University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 5

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due date: **Wednesday, 28 November 2018** at the beginning of class, or before that by email or canvas.
Please solve **at most 4 of the 6 exercises**!

1 Exercise 1

1.1 Problem

A *point* shall mean an element of $\mathbb{Z}^2$, that is, a pair of integers. We depict these points as lattice points on the Cartesian plane, and add and subtract them as vectors.

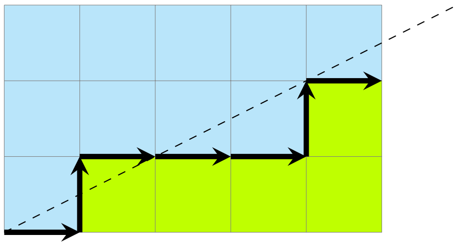
Recall the notion of a *lattice path*, defined in §6.1 (class notes from 2018-11-12) and (equivalently) in UMN Spring 2018 Math 4707 Midterm 1. (Lattice paths have up-steps and right-steps.)

Fix a positive integer k.

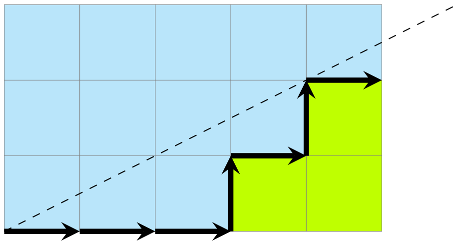
We say that a point $(x, y) \in \mathbb{Z}^2$ is *off-limits* if $ky > x$. (Thus, the off-limits points are the ones that lie strictly above the $x = ky$ diagonal in Cartesian coordinates.)

A lattice path $(v_0, v_1, \ldots, v_n)$ is said to be *k-legal* if none of the points $v_0, v_1, \ldots, v_n$ is off-limits. Equivalently, a lattice path $(v_0, v_1, \ldots, v_n)$ is k-legal if each point $(x, y) \in \{v_0, v_1, \ldots, v_n\}$ satisfies $x \geq ky$.

For example, the lattice path from $(0,0)$ to $(5,2)$ drawn in the picture



[1] is not 2-legal, since it contains the off-limits point $(1,1)$. Meanwhile, the lattice path from $(0,0)$ to $(5,2)$ drawn in the picture



is 2-legal.

For any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we let $L_{n,m,k}$ be the number of all k-legal lattice paths from $(0,0)$ to (n,m).

(a) Prove that $L_{n,m,k} = L_{n-1,m,k} + L_{n,m-1,k}$ for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n \geq km$ and $(n,m) \neq (0,0)$.

(b) Prove that

$$L_{n,m,k} = \binom{n+m}{m} - k\binom{n+m}{m-1}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq km - 1$.

[The requirement $n \geq km - 1$ as opposed to $n \geq km$ is not a typo; the equality still holds for $n = km - 1$, albeit for fairly simple reasons.]

(c) Prove that $L_{n,m,k} = \dfrac{n+1-km}{n+1}\dbinom{n+m}{m}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq km - 1$.

(d) Prove that

$$L_{km,m,k} = \frac{1}{km+1}\binom{(k+1)m}{m} = \frac{1}{(k+1)m+1}\binom{(k+1)m+1}{m}$$

for any $m \in \mathbb{N}$.

1.2 Remark

This exercise generalizes Exercise 2 from UMN Spring 2018 Math 4707 Midterm 2 (except that I've added an extra equality to part **(d)**). You are free to solve it by copypasting the solution of the latter (download its TeX source – alas, computer-generated), or by referencing the solution of the latter and pointing out what changes are necessary and where.

[1]Formally speaking, this lattice path is the list $((0,0),(1,0),(1,1),(2,1),(3,1),(4,1),(4,2),(5,2))$.

1.3 Solution

[...]

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2 Exercise 2

2.1 Problem

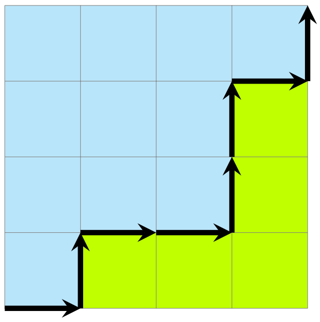
We shall abbreviate "lattice path" as "LP".

Recall that an LP is said to be *legal* if it is k-legal for $k = 1$.

Recall also that C_n denotes the n-th Catalan number (that is, $\frac{1}{n+1}\binom{2n}{n}$) for any $n \in \mathbb{N}$.

If $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ is an LP, then an *inversion* of $\mathbf{v}$ means a pair $(i, j) \in [n]^2$ such that $i < j$ and such that the i-th step of $\mathbf{v}$ is an up-step (i.e., we have $v_i - v_{i-1} = (0, 1)$) but the j-th step of $\mathbf{v}$ is a right-step (i.e., we have $v_j - v_{j-1} = (1, 0)$).

For example, the LP depicted in



(1)

has the 5 inversions $(2, 3)$ and $(2, 4)$ and $(2, 7)$ and $(5, 7)$ and $(6, 7)$ (and no others). It is easy to see that these inversions correspond to the 5 green squares under the LP in the above picture; more generally, any LP $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ from a point s to a point t subdivides its "bounding box" (i.e., the rectangle with opposing corners s and t) into two parts, and the area of the part under the LP is exactly the number of inversions of $\mathbf{v}$.

If $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ is a LP, then $\operatorname{inv} \mathbf{v}$ denotes the number of inversions of $\mathbf{v}$.

Now, let $n \in \mathbb{N}$. Prove that

$$\sum_{\substack{\mathbf{v} \text{ is a legal LP} \\ \text{from } (0,0) \text{ to } (n,n)}} (-1)^{\operatorname{inv} \mathbf{v}} = \begin{cases} C_{(n-1)/2}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

[**Hint:** What happens to a legal LP from $(0, 0)$ to (n, n) if we swap its 2-nd and 3-rd steps? If the answer is "nothing", then what if we swap its 4-th and 5-th steps? If nothing again, its 6-th and 7-th steps?]

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $x \in \mathbb{Q}$ and $p \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{p} C_k \dbinom{x-2k}{p-k} = \dbinom{x+1}{p}.$$

[**Hint:** By the "polynomial identity trick", it suffices to prove this in the case when $x+1 \geq 2p$.]

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k C_k \dbinom{n+k}{2k} = 0.$$

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. A *deranged involution* of $[2n]$ shall mean a fixed-point-free involution $\sigma : [2n] \to [2n]$ such that every $i \in [n]$ satisfies $\sigma(2i-1) \neq 2i$.

(For example, the permutation of $[6]$ whose one-line notation is $[4,5,6,1,2,3]$ is a deranged involution, but the permutation of $[6]$ whose one-line notation is $[6,5,4,3,2,1]$ is not.)

Let a_n be the number of deranged involutions of $[2n]$.

(a) Prove that $a_0 = 1$ and $a_1 = 0$ and $a_n = 2(n-1)(a_{n-1} + a_{n-2})$ for all $n \geq 2$.

(b) Prove that

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \dbinom{n}{k} (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)) \qquad \text{for all } n \in \mathbb{N}.$$

5.2 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. Set $h(\sigma) = \sum_{i \in [n]} |\sigma(i) - i|$.

(a) Prove that

$$h(\sigma) = 2 \sum_{\substack{i \in [n];\\ \sigma(i) > i}} (\sigma(i) - i) = 2 \sum_{\substack{i \in [n];\\ \sigma(i) < i}} (i - \sigma(i)).$$

(b) Prove that $h(s_i \circ \sigma) \leq h(\sigma) + 2$ for each $i \in [n-1]$.

(c) Prove that

$$h(\sigma)/2 \leq \ell(\sigma) \leq h(\sigma).$$

[**Hint:** As mentioned, you are free to use previous homework sets and midterms.]

6.2 Solution

[...]