N-1} f_N$ holds in this case.
- If $N = 1$, then $f_{-N} = f_{-1} = 1$ and $f_N = f_1 = 1$ and $(-1)^{N-1} = (-1)^{1-1} = 1$. Thus, $f_{-N} = (-1)^{N-1} f_N$ holds in this case.

Thus, $f_{-N} = (-1)^{N-1} f_N$ holds if $N \leq 1$.

Solving this equation for f_{-N} , we obtain

$$\begin{aligned}
 f_{-N} &= \underbrace{f_{-N+2}}_{=f_{-(N-2)}=(-1)^{(N-2)-1}f_{N-2}} - \underbrace{f_{-N+1}}_{=f_{-(N-1)}=(-1)^{(N-1)-1}f_{N-1}} \\
 &= \underbrace{(-1)^{(N-2)-1}}_{\substack{=(-1)^{N-1} \\ \text{(since } (N-2)-1 \equiv N-1 \pmod{2})}} f_{N-2} - \underbrace{(-1)^{(N-1)-1}}_{=-(-1)^{N-1}} f_{N-1} \\
 &= (-1)^{N-1} f_{N-2} - \left(-(-1)^{N-1} \right) f_{N-1} = (-1)^{N-1} \underbrace{(f_{N-2} + f_{N-1})}_{\substack{=f_N \\ \text{(by (19))}}} = (-1)^{N-1} f_N.
 \end{aligned}$$

Thus, $f_{-N} = (-1)^{N-1} f_N$ is proven. In other words, Claim 1 holds for $n = N$. This completes the induction step. Thus, Claim 1 is proven by strong induction.]

Now, we still have to solve part (a) of the exercise. So let $n \in \mathbb{Z}$. We must prove that $f_{-n} = (-1)^{n-1} f_n$. We are in one of the following two cases:

Case 1: We have $n \geq 0$.

Case 2: We have $n < 0$.

Let us first consider Case 1. In this case, we have $n \geq 0$. Thus, $n \in \mathbb{N}$ (since $n \in \mathbb{Z}$). Hence, Claim 1 yields $f_{-n} = (-1)^{n-1} f_n$. Thus, $f_{-n} = (-1)^{n-1} f_n$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $n < 0$. Hence, $-n > 0$ and thus $-n \in \mathbb{N}$ (since $-n \in \mathbb{Z}$). Hence, Claim 1 (applied to $-n$ instead of n) yields $f_{-(-n)} = (-1)^{-n-1} f_{-n}$. Solving this equation for f_{-n} , we find

$$\begin{aligned}
 f_{-n} &= \frac{1}{\underbrace{(-1)^{-n-1}}_{\substack{=(-1)^{n+1}=(-1)^{n-1} \\ \text{(since } n+1 \equiv n-1 \pmod{2})}}} \underbrace{f_{-(-n)}}_{=f_n} = (-1)^{n-1} f_n.
 \end{aligned}$$

Hence, $f_{-n} = (-1)^{n-1} f_n$ is proven in Case 2.

We have now proven $f_{-n} = (-1)^{n-1} f_n$ in each of the two Cases 1 and 2. Thus, $f_{-n} = (-1)^{n-1} f_n$ always holds. This solves part (a) of the exercise.

(b) We shall use the following variant induction principle, known to some as “two-sided induction”, which can be used for proving claims about all integers:

Theorem 4.1. Let $g \in \mathbb{Z}$.

Let $\mathbb{Z}_{\geq g}$ be the set $\{g, g+1, g+2, \dots\}$ (that is, the set of all integers that are $\geq g$).

Let $\mathbb{Z}_{\leq g}$ be the set $\{g, g-1, g-2, \dots\}$ (that is, the set of all integers that are $\leq g$).

For each $n \in \mathbb{Z}$, let $\mathcal{A}(n)$ be a logical statement.

Assume the following:

Assumption 1: The statement $\mathcal{A}(g)$ holds.

Assumption 2: If $k \in \mathbb{Z}_{\geq g}$ is such that $\mathcal{A}(k)$ holds, then $\mathcal{A}(k+1)$ also holds.

Assumption 3: If $k \in \mathbb{Z}_{\leq g}$ is such that $\mathcal{A}(k)$ holds, then $\mathcal{A}(k-1)$ also holds.

Then, $\mathcal{A}(n)$ holds for each $n \in \mathbb{Z}$.

Theorem 4.1 is Theorem 2.149 in [Grinbe16] (except that I have renamed m as k here). That said, its truth should be intuitively clear: Roughly speaking, it says that a claim about all integers can be proven by first verifying it for **some** integer g and then showing that it “spreads” both to the right and to the left along the number line.

Now, let us solve part **(b)** of our exercise.

For each $n \in \mathbb{Z}$, we let $\mathcal{A}(n)$ be the statement

$$(f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1} \text{ for all } m \in \mathbb{Z}). \quad (20)$$

Let $g = 0$. Then, the statement $\mathcal{A}(0)$ holds⁶. In other words, the statement $\mathcal{A}(g)$ holds (since $g = 0$). Hence, Assumption 1 of Theorem 4.1 is satisfied.

Let us now prove that the other two assumptions of Theorem 4.1 are satisfied as well:

[*Proof of Assumption 2:* Let $k \in \mathbb{Z}_{\geq g}$ be such that $\mathcal{A}(k)$ holds. We must prove that $\mathcal{A}(k+1)$ also holds.

We have assumed that $\mathcal{A}(k)$ holds. In other words, we have

$$(f_{k+m+1} = f_k f_m + f_{k+1} f_{m+1} \text{ for all } m \in \mathbb{Z}). \quad (21)$$

Now, let $m \in \mathbb{Z}$. Then, (17) (applied to $n = m + 2$) yields

$$f_{m+2} = \underbrace{f_{(m+2)-1}}_{=f_{m+1}} + \underbrace{f_{(m+2)-2}}_{=f_m} = f_{m+1} + f_m. \quad (22)$$

The same argument (with m replaced by k) yields

$$f_{k+2} = f_{k+1} + f_k. \quad (23)$$

But we can apply (21) to $m + 1$ instead of m . Thus, we obtain

$$\begin{aligned} f_{k+(m+1)+1} &= f_k f_{m+1} + f_{k+1} \underbrace{f_{(m+1)+1}}_{\substack{=f_{m+2}=f_{m+1}+f_m \\ \text{(by (22))}}} = f_k f_{m+1} + \underbrace{f_{k+1} (f_{m+1} + f_m)}_{=f_{k+1} f_{m+1} + f_{k+1} f_m} \\ &= \underbrace{f_k f_{m+1} + f_{k+1} f_{m+1}}_{=(f_{k+1}+f_k) f_{m+1}} + f_{k+1} f_m = \underbrace{(f_{k+1} + f_k)}_{\substack{=f_{k+2} \\ \text{(by (23))}}} f_{m+1} + f_{k+1} f_m \\ &= f_{k+2} f_{m+1} + f_{k+1} f_m = f_{k+1} f_m + \underbrace{f_{k+2}}_{=f_{(k+1)+1}} f_{m+1} = f_{k+1} f_m + f_{(k+1)+1} f_{m+1}. \end{aligned}$$

In view of $k + (m + 1) + 1 = (k + 1) + m + 1$, this rewrites as

$$f_{(k+1)+m+1} = f_{k+1} f_m + f_{(k+1)+1} f_{m+1}.$$

Now, forget that we fixed m . We thus have shown that $f_{(k+1)+m+1} = f_{k+1} f_m + f_{(k+1)+1} f_{m+1}$ for all $m \in \mathbb{Z}$. In other words, $\mathcal{A}(k+1)$ holds. This completes the proof of Assumption 2.]

[*Proof of Assumption 3:* Let $k \in \mathbb{Z}_{\leq g}$ be such that $\mathcal{A}(k)$ holds. We must prove that $\mathcal{A}(k-1)$ also holds.

⁶*Proof.* For each $m \in \mathbb{Z}$, we have

$$\underbrace{f_0}_{=0} f_m + \underbrace{f_{0+1}}_{=f_1=1} f_{m+1} = 0 f_m + 1 f_{m+1} = f_{m+1},$$

so that $f_{0+m+1} = f_{m+1} = f_0 f_m + f_{0+1} f_{m+1}$. In other words, we have $f_{0+m+1} = f_0 f_m + f_{0+1} f_{m+1}$ for all $m \in \mathbb{Z}$. But this is precisely the statement $\mathcal{A}(0)$. Hence, the statement $\mathcal{A}(0)$ holds.

We have assumed that $\mathcal{A}(k)$ holds. In other words, we have

$$(f_{k+m+1} = f_k f_m + f_{k+1} f_{m+1} \text{ for all } m \in \mathbb{Z}). \quad (24)$$

Now, let $m \in \mathbb{Z}$. Then, (17) (applied to $n = m + 1$) yields

$$f_{m+1} = \underbrace{f_{(m+1)-1}}_{=f_m} + \underbrace{f_{(m+1)-2}}_{=f_{m-1}} = f_m + f_{m-1}. \quad (25)$$

The same argument (with m replaced by k) yields

$$f_{k+1} = f_k + f_{k-1}. \quad (26)$$

But we can apply (24) to $m - 1$ instead of m . Thus, we obtain

$$\begin{aligned} f_{k+(m-1)+1} &= f_k f_{m-1} + \underbrace{f_{k+1}}_{=f_k+f_{k-1} \text{ (by (26))}} \underbrace{f_{(m-1)+1}}_{=f_m} = f_k f_{m-1} + \underbrace{(f_k + f_{k-1}) f_m}_{=f_k f_m + f_{k-1} f_m} \\ &= \underbrace{f_k f_{m-1} + f_k f_m}_{=f_k(f_m + f_{m-1})} + f_{k-1} f_m = \underbrace{f_k}_{=f_{(k-1)+1}} \underbrace{(f_m + f_{m-1})}_{=f_{m+1} \text{ (by (25))}} + f_{k-1} f_m \\ &= f_{(k-1)+1} f_{m+1} + f_{k-1} f_m = f_{k-1} f_m + f_{(k-1)+1} f_{m+1}. \end{aligned}$$

In view of $k + (m - 1) + 1 = (k - 1) + m + 1$, this rewrites as

$$f_{(k-1)+m+1} = f_{k-1} f_m + f_{(k-1)+1} f_{m+1}.$$

Now, forget that we fixed m . We thus have shown that $f_{(k-1)+m+1} = f_{k-1} f_m + f_{(k-1)+1} f_{m+1}$ for all $m \in \mathbb{Z}$. In other words, $\mathcal{A}(k - 1)$ holds. This completes the proof of Assumption 3.]

Thus, we have shown that all three assumptions of Theorem 4.1 are satisfied. Hence, Theorem 4.1 yields that $\mathcal{A}(n)$ holds for each $n \in \mathbb{Z}$. In view of the definition of $\mathcal{A}(n)$, this rewrites as follows: For each $n \in \mathbb{Z}$, we have

$$(f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1} \text{ for all } m \in \mathbb{Z}).$$

This solves part **(b)** of the exercise.

(c) First solution to part (c): We shall use the symbol “ $\stackrel{(17)}{=}$ ” for “equals, because of the recurrence equation (17)”. For example, $f_5 \stackrel{(17)}{=} f_4 + f_3$ and $f_3 + f_2 \stackrel{(17)}{=} f_4$ and $f_{k+5} \stackrel{(17)}{=} f_{k+4} + f_{k+3}$ for every $k \in \mathbb{N}$.

Let $n \in \mathbb{Z}$. Then,

$$\begin{aligned}
 f_{n-4} + \underbrace{f_{n+4}}_{\stackrel{(17)}{=} f_{n+3}+f_{n+2}}} &= f_{n-4} + \underbrace{f_{n+3}}_{\stackrel{(17)}{=} f_{n+2}+f_{n+1}}} + \underbrace{f_{n+2}}_{\stackrel{(17)}{=} f_{n+1}+f_n}} \\
 &= f_{n-4} + f_{n+2} + f_{n+1} + f_{n+1} + f_n = f_{n-4} + \underbrace{f_{n+2}}_{\stackrel{(17)}{=} f_{n+1}+f_n}} + 2f_{n+1} + f_n \\
 &= f_{n-4} + f_{n+1} + f_n + 2f_{n+1} + f_n = f_{n-4} + 2f_n + 3 \underbrace{f_{n+1}}_{\stackrel{(17)}{=} f_n+f_{n-1}}} \\
 &= f_{n-4} + 2f_n + 3(f_n + f_{n-1}) = f_{n-4} + 5f_n + 3 \underbrace{f_{n-1}}_{\stackrel{(17)}{=} f_{n-2}+f_{n-3}}} \\
 &= f_{n-4} + 5f_n + 3(f_{n-2} + f_{n-3}) = \underbrace{f_{n-4} + f_{n-3}}_{\stackrel{(17)}{=} f_{n-2}}} + 5f_n + 3f_{n-2} + 2f_{n-3} \\
 &= f_{n-2} + 5f_n + 3f_{n-2} + 2f_{n-3} = 5f_n + 2f_{n-2} + 2 \underbrace{(f_{n-2} + f_{n-3})}_{\stackrel{(17)}{=} f_{n-1}}} \\
 &= 5f_n + 2f_{n-2} + 2f_{n-1} = 5f_n + 2 \underbrace{(f_{n-1} + f_{n-2})}_{\stackrel{(17)}{=} f_n}} \\
 &= 5f_n + 2f_n = 7f_n.
 \end{aligned}$$

This solves part (c) of the exercise.

Second solution to part (c): The following nice argument was found by Henry Twiss.

Let $n \in \mathbb{Z}$. It is straightforward to find (using the definition of the Fibonacci numbers) that $f_{-4} = -3$ and $f_{-5} = 5$ and $f_3 = 2$ and $f_4 = 3$. Part (b) of the exercise (applied to $m = -5$) yields

$$f_{n+(-5)+1} = f_n \underbrace{f_{-5}}_{=5} + f_{n+1} \underbrace{f_{(-5)+1}}_{=f_{-4}=-3} = f_n \cdot 5 + f_{n+1} \cdot (-3) = 5f_n - 3f_{n+1}.$$

Part (b) of the exercise (applied to $m = 3$) yields

$$f_{n+3+1} = f_n \underbrace{f_3}_{=2} + f_{n+1} \underbrace{f_{3+1}}_{=f_4=3} = f_n \cdot 2 + f_{n+1} \cdot 3 = 2f_n + 3f_{n+1}.$$

Adding the preceding two equalities together, we find

$$f_{n+(-5)+1} + f_{n+3+1} = (5f_n - 3f_{n+1}) + (2f_n + 3f_{n+1}) = 7f_n.$$

Hence,

$$7f_n = \underbrace{f_{n+(-5)+1}}_{=f_{n-4}} + \underbrace{f_{n+3+1}}_{=f_{n+4}} = f_{n-4} + f_{n+4}.$$

This solves part (c) of the exercise.

4.3 REMARK

Part (c) of the exercise is part of a sequence of identities:

$$\begin{aligned}
 1f_n &= f_n; \\
 2f_n &= f_{n-2} + f_{n+1}; \\
 3f_n &= f_{n-2} + f_{n+2}; \\
 4f_n &= f_{n-2} + f_n + f_{n+2}; \\
 5f_n &= f_{n-4} + f_{n-1} + f_{n+3}; \\
 6f_n &= f_{n-4} + f_{n+1} + f_{n+3}; \\
 7f_n &= f_{n-4} + f_{n+4}; \\
 &\vdots
 \end{aligned}$$

(for all $n \in \mathbb{Z}$). For each positive integer k , the k -th identity in this sequence has the form

$$kf_n = f_{n-a_1} + f_{n-a_2} + \cdots + f_{n-a_m}$$

for some integers $a_1 < a_2 < \cdots < a_m$ such that no two of the integers a_1, a_2, \dots, a_m are consecutive (i.e., the subset $\{a_1, a_2, \dots, a_m\}$ of \mathbb{Z} is lacunar). There exists exactly one such identity for each positive integer k . For a proof that this sequence of identities exists, see the following note:

- Darij Grinberg, *Zeckendorf family identities generalized*, arXiv preprint [arXiv:1103.4507v2](https://arxiv.org/abs/1103.4507v2).

(Note, however, that I only consider Fibonacci numbers f_i with $i > 0$ in this note; but extending all the results to arbitrary integers is not difficult.)

See also Section 9.3 of the last lecture (2018-05-02) of Spring 2018 Math 4707 for a short introduction.

5 EXERCISE 5

5.1 PROBLEM

Let $n \in \mathbb{N}$ and $p \in \{0, 1, \dots, n\}$. A p -derangement of $[n]$ shall mean a permutation σ of $[n]$ such that every $i \in [n - p]$ satisfies $\sigma(i) \neq i + p$. Compute the number of all p -derangements of $[n]$ as a sum of the form $\sum_{i=0}^{n-p} \cdots$.

[Hint: The case $p = 1$ was Exercise 6 on Spring 2018 Math 4707 homework set #2.]

5.2 SOLUTION (SKETCHED)

This exercise is a mild generalization of Exercise 6 on Spring 2018 Math 4707 homework set #2, and we can solve it by a straightforward adaptation of the solution of the latter exercise.

Let us first state two lemmas:

Lemma 5.1. Let $k \in \mathbb{N}$. Let S be a finite set. Let A_1, A_2, \dots, A_k be k subsets of S . Then,

$$\left| S \setminus \bigcup_{i=1}^k A_i \right| = \sum_{I \subseteq [k]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|.$$

Here, the “empty” intersection $\bigcap_{i \in \emptyset} A_i$ is understood to mean the set S .

Lemma 5.2. Let $n \in \mathbb{N}$. Let I be a subset of $[n]$. Let h_i be an element of $[n]$ for each $i \in I$. Assume that the h_i for different $i \in I$ are distinct. Then,

$$|\{\sigma \in S_n \mid \sigma(i) = h_i \text{ for all } i \in I\}| = (n - |I|)!.$$

Lemma 5.1 and Lemma 5.2 were proven during our solution of Exercise 6 on Spring 2018 Math 4707 homework set #2; thus, we have no need to prove them here again.

From Lemma 5.2, we can easily derive the following:

Lemma 5.3. Let $n \in \mathbb{N}$ and $p \in \{0, 1, \dots, n\}$. Let I be a subset of $[n - p]$. Then,

$$|\{\sigma \in S_n \mid \sigma(i) = i + p \text{ for all } i \in I\}| = (n - |I|)!.$$

Proof of Lemma 5.3 (sketched). We have $I \subseteq [n - p] \subseteq [n]$. For each $i \in I$, we have $i + p \in [n]$ ⁷. Moreover, the $i + p$ for different $i \in I$ are distinct. Hence, Lemma 5.2 (applied to $h_i = i + p$) shows that $|\{\sigma \in S_n \mid \sigma(i) = i + p \text{ for all } i \in I\}| = (n - |I|)!$. This proves Lemma 5.3. \square

We can now solve the exercise. Indeed, we have $n - p \in \mathbb{N}$ (since $p \in \{0, 1, \dots, n\}$). For each $i \in [n - p]$, we define a subset A_i of S_n by

$$A_i = \{\sigma \in S_n \mid \sigma(i) = i + p\}. \quad (27)$$

Then, for each subset I of $[n - p]$, we have

$$\begin{aligned} \bigcap_{i \in I} A_i &= \bigcap_{i \in I} \{\sigma \in S_n \mid \sigma(i) = i + p\} && \text{(by (27))} \\ &= \{\sigma \in S_n \mid \sigma(i) = i + p \text{ for all } i \in I\} \end{aligned}$$

and therefore

$$\left| \bigcap_{i \in I} A_i \right| = |\{\sigma \in S_n \mid \sigma(i) = i + p \text{ for all } i \in I\}| = (n - |I|)! \quad (28)$$

(by Lemma 5.3).

On the other hand, the p -derangements of $[n]$ are exactly the permutations σ of $[n]$ such

⁷*Proof.* Let $i \in I$. Thus, $i \in I \subseteq [n - p] = \{1, 2, \dots, n - p\}$, so that $i + p \in \{p + 1, p + 2, \dots, n\} \subseteq [n]$, qed.

that every $i \in [n - p]$ satisfies $\sigma(i) \neq i + p$ (by the definition of a p -derangement). Thus,

$$\begin{aligned}
& \{p\text{-derangements of } [n]\} \\
&= \{\text{permutations } \sigma \text{ of } [n] \text{ such that every } i \in [n - p] \text{ satisfies } \sigma(i) \neq i + p\} \\
&= \left\{ \sigma \in S_n \mid \underbrace{\sigma(i) \neq i + p \text{ for all } i \in [n - p]}_{\iff (\text{not } (\sigma(i) = i + p \text{ for some } i \in [n - p]))} \right\} \\
&= \{\sigma \in S_n \mid \text{not } (\sigma(i) = i + p \text{ for some } i \in [n - p])\} \\
&= S_n \setminus \underbrace{\{\sigma \in S_n \mid \sigma(i) = i + p \text{ for some } i \in [n - p]\}}_{= \bigcup_{i \in [n-p]} \{\sigma \in S_n \mid \sigma(i) = i + p\}} \\
&= S_n \setminus \underbrace{\bigcup_{i \in [n-p]} \{\sigma \in S_n \mid \sigma(i) = i + p\}}_{\substack{= A_i \\ (\text{by (27))}}}_{= \bigcup_{i=1}^{n-p}} A_i = S_n \setminus \bigcup_{i=1}^{n-p} A_i.
\end{aligned}$$

Hence,

$$\begin{aligned}
& |\{p\text{-derangements of } [n]\}| \\
&= \left| S_n \setminus \bigcup_{i=1}^{n-p} A_i \right| = \sum_{I \subseteq [n-p]} (-1)^{|I|} \underbrace{\left| \bigcap_{i \in I} A_i \right|}_{\substack{= (n-|I|)! \\ (\text{by (28))}}} \\
&\quad (\text{by Lemma 5.1, applied to } S = S_n \text{ and } k = n - p) \\
&= \sum_{\underbrace{I \subseteq [n-p]}_{\substack{= \sum_{k=0}^{n-p} \sum_{\substack{I \subseteq [n-p]; \\ |I|=k}} \\ (\text{since } |I| \in \{0, 1, \dots, n-p\} \\ \text{for each } I \subseteq [n-p])}}} (-1)^{|I|} (n - |I|)! = \sum_{k=0}^{n-p} \sum_{\substack{I \subseteq [n-p]; \\ |I|=k}} \underbrace{(-1)^{|I|}}_{= (-1)^k \text{ (since } |I|=k)} \underbrace{(n - |I|)!}_{= (n-k)! \text{ (since } |I|=k)} \\
&= \sum_{k=0}^{n-p} \underbrace{\sum_{\substack{I \subseteq [n-p]; \\ |I|=k}} (-1)^k (n - k)!}_{= |\{I \subseteq [n-p] \mid |I|=k\}| \cdot (-1)^k (n-k)!} \\
&= \sum_{k=0}^{n-p} \underbrace{|\{I \subseteq [n-p] \mid |I|=k\}|}_{= (\text{the number of all } k\text{-element subsets } I \text{ of } [n-p])} \cdot (-1)^k (n - k)! \\
&\quad = \sum_{k=0}^{n-p} \underbrace{\binom{n-p}{k}}_{(\text{since } n-p \in \mathbb{N})} \cdot (-1)^k (n - k)!.
\end{aligned}$$

This solves the exercise.

6 EXERCISE 6

6.1 PROBLEM

Let n and k be positive integers. A k -smord will mean a k -tuple $(a_1, a_2, \dots, a_k) \in [n]^k$ such that no two consecutive entries of this k -tuple are equal (i.e., we have $a_i \neq a_{i+1}$ for all $i \in [k-1]$). For example, $(4, 1, 4, 2, 6)$ is a 5-smord (when $n \geq 6$), but $(1, 4, 4, 2, 6)$ is not.

It is easy to see that the number of k -smords is $n(n-1)^{k-1}$. (See, e.g., Exercise 5 on Math 4990 Fall 2017 homework set #3.)

A *double k -smord* shall mean a pair $((a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k))$ of two k -smords (a_1, a_2, \dots, a_k) and (b_1, b_2, \dots, b_k) such that every $i \in [k]$ satisfies $a_i \neq b_i$.

Prove that the number of double k -smords is $n(n-1)(n^2-3n+3)^{k-1}$.

6.2 REMARK

“Smord” is short for “Smirnov word” (which is how these tuples are sometimes called).

Double k -smords can also be regarded as $2 \times k$ -matrices with entries lying in $[n]$ and with the property that no two adjacent entries are equal. (The double k -smord

$((a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k))$ thus corresponds to the $2 \times k$ -matrix $\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}$.)

6.3 SOLUTION (SKETCHED)

The exercise will quickly follow by induction on k , once we have shown the following claim:

Claim 1: Let $k > 1$ be an integer. Let $((a_1, a_2, \dots, a_{k-1}), (b_1, b_2, \dots, b_{k-1}))$ be a double $(k-1)$ -smord. Then, there are precisely $n^2 - 3n + 3$ pairs $(a_k, b_k) \in [n] \times [n]$ such that $((a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k))$ is a double k -smord.

[*Proof of Claim 1 (sketched):* Since $((a_1, a_2, \dots, a_{k-1}), (b_1, b_2, \dots, b_{k-1}))$ is a double $(k-1)$ -smord, we have $a_i \neq b_i$ for every $i \in [k-1]$. Thus, in particular, $a_{k-1} \neq b_{k-1}$. Also, $(a_1, a_2, \dots, a_{k-1})$ and $(b_1, b_2, \dots, b_{k-1})$ are $(k-1)$ -smords⁸; thus, each $i \in [k-2]$ satisfies $a_i \neq a_{i+1}$ and $b_i \neq b_{i+1}$.

For any pair $(a_k, b_k) \in [n] \times [n]$, we have the following logical equivalence:

$$\begin{aligned} & (((a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k)) \text{ is a double } k\text{-smord}) \\ \iff & (a_k \neq a_{k-1} \text{ and } b_k \neq b_{k-1} \text{ and } a_k \neq b_k) \end{aligned}$$

(because we already know that $a_i \neq b_i$ for every $i \in [k-1]$, and that each $i \in [k-2]$ satisfies $a_i \neq a_{i+1}$ and $b_i \neq b_{i+1}$).

Thus, it remains to prove that there are precisely $n^2 - 3n + 3$ pairs $(a_k, b_k) \in [n] \times [n]$ satisfying

$$(a_k \neq a_{k-1} \text{ and } b_k \neq b_{k-1} \text{ and } a_k \neq b_k).$$

To prove this, we classify these pairs into two types:

⁸since $((a_1, a_2, \dots, a_{k-1}), (b_1, b_2, \dots, b_{k-1}))$ is a double $(k-1)$ -smord

- *Type I* will consist of those pairs (a_k, b_k) for which $a_k = b_{k-1}$.
- *Type II* will consist of those pairs (a_k, b_k) for which $a_k \neq b_{k-1}$.

The number of pairs of Type I is $n - 1$. In fact, defining such a pair boils down to choosing some b_k that is distinct from b_{k-1} ⁹, and there are exactly $n - 1$ choices for this b_k .

The number of pairs of Type II is $(n - 2)^2$. In fact, defining such a pair boils down to choosing some a_k that is distinct from both b_{k-1} and a_{k-1} (so that $a_k \neq a_{k-1}$ and $a_k \neq b_{k-1}$ are satisfied) and then choosing some b_k that is distinct from both a_k and b_{k-1} (so that $a_k \neq b_k$ and $b_k \neq b_{k-1}$ are satisfied), and there are exactly $(n - 2)^2$ ways to make these two choices¹⁰.

So we conclude that the number of all pairs $(a_k, b_k) \in [n] \times [n]$ satisfying

$$(a_k \neq a_{k-1} \text{ and } b_k \neq b_{k-1} \text{ and } a_k \neq b_k)$$

equals

$$\underbrace{(\text{the number of all pairs of Type I})}_{=n-1} + \underbrace{(\text{the number of all pairs of Type II})}_{=(n-2)^2} \\ = (n - 1) + (n - 2)^2 = n^2 - 3n + 3.$$

This proves Claim 1.]

From Claim 1, the exercise follows.

6.4 REMARK

In the language of graph theory, the exercise is saying that the chromatic polynomial of the $2 \times k$ rectangular grid graph is $x(x - 1)(x^2 - 3x + 3)^{k-1}$. More generally, one can try to compute the chromatic polynomial of an arbitrary $d \times k$ rectangular grid graph (i.e., the number of $d \times k$ -matrices with entries lying in $[n]$ and with the property that no two adjacent entries are equal). This corresponds to counting k -smords when $d = 1$, and to counting double k -smords when $d = 2$. However, for $d > 2$, there don't seem to be explicit answers like we have for $d = 1$ and $d = 2$. For a computer-aided study of the $d > 2$ cases, see:

- Shalosh B. Ekhad, Jocelyn Quaintance, Doron Zeilberger, *Automatic Generation of Generating Functions for Chromatic Polynomials for Grid Graphs (and more general creatures) of Fixed (but arbitrary!) Width*, arXiv:1103.6206.

REFERENCES

- [Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
<http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf>

⁹Indeed, this is sufficient, because if we have $a_k = b_{k-1}$ and $b_k \neq b_{k-1}$, then we obtain $a_k = b_{k-1} \neq a_{k-1}$ and $b_k \neq b_{k-1}$ and $a_k = b_{k-1} \neq b_k$.

¹⁰Indeed, there are $n - 2$ ways to choose a_k (because it needs to be distinct from b_{k-1} and a_{k-1} , which are two distinct numbers because $a_{k-1} \neq b_{k-1}$), and then there are $n - 2$ ways to choose b_k (because it needs to be distinct from both a_k and b_{k-1} , which are two distinct numbers because $a_k \neq b_{k-1}$).

The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see <https://github.com/darijgr/detnotes/releases/tag/2019-01-10> .