University of Minnesota, School of Mathematics

# Math 5705: Enumerative Combinatorics, Fall 2018: Midterm 1

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due date: **Wednesday, 24 October 2018** at the beginning of class, or before that by email or canvas.
Please solve **at most 4 of the 6 exercises**!
Beware: **Collaboration is not allowed** on midterms!

## 1 Exercise 1

### 1.1 Problem

Let $A$ and $B$ be two finite sets, and let $f : A \to B$ be a map.

**(a)** Prove that the number of maps $g : B \to A$ satisfying $f \circ g \circ f = f$ is

$$|A|^{|B \setminus f(A)|} \prod_{b \in f(A)} \left| f^{-1}(b) \right|.$$

(Here and in the following, $f(A)$ denotes the set $\{f(a) \mid a \in A\}$, whereas $f^{-1}(b)$ denotes the set $\{a \in A \mid f(a) = b\}$.)

**(b)** Prove that the number of maps $g : B \to A$ satisfying $f \circ g \circ f = f$ and $g \circ f \circ g = g$ is

$$|f(A)|^{|B \setminus f(A)|} \prod_{b \in f(A)} \left| f^{-1}(b) \right|.$$

[**Hint:** For part **(a)**, observe that

$$|A|^{|B \setminus f(A)|} \prod_{b \in f(A)} \left|f^{-1}(b)\right| = \prod_{b \in B} \begin{cases} \left|f^{-1}(b)\right|, & \text{if } b \in f(A); \\ |A|, & \text{if } b \notin f(A). \end{cases}$$

What does this suggest about the construction of such maps $g$?]

### 1.2 Remark

The maps $g$ in part **(a)** are called "generalized inverses" of $f$. The maps $g$ in part **(b)** are called "reflexive generalized inverses" of $f$. Note that one consequence of part **(b)** is that there is always at least one reflexive generalized inverse of $f$ (unless $A$ is empty).

One can similarly define generalized inverses for linear maps between vector spaces; the resulting notion is much more well-known and has books devoted to it (see the Wikipedia for an overview).

### 1.3 Solution

[...]

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## 2 Exercise 2

### 2.1 Problem

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Prove that

$$(n+m) \sum_{j=0}^{m} (-1)^j \frac{\binom{m}{j}}{\binom{n+j}{j}} = n.$$

[**Hint:** The fraction on the left hand side has too many $j$'s. Try to simplify it to get the number of $j$'s down to just 1 (not counting the exponent in $(-1)^j$).]

### 2.2 Solution

[...]

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# 3 Exercise 3

## 3.1 Problem

Let $n$ be a positive integer. Let $a_1, a_2, \ldots, a_n$ be $n$ integers. Let $F : \mathbb{Z} \to \mathbb{R}$ be any function. Prove that

$$F\left(\max\left\{a_1, a_2, \ldots, a_n\right\}\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} F\left(\min\left\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\right\}\right).$$

[**Hint:** This generalizes Exercise 5 on Spring 2018 Math 4707 homework set #2. Will some of the solutions given there still apply to this generalization?]

## 3.2 Solution

[...]

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# 4 Exercise 4

## 4.1 Problem

Recall once again the *Fibonacci sequence* $(f_0, f_1, f_2, \ldots)$, which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 2. \tag{1}$$

Now, let us define $f_n$ for **negative** integers $n$ as well, by "applying (1) backwards": This means that we set $f_{n-2} = f_n - f_{n-1}$ for all integers $n \leq 1$. This allows us to recursively compute $f_{-1}, f_{-2}, f_{-3}, \ldots$ (in this order). For example,

$$\begin{aligned} f_{-1} &= f_1 - f_0 = 1 - 0 = 1; \\ f_{-2} &= f_0 - f_{-1} = 0 - 1 = -1; \\ f_{-3} &= f_{-1} - f_{-2} = 1 - (-1) = 2, \end{aligned}$$

etc.

**(a)** Prove that $f_{-n} = (-1)^{n-1} f_n$ for each $n \in \mathbb{Z}$.

**(b)** Prove that $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

**(c)** Prove that $7 f_n = f_{n-4} + f_{n+4}$ for all $n \in \mathbb{Z}$.

[**Hint:** This is **not** an exercise about the combinatorial interpretations (domino tilings, lacunar subsets, etc.) of Fibonacci numbers. Make sure that your proofs cover all integers, not just elements of $\mathbb{N}$.]

## 4.2 Solution

[...]

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# 5 Exercise 5

## 5.1 Problem

Let $n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. A *$p$-derangement of* $[n]$ shall mean a permutation $\sigma$ of $[n]$ such that every $i \in [n-p]$ satisfies $\sigma(i) \neq i+p$. Compute the number of all $p$-derangements of $[n]$ as a sum of the form $\sum_{i=0}^{n-p} \cdots$.

[**Hint:** The case $p = 1$ was Exercise 6 on Spring 2018 Math 4707 homework set #2.]

## 5.2 Solution

[...]

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# 6 Exercise 6

## 6.1 Problem

Let $n$ and $k$ be positive integers. A *$k$-smord* will mean a $k$-tuple $(a_1, a_2, \ldots, a_k) \in [n]^k$ such that no two consecutive entries of this $k$-tuple are equal (i.e., we have $a_i \neq a_{i+1}$ for all $i \in [k-1]$). For example, $(4, 1, 4, 2, 6)$ is a 5-smord (when $n \geq 6$), but $(1, 4, 4, 2, 6)$ is not.

It is easy to see that the number of $k$-smords is $n(n-1)^{k-1}$. (See, e.g., Exercise 5 on Math 4990 Fall 2017 homework set #3.)

A *double $k$-smord* shall mean a pair $((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k))$ of two $k$-smords $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ such that every $i \in [k]$ satisfies $a_i \neq b_i$.

Prove that the number of double $k$-smords is $n(n-1)(n^2 - 3n + 3)^{k-1}$.

## 6.2 Remark

"Smord" is short for "Smirnov word" (which is how these tuples are sometimes called).

Double $k$-smords can also be regarded as $2 \times k$-matrices with entries lying in $[n]$ and with the property that no two adjacent entries are equal. (The double $k$-smord $((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k))$ thus corresponds to the $2 \times k$-matrix $\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}$.)

## 6.3 Solution

[...]