University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 4 (preliminary version)

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Let i and j be two elements of $[n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$. Let Q be the set of all $k \in \{i+1, i+2, \ldots, j-1\}$ satisfying $\sigma(i) > \sigma(k) > \sigma(j)$. Prove that

$$\ell(\sigma \circ t_{i,j}) = \ell(\sigma) - 2|Q| - 1.$$

1.2 Remark

This exercise implies that, in particular, $\ell(\sigma \circ t_{i,j}) < \ell(\sigma)$; this answers the question on page 213 of the notes from class (2018-10-22).

1.3 Solution

Jacob Elafandi gives a somewhat laborious but simple solution in [Elafan18].

I give a different solution in [Grinbe16, Exercise 5.20].

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $\pi \in S_n$.

(a) Prove that

$$\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} (\pi(j)-\pi(i)) = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} (i-j).$$

(b) Prove that

$$\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} (\pi(j)-\pi(i)) = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} (j-i).$$

2.2 Solution

We shall use the following fact:

Proposition 2.1. *Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. Let $a_1, a_2, \ldots, a_n$ be any n numbers. (Here, "number" means "real number" or "complex number" or "rational number", as you prefer; this makes no difference.) Prove that*

$$\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}} (a_j - a_i) = \sum_{i=1}^{n} a_i (i-\sigma(i)).$$

[Here, the summation sign "$\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}}$" means "$\sum_{\substack{(i,j)\in\{1,2,\ldots,n\}^2;\\ i<j \text{ and } \sigma(i)>\sigma(j)}}$"; this is a sum over all inversions of σ.]

Proposition 2.1 is [Grinbe16, Exercise 5.23]. For a different proof of it, see [Gorski18, Exercise 4].

Now, let us solve the exercise. We have $\pi \in S_n$. In other words, π is a permutation of $[n]$. In other words, π is a bijection $[n] \to [n]$. Hence, we can substitute $\pi(i)$ for i in the sum $\sum_{i\in[n]} i^2$. We thus obtain

$$\sum_{i\in[n]} i^2 = \sum_{i\in[n]} (\pi(i))^2. \tag{1}$$

(a) Proposition 2.1 (applied to $\sigma = \pi$ and $a_k = \pi(k) + k$) yields

$$\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} ((\pi(j)+j)-(\pi(i)+i)) = \underbrace{\sum_{i=1}^{n}}_{=\sum_{i\in[n]}} \underbrace{(\pi(i)+i)(i-\pi(i))}_{\substack{=(i+\pi(i))(i-\pi(i))\\ =i^2-(\pi(i))^2\\ \text{(since } (x+y)(x-y)=x^2-y^2\\ \text{for any two numbers } x \text{ and } y)}}$$
$$= \sum_{i\in[n]} \left(i^2 - (\pi(i))^2\right) = \sum_{i\in[n]} i^2 - \sum_{i\in[n]} (\pi(i))^2 = 0$$

(by (1)). Hence,

$$\begin{aligned}
0 &= \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} \underbrace{\left(\left(\pi\left(j\right)+j\right)-\left(\pi\left(i\right)+i\right)\right)}_{=(\pi(j)-\pi(i))-(i-j)} \\
&= \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} \left(\left(\pi\left(j\right)-\pi\left(i\right)\right)-\left(i-j\right)\right) = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} \left(\pi\left(j\right)-\pi\left(i\right)\right) - \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} \left(i-j\right).
\end{aligned}$$

Adding $\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} (i-j)$ to both sides of this equality, we obtain

$$\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} \left(i-j\right) = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)>\pi(j)}} \left(\pi\left(j\right)-\pi\left(i\right)\right).$$

This solves part **(a)** of the exercise.

(b) Let w_0 denote the permutation in S_n which sends each $k\in[n]$ to $n+1-k$. Define a permutation $\sigma\in S_n$ by $\sigma=w_0\circ\pi$. Thus, each $k\in[n]$ satisfies

$$\underbrace{\sigma}_{=w_0\circ\pi}\left(k\right) = \left(w_0\circ\pi\right)\left(k\right) = w_0\left(\pi\left(k\right)\right) = n+1-\pi\left(k\right) \qquad (2)$$

(by the definition of w_0).

For any $(i,j)\in[n]^2$, we have the following chain of logical equivalences:

$$\begin{aligned}
&\left(\underbrace{\sigma\left(i\right)}_{\substack{=n+1-\pi(i)\\ \text{(by (2)}\\ \text{(applied to } k=i))}} > \underbrace{\sigma\left(j\right)}_{\substack{=n+1-\pi(j)\\ \text{(by (2)}\\ \text{(applied to } k=j))}}\right) \iff \left(n+1-\pi\left(i\right) > n+1-\pi\left(j\right)\right) \\
&\qquad \iff \left(\pi\left(i\right)<\pi\left(j\right)\right).
\end{aligned}$$

Thus, for any $(i,j)\in[n]^2$, the condition $(\sigma(i)>\sigma(j))$ is equivalent to $(\pi(i)<\pi(j))$. Hence, the summation sign "$\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}}$" can be rewritten as "$\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}}$". In other words, we have

$$\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}} = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}}$$

(an equality between summation signs). Now, part **(a)** of the exercise (applied to σ instead of π) yields

$$\begin{aligned}
\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}} \left(\sigma\left(j\right)-\sigma\left(i\right)\right) &= \underbrace{\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}}}_{=\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}}} \underbrace{\left(i-j\right)}_{=-(j-i)} = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(-\left(j-i\right)\right) \\
&= -\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(j-i\right).
\end{aligned}$$

Comparing this with

$$\underbrace{\sum_{\substack{1\leq i<j\leq n;\\ \sigma(i)>\sigma(j)}}}_{=\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}}} \left(\underbrace{\sigma\left(j\right)}_{\substack{=n+1-\pi(j)\\ \text{(by (2)}\\ \text{(applied to } k=j))}} - \underbrace{\sigma\left(i\right)}_{\substack{=n+1-\pi(i)\\ \text{(by (2)}\\ \text{(applied to } k=i))}} \right)$$
$$= \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \underbrace{\left(\left(n+1-\pi\left(j\right)\right)-\left(n+1-\pi\left(i\right)\right)\right)}_{=-(\pi(j)-\pi(i))} = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(-\left(\pi\left(j\right)-\pi\left(i\right)\right)\right)$$
$$= -\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(\pi\left(j\right)-\pi\left(i\right)\right),$$

we obtain

$$-\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(\pi\left(j\right)-\pi\left(i\right)\right) = -\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(j-i\right).$$

Thus,

$$\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(\pi\left(j\right)-\pi\left(i\right)\right) = \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} \left(j-i\right).$$

This solves part **(b)** of the exercise.

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3 Exercise 3

3.1 Problem

Let n be a positive integer. For each $p\in\mathbb{Z}$, we let

$$D_{n,p} = \left\{\sigma\in S_n \mid \sigma \text{ has exactly } p \text{ descents}\right\}.$$

(Recall that a *descent* of a permutation $\sigma\in S_n$ denotes an element $k\in[n-1]$ satisfying $\sigma\left(k\right)>\sigma\left(k+1\right)$.)

Let $p\in\mathbb{Z}$. Prove that $\left|D_{n,p}\right| = \left|D_{n,n-1-p}\right|$.

3.2 Solution sketch

We have $n-1\in\mathbb{N}$ (since n is a positive integer).

Recall that if $\sigma\in S_n$ is a permutation, then $\operatorname{Des}\sigma$ denotes the set of all descents of σ.

Let w_0 denote the permutation in S_n which sends each $k\in[n]$ to $n+1-k$.

Let $\pi \in S_n$. Thus, for each $k \in [n-1]$, we have the following chain of equivalences:

$$\begin{aligned}
(k \in \operatorname{Des}(w_0 \circ \pi)) &\iff (k \text{ is a descent of } w_0 \circ \pi) \\
&\iff \left(\underbrace{(w_0 \circ \pi)(k)}_{\substack{= w_0(\pi(k)) = n+1-\pi(k) \\ \text{(by the definition of } w_0)}} > \underbrace{(w_0 \circ \pi)(k+1)}_{\substack{= w_0(\pi(k+1)) = n+1-\pi(k+1) \\ \text{(by the definition of } w_0)}} \right) \\
&\iff (n+1-\pi(k) > n+1-\pi(k+1)) \\
&\iff (\pi(k) < \pi(k+1)) \iff (\pi(k) \leq \pi(k+1)) \\
&\qquad \text{(since } \pi(k) = \pi(k+1) \text{ can never hold (because } \pi \in S_n)) \\
&\iff (\text{not } \pi(k) > \pi(k+1)) \iff (k \text{ is not a descent of } \pi) \\
&\iff (k \notin \operatorname{Des} \pi).
\end{aligned}$$

In other words, the elements of $\operatorname{Des}(w_0 \circ \pi)$ are precisely the elements of $[n-1]$ that don't belong to $\operatorname{Des} \pi$. In other words, the set $\operatorname{Des}(w_0 \circ \pi)$ is the complement of the set $\operatorname{Des} \pi$ in $[n-1]$. Thus,

$$|\operatorname{Des}(w_0 \circ \pi)| = \underbrace{|[n-1]|}_{\substack{= n-1 \\ (\text{since } n-1 \in \mathbb{N})}} - |\operatorname{Des} \pi| = n-1-|\operatorname{Des} \pi|. \tag{3}$$

Now, forget that we fixed π. We thus have proven (3) for each $\pi \in S_n$.

Now, let $\pi \in D_{n,p}$. Then, π has exactly p descents[1]. In other words, $|\operatorname{Des} \pi| = p$. Thus, (3) yields $|\operatorname{Des}(w_0 \circ \pi)| = n-1-\underbrace{|\operatorname{Des} \pi|}_{=p} = n-1-p$. In other words, the permutation $w_0 \circ \pi$ has exactly $n-1-p$ descents. In other words, $w_0 \circ \pi \in D_{n,n-1-p}$ (since the definition of $D_{n,n-1-p}$ yields $D_{n,n-1-p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } n-1-p \text{ descents}\}$).

Now, forget that we fixed π. We thus have proven that $w_0 \circ \pi \in D_{n,n-1-p}$ for each $\pi \in D_{n,p}$. Thus, the map

$$\begin{aligned} D_{n,p} &\to D_{n,n-1-p}, \\ \pi &\mapsto w_0 \circ \pi \end{aligned} \tag{4}$$

is well-defined. The same argument (but with p replaced by $n-1-p$) shows that the map

$$\begin{aligned} D_{n,n-1-p} &\to D_{n,n-1-(n-1-p)}, \\ \pi &\mapsto w_0 \circ \pi \end{aligned}$$

is well-defined. In other words, the map

$$\begin{aligned} D_{n,n-1-p} &\to D_{n,p}, \\ \pi &\mapsto w_0 \circ \pi \end{aligned} \tag{5}$$

is well-defined (since $n-1-(n-1-p) = p$). But $w_0 \circ w_0 = \operatorname{id}$ (since each $k \in [n]$ satisfies

$$\begin{aligned} (w_0 \circ w_0)(k) &= w_0(w_0(k)) = n+1-(n+1-k) \qquad \text{(by the definition of } w_0) \\ &= k = \operatorname{id}(k) \end{aligned}$$

). Thus, the two maps (4) and (5) are mutually inverse. Hence, these two maps are bijections. Thus, we have found a bijection from $D_{n,p}$ to $D_{n,n-1-p}$. Hence, $|D_{n,p}| = |D_{n,n-1-p}|$. This solves the exercise.

[1] since $\pi \in D_{n,p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents}\}$

3.3 Remark

1. A similar solution could have been obtained by using the permutation $\pi \circ w_0$ instead of $w_0 \circ \pi$. Indeed, similarly to (3), we also have

$$|\operatorname{Des}(\pi \circ w_0)| = n - 1 - |\operatorname{Des} \pi| \qquad \text{for each } \pi \in S_n.$$

To prove this, we would have to show that

$$\operatorname{Des}(\pi \circ w_0) = \{n - k \mid k \in [n-1] \setminus \operatorname{Des} \pi\}$$

(which is only a tad more complicated than proving that $\operatorname{Des}(w_0 \circ \pi) = [n-1] \setminus \operatorname{Des} \pi$).

2. I have snuck a correction into the exercise: It used to only require $n \in \mathbb{N}$, but now it requires n to be a positive integer. Indeed, the claim fails for $n = 0$. Sorry!

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Let $S = \{s_1 < s_2 < \cdots < s_k\}$ be a subset of $[n-1]$. Set $s_0 = 0$ and $s_{k+1} = n$. For each $i \in [k+1]$, set $d_i = s_i - s_{i-1}$. (You might remember this construction from the definition of the map D in the solution to Exercise 1 on homework set #0.)

(a) Prove that

$$|\{\sigma \in S_n \mid \operatorname{Des} \sigma \subseteq S\}| = \binom{n}{d_1, d_2, \ldots, d_{k+1}}.$$

(The term on the right hand side is a multinomial coefficient. The $\operatorname{Des} \sigma$ on the left hand side denotes the descent set of σ, that is, the set of all descents of σ.)

(b) Prove that

$$|\{\sigma \in S_n \mid \operatorname{Des} \sigma = S\}| = \sum_{T \subseteq S} (-1)^{|S|-|T|} |\{\sigma \in S_n \mid \operatorname{Des} \sigma \subseteq T\}|.$$

4.2 Solution sketch

(a) A permutation $\sigma \in S_n$ satisfies $\operatorname{Des} \sigma \subseteq S$ if and only if it is strictly increasing on each of the $k+1$ intervals

$$[s_0 + 1, s_1], \qquad [s_1 + 1, s_2], \qquad [s_2 + 1, s_3], \qquad \ldots, \qquad [s_k + 1, s_{k+1}].$$

Hence, a permutation $\sigma \in S_n$ satisfying $\operatorname{Des} \sigma \subseteq S$ is uniquely determined by the images

$$\sigma([s_0 + 1, s_1]), \qquad \sigma([s_1 + 1, s_2]), \qquad \sigma([s_2 + 1, s_3]), \qquad \ldots, \qquad \sigma([s_k + 1, s_{k+1}])$$

of these $k+1$ intervals (indeed, once these images are known, we can use the strict increasingness of σ on these intervals to reconstruct each value of σ). These images must be disjoint subsets of $[n]$ (since σ is injective) and have the same sizes as the $k+1$ intervals themselves (for the same reason); these sizes are

$$s_1 - s_0 = d_1, \qquad s_2 - s_1 = d_2, \qquad s_3 - s_2 = d_3, \qquad \ldots, \qquad s_{k+1} - s_k = d_{k+1}.$$

Thus, every permutation $\sigma \in S_n$ satisfying $\operatorname{Des} \sigma \subseteq S$ can be constructed by the following algorithm:

- We choose a d_1-element subset of $[n]$ to be the image $\sigma([s_0+1, s_1])$. This subset can be chosen in $\binom{n}{d_1}$ ways.
- Next, we choose a d_2-element subset of $[n]$ to be the image $\sigma([s_1+1, s_2])$, requiring that it be disjoint from the already chosen subset $\sigma([s_0+1, s_1])$. This subset can be chosen in $\binom{n-d_1}{d_2}$ ways (because by requiring it to be disjoint from the d_1-element subset $\sigma([s_0+1, s_1])$, we are forcing it to be a d_2-element subset of the $(n-d_1)$-element set $[n] \setminus \sigma([s_0+1, s_1])$).
- Next, we choose a d_3-element subset of $[n]$ to be the image $\sigma([s_2+1, s_3])$, requiring that it be disjoint from the already chosen subsets $\sigma([s_0+1, s_1])$ and $\sigma([s_1+1, s_2])$. This subset can be chosen in $\binom{n-d_1-d_2}{d_3}$ ways (because by requiring it to be disjoint from the d_1-element subset $\sigma([s_0+1, s_1])$ and the d_2-element subset $\sigma([s_1+1, s_2])$, we are forcing it to be a d_3-element subset of the $(n-d_1-d_2)$-element set $[n] \setminus \sigma([s_0+1, s_1]) \setminus \sigma([s_1+1, s_2])$ [2]).
- And so on, until all $k+1$ images

  $$\sigma([s_0+1, s_1]), \qquad \sigma([s_1+1, s_2]), \qquad \sigma([s_2+1, s_3]), \qquad \ldots, \qquad \sigma([s_k+1, s_{k+1}])$$

  are chosen. As we know, at this point, σ is uniquely determined.

The total number of ways in which this construction can be carried out is

$$\binom{n}{d_1}\binom{n-d_1}{d_2}\binom{n-d_1-d_2}{d_3}\cdots\binom{n-d_1-d_2-\cdots-d_k}{d_{k+1}}$$
$$= \prod_{i=0}^{k}\binom{n-d_1-d_2-\cdots-d_i}{d_{i+1}} = \prod_{i=1}^{k+1}\binom{n-d_1-d_2-\cdots-d_{i-1}}{d_i} = \binom{n}{d_1, d_2, \ldots, d_{k+1}}$$

(by the first equation in Proposition 2.38 in the class notes (2018-10-03)). Thus, the number of permutations $\sigma \in S_n$ satisfying $\operatorname{Des}\sigma \subseteq S$ is $\binom{n}{d_1, d_2, \ldots, d_{k+1}}$. This solves part **(a)** of the exercise.

(b) We need the following result:

Proposition 4.1. *Let G be a finite set. Let S be a subset of G. Then,*

$$\sum_{\substack{I \subseteq G; \\ S \subseteq I}} (-1)^{|I|} = (-1)^{|S|} [G = S].$$

Proposition 4.1 was proven during the solution of Exercise 6 on homework set #3.

[2] Of course, we are tacitly using the fact that the two already chosen subsets $\sigma([s_0+1, s_1])$ and $\sigma([s_1+1, s_2])$ are disjoint (so that the set $[n] \setminus \sigma([s_0+1, s_1]) \setminus \sigma([s_1+1, s_2])$ really is a $(n-d_1-d_2)$-element set).

We have

$$\begin{aligned}
&\sum_{T\subseteq S}(-1)^{|S|-|T|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma\subseteq T\}\right| \\
&=\sum_{I\subseteq S}(-1)^{|S|-|I|}\underbrace{\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma\subseteq I\}\right|}_{=\sum_{U\subseteq I}\left|\{\sigma\in S_n\mid\operatorname{Des}\sigma=U\}\right|} \\
&\qquad \text{(here, we have renamed the summation index } T \text{ as } I\text{)} \\
&=\sum_{I\subseteq S}(-1)^{|S|-|I|}\sum_{U\subseteq I}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=U\}\right| \\
&=\underbrace{\sum_{I\subseteq S}\sum_{U\subseteq I}}_{=\sum_{U\subseteq S}\sum_{\substack{I\subseteq S;\\ U\subseteq I}}}\underbrace{(-1)^{|S|-|I|}}_{=(-1)^{|S|}(-1)^{|I|}}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=U\}\right| \\
&=\sum_{U\subseteq S}\sum_{\substack{I\subseteq S;\\ U\subseteq I}}(-1)^{|S|}(-1)^{|I|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=U\}\right| \\
&=\sum_{U\subseteq S}\underbrace{\left(\sum_{\substack{I\subseteq S;\\ U\subseteq I}}(-1)^{|I|}\right)}_{\substack{=(-1)^{|U|}[S=U]\\ \text{(by Proposition 4.1,}\\ \text{applied to } S \text{ and } U \text{ instead of } G \text{ and } S)}}(-1)^{|S|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=U\}\right| \\
&=\sum_{U\subseteq S}(-1)^{|U|}[S=U](-1)^{|S|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=U\}\right| \\
&=\sum_{\substack{U\subseteq S;\\ U\neq S}}(-1)^{|U|}\underbrace{[S=U]}_{\substack{=0\\ \text{(since } U\neq S)}}(-1)^{|S|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=U\}\right| \\
&\qquad+(-1)^{|S|}[S=S](-1)^{|S|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=S\}\right| \\
&\qquad \text{(here, we have split off the addend for } U=S \text{ from the sum)} \\
&=\underbrace{\sum_{\substack{U\subseteq S;\\ U\neq S}}(-1)^{|U|}0(-1)^{|S|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=U\}\right|}_{=0} \\
&\qquad+(-1)^{|S|}\underbrace{[S=S]}_{=1}(-1)^{|S|}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=S\}\right| \\
&=\underbrace{(-1)^{|S|}(-1)^{|S|}}_{=\left((-1)^{|S|}\right)^2=1}\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=S\}\right|=\left|\{\sigma\in S_n \mid \operatorname{Des}\sigma=S\}\right|.
\end{aligned}$$

This solves part **(b)** of the exercise.

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. We shall follow the convention that $t_{i,i}$ denotes the identity permutation $\operatorname{id} \in S_n$ for each $i \in [n]$.

Let $\sigma \in S_n$.

It is known that there is a unique n-tuple $(i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n]$ satisfying $\sigma = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n}$. (See [Grinbe16, Exercise 5.9] for the proof of this fact, or – easier – do it on your own.) Consider this n-tuple. (It is sometimes called the *transposition code* of σ.)

For each $k \in \{0, 1, \ldots, n\}$, we define a permutation $\sigma_k \in S_n$ by $\sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k}$. Note that this permutation σ_k leaves each of the numbers $k+1, k+2, \ldots, n$ unchanged (since all of $i_1, i_2, \ldots, i_k$, as well as $1, 2, \ldots, k$, are $\leq k$).

For each $k \in [n]$, let $m_k = \sigma_k(k)$.

(a) Show that $m_k \in [k]$ for all $k \in [n]$.

(b) Show that $\sigma_k(i_k) = k$ for all $k \in [n]$.

(c) Show that $\sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}$.

(d) Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be any $2n$ numbers. Prove that

$$\sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} x_k y_{\sigma(k)} = \sum_{k=1}^{n} (x_{i_k} - x_k)(y_{m_k} - y_k).$$

(e) Now assume that the numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real and satisfy $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. Conclude that

$$\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\sigma(k)}.$$

5.2 Remark

This exercise is part of [Grinbe16, Exercise 5.25].

Parts **(a)** and **(c)**, combined, show that $(m_1, m_2, \ldots, m_n)$ is the transposition code of σ^{-1}.

Part **(e)** of the exercise is known as the *rearrangement inequality*. The proof in this exercise is far from its easiest proof, but has the advantage of "manifest positivity" – i.e., it gives an explicit formula for the difference between the two sides as a sum of products of nonnegative numbers.

5.3 Solution sketch

Let us first notice that any two elements $u, v \in [n]$ and any permutation $\pi \in S_n$ satisfy

$$t_{\pi(u),\pi(v)} \circ \pi = \pi \circ t_{u,v}. \tag{6}$$

[*Proof of* (6)*:* Let $u, v \in [n]$ and $\pi \in S_n$. Fix $k \in [n]$. We shall prove that $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$.

Indeed, we are in one of the following three cases:

Case 1: We have $k = u$.

Case 2: We have $k = v$.

Case 3: We have neither $k = u$ nor $k = v$.

Let us first consider Case 1. In this case, we have $k = u$. Thus, $t_{u,v}(k) = t_{u,v}(u) = v$ (independently of whether $u = v$ or $u \neq v$). Also, from $k = u$, we obtain

$$\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = \left(t_{\pi(u),\pi(v)} \circ \pi\right)(u) = t_{\pi(u),\pi(v)}(\pi(u)) = \pi(v)$$

(again, independently of whether $\pi(u) = \pi(v)$ holds or not). Comparing this with

$$(\pi \circ t_{u,v})(k) = \pi(t_{u,v}(k)) = \pi(v) \qquad (\text{since } t_{u,v}(k) = v),$$

we obtain $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$. Hence, $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$ is proven in Case 1.

The argument in Case 2 is analogous, and we leave it to the reader.

Let us now consider Case 3. In this case, we have neither $k = u$ nor $k = v$. Thus, $t_{u,v}(k) = k$ (independently of whether $u = v$ or $u \neq v$). Also, recall that we have neither $k = u$ nor $k = v$. Thus, we have neither $\pi(k) = \pi(u)$ nor $\pi(k) = \pi(v)$ (since the map π is injective (because $\pi \in S_n$)). Hence, $t_{\pi(u),\pi(v)}(\pi(k)) = \pi(k)$ (again, independently of whether $\pi(u) = \pi(v)$ holds or not). Now,

$$\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = t_{\pi(u),\pi(v)}(\pi(k)) = \pi(k).$$

Comparing this with

$$(\pi \circ t_{u,v})(k) = \pi(t_{u,v}(k)) = \pi(k) \qquad (\text{since } t_{u,v}(k) = k),$$

we obtain $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$. Hence, $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$ is proven in Case 3.

We have now proven $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$ in each of the three Cases 1, 2 and 3. Thus, $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$ always holds.

Forget now that we fixed k. We thus have shown that $\left(t_{\pi(u),\pi(v)} \circ \pi\right)(k) = (\pi \circ t_{u,v})(k)$ for each $k \in [n]$. In other words, $t_{\pi(u),\pi(v)} \circ \pi = \pi \circ t_{u,v}$. Thus, (6) is proven.]

Recall that $(i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n]$. Thus,

$$i_j \in [j] \qquad \text{for each } j \in [n]. \tag{7}$$

The definition of σ_0 shows that

$$\sigma_0 = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{0,i_0} = (\text{composition of } 0 \text{ permutations}) = \text{id}.$$

The definition of σ_n shows that

$$\sigma_n = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n} = \sigma.$$

(a) Let $k \in [n]$. Then, from (7), we conclude that each $j \in [k]$ satisfies $i_j \in [j] \subseteq [k]$ (since $j \leq k$). Hence, the k numbers $i_1, i_2, \ldots, i_k$ all belong to $[k]$. The same holds for the k numbers $1, 2, \ldots, k$. Thus, the k permutations $t_{1,i_1}, t_{2,i_2}, \ldots, t_{k,i_k}$ all preserve the set $[k]$ [3]. Hence, their composition $t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k}$ preserves the set $[k]$ as well[4]. In view of

[3]We say that a map $\tau : [n] \to [n]$ *preserves* a subset S of $[n]$ if and only if it satisfies $\tau(S) \subseteq S$. This does **not** mean that $\tau(s) = s$ for each $s \in S$; it only means that τ sends each element of S to a (possibly different) element of S.

[4]Here, we are using the following fact: If S is a subset of $[n]$, and if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are k maps from $[n]$ to $[n]$ that all preserve the set S, then the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ of these k maps must preserve the set S as well. (This is easy to prove by induction on k.)

$\sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k}$, this rewrites as follows: The map σ_k preserves the set $[k]$. In other words, $\sigma_k([k]) \subseteq [k]$. Now, $k \in [k]$, so that $\sigma_k(k) \in \sigma_k([k]) \subseteq [k]$. Hence, $m_k = \sigma_k(k) \in [k]$. This solves part **(a)** of the exercise.

(b) Let $k \in [n]$. Then, from (7), we conclude that each $j \in [k-1]$ satisfies $i_j \in [j] \subseteq [k-1]$ (since $j \leq k-1$). Hence, the $k-1$ numbers $i_1, i_2, \ldots, i_{k-1}$ all belong to $[k-1]$. The same holds for the $k-1$ numbers $1, 2, \ldots, k-1$. Thus, the $k-1$ permutations $t_{1,i_1}, t_{2,i_2}, \ldots, t_{k-1,i_{k-1}}$ all leave each of the numbers $k, k+1, \ldots, n$ unchanged. Hence, their composition $t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}$ leaves each of the numbers $k, k+1, \ldots, n$ unchanged. In particular, it thus leaves the number k unchanged. In other words,

$$\left(t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}\right)(k) = k.$$

The definition of σ_k yields

$$\sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} = \left(t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}\right) \circ t_{k,i_k}.$$

Hence,

$$\sigma_k(i_k) = \left(\left(t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}\right) \circ t_{k,i_k}\right)(i_k) = \left(t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}\right)\left(\underbrace{t_{k,i_k}(i_k)}_{=k}\right)$$
$$= \left(t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}\right)(k) = k.$$

This solves part **(b)** of the exercise.

(c) We shall show that

$$\sigma_p^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{p,m_p} \qquad \text{for each } p \in \{0, 1, \ldots, n\}. \tag{8}$$

[*Proof of* (8)*:* We shall prove (8) by induction on p:

Induction base: In the case of $p = 0$, the equality (8) holds, since σ_0 is defined as an empty composition whereas the right hand side of (8) also is an empty composition in this case. This completes the induction base.

Induction step: Let $k \in [n]$. Assume that (8) holds for $p = k-1$. We must prove that (8) holds for $p = k$.

We have assumed that (8) holds for $p = k-1$. That is, we have

$$\sigma_{k-1}^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{k-1,m_{k-1}}.$$

Part **(b)** of the exercise yields $\sigma_k(i_k) = k$, whereas the definition of m_k yields $\sigma_k(k) = m_k$. But (6) (applied to $\pi = \sigma_k$, $u = i_k$ and $v = k$) yields

$$t_{\sigma_k(i_k),\sigma_k(k)} \circ \sigma_k = \sigma_k \circ t_{i_k,k}.$$

In view of $\sigma_k(i_k) = k$ and $\sigma_k(k) = m_k$, this rewrites as

$$t_{k,m_k} \circ \sigma_k = \sigma_k \circ \underbrace{t_{i_k,k}}_{=t_{k,i_k}} = \sigma_k \circ t_{k,i_k}. \tag{9}$$

We have $\sigma_{k-1} = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}$ (by the definition of σ_{k-1}). Now, the definition of σ_k yields

$$\sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} = \underbrace{\left(t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}\right)}_{=\sigma_{k-1}} \circ t_{k,i_k} = \sigma_{k-1} \circ t_{k,i_k}. \tag{10}$$

Solving this equation for σ_{k-1}, we obtain

$$\sigma_{k-1} = \sigma_k \circ \underbrace{t_{k,i_k}^{-1}}_{=t_{k,i_k}} = \sigma_k \circ t_{k,i_k} = t_{k,m_k} \circ \sigma_k \qquad \text{(by (9))}. \tag{11}$$

Solving this equation for σ_k, we find

$$\sigma_k = \underbrace{t_{k,m_k}^{-1}}_{=t_{k,m_k}} \circ \sigma_{k-1} = t_{k,m_k} \circ \sigma_{k-1}.$$

Hence,

$$\begin{aligned} \sigma_k^{-1} &= \left(t_{k,m_k} \circ \sigma_{k-1}\right)^{-1} = \underbrace{\sigma_{k-1}^{-1}}_{=t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{k-1,m_{k-1}}} \circ \underbrace{t_{k,m_k}^{-1}}_{=t_{k,m_k}} \\ &= \left(t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{k-1,m_{k-1}}\right) \circ t_{k,m_k} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{k,m_k}. \end{aligned}$$

In other words, (8) holds for $p = k$. This completes the induction step. Thus, (8) is proven by induction.]

Applying (8) to $p = n$, we obtain $\sigma_n^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}$. In view of $\sigma_n = \sigma$, this rewrites as $\sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}$. This solves part **(c)** of the exercise.

(d) For each permutation $\tau \in S_n$, we define a number $z(\tau)$ by

$$z(\tau) = \sum_{k=1}^{n} x_k y_{\tau(k)}.$$

We shall show that

$$z(\sigma_{p-1}) - z(\sigma_p) = \left(x_{i_p} - x_p\right)\left(y_{m_p} - y_p\right) \qquad \text{for each } p \in [n]. \tag{12}$$

[*Proof of* (12)*:* Let $p \in [n]$. Applying (10) to $k = p$, we obtain $\sigma_p = \sigma_{p-1} \circ t_{p,i_p}$. Hence, if $p = i_p$, then (12) holds[5]. Thus, for the rest of this proof, we WLOG assume that $p \neq i_p$. Hence, t_{p,i_p} is an actual transposition (not the identity map).

From $\sigma_p = \sigma_{p-1} \circ t_{p,i_p}$, we obtain

$$\sigma_p(p) = \left(\sigma_{p-1} \circ t_{p,i_p}\right)(p) = \sigma_{p-1}\left(\underbrace{t_{p,i_p}(p)}_{=i_p}\right) = \sigma_{p-1}(i_p),$$

so that

$$\sigma_{p-1}(i_p) = \sigma_p(p) = m_p \tag{13}$$

(since the definition of m_p yields $m_p = \sigma_p(p)$).

From $\sigma_p = \sigma_{p-1} \circ t_{p,i_p}$, we also obtain

$$\sigma_p(i_p) = \left(\sigma_{p-1} \circ t_{p,i_p}\right)(i_p) = \sigma_{p-1}\left(\underbrace{t_{p,i_p}(i_p)}_{=p}\right) = \sigma_{p-1}(p),$$

[5] *Proof.* Assume that $p = i_p$. Thus, $i_p = p$, so that $x_{i_p} - x_p = x_p - x_p = 0$. Hence, the right hand side of (12) equals 0. Also, $\sigma_p = \sigma_{p-1} \circ \underbrace{t_{p,i_p}}_{\substack{=\mathrm{id} \\ (\text{since } p = i_p)}} = \sigma_{p-1}$, so that $z(\sigma_{p-1}) - z(\sigma_p) = z(\sigma_{p-1}) - z(\sigma_{p-1}) = 0$. Thus, the left hand side of (12) equals 0 as well. Hence, the equality (12) holds (since both its right hand side and its left hand side equal 0).

so that

$$\sigma_{p-1}(p) = \sigma_p(i_p) = p \tag{14}$$

(by part **(b)** of the exercise, applied to $k = p$).

Every $k \in [n]$ satisfying $k \neq p$ and $k \neq i_p$ satisfies

$$\sigma_{p-1}(k) = \sigma_p(k) \tag{15}$$

[6]. Now, the definition of $z(\sigma_{p-1})$ yields

$$\begin{aligned}
z(\sigma_{p-1}) &= \sum_{k=1}^{n} x_k y_{\sigma_{p-1}(k)} = x_p \underbrace{y_{\sigma_{p-1}(p)}}_{\substack{=y_p \\ (\text{by } (14))}} + x_{i_p} \underbrace{y_{\sigma_{p-1}(i_p)}}_{\substack{=y_{m_p} \\ (\text{by } (13))}} + \sum_{\substack{k\in[n]; \\ k\neq p \text{ and } k\neq i_p}} x_k \underbrace{y_{\sigma_{p-1}(k)}}_{\substack{=y_{\sigma_p(k)} \\ (\text{by } (15))}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split the addends for } k=p \text{ and} \\ \text{for } k=i_p \text{ from the sum (and these are} \\ \text{two distinct addends, since } p\neq i_p)\end{array}\right) \\
&= x_p y_p + x_{i_p} y_{m_p} + \sum_{\substack{k\in[n]; \\ k\neq p \text{ and } k\neq i_p}} x_k y_{\sigma_p(k)}.
\end{aligned}$$

On the other hand, the definition of $z(\sigma_p)$ yields

$$\begin{aligned}
z(\sigma_p) &= \sum_{k=1}^{n} x_k y_{\sigma_p(k)} = x_p \underbrace{y_{\sigma_p(p)}}_{\substack{=y_{m_p} \\ (\text{since } \sigma_p(p)=m_p)}} + x_{i_p} \underbrace{y_{\sigma_p(i_p)}}_{\substack{=y_p \\ (\text{since } \sigma_p(i_p)=p)}} + \sum_{\substack{k\in[n]; \\ k\neq p \text{ and } k\neq i_p}} x_k y_{\sigma_p(k)} \\
&\qquad \left(\begin{array}{c}\text{here, we have split the addends for } k=p \text{ and} \\ \text{for } k=i_p \text{ from the sum (and these are} \\ \text{two distinct addends, since } p\neq i_p)\end{array}\right) \\
&= x_p y_{m_p} + x_{i_p} y_p + \sum_{\substack{k\in[n]; \\ k\neq p \text{ and } k\neq i_p}} x_k y_{\sigma_p(k)}.
\end{aligned}$$

Subtracting this equality from the preceding equality, we obtain

$$\begin{aligned}
& z(\sigma_{p-1}) - z(\sigma_p) \\
&= \left(x_p y_p + x_{i_p} y_{m_p} + \sum_{\substack{k\in[n]; \\ k\neq p \text{ and } k\neq i_p}} x_k y_{\sigma_p(k)} \right) - \left(x_p y_{m_p} + x_{i_p} y_p + \sum_{\substack{k\in[n]; \\ k\neq p \text{ and } k\neq i_p}} x_k y_{\sigma_p(k)} \right) \\
&= x_p y_p + x_{i_p} y_{m_p} - x_p y_{m_p} - x_{i_p} y_p = \left(x_{i_p} - x_p\right)\left(y_{m_p} - y_p\right).
\end{aligned}$$

This proves (12).]

[6] *Proof:* Let $k \in [n]$ be such that $k \neq p$ and $k \neq i_p$. Thus, $t_{p,i_p}(k) = k$. But $\sigma_p = \sigma_{p-1} \circ t_{p,i_p}$; hence,

$$\sigma_p(k) = \left(\sigma_{p-1} \circ t_{p,i_p}\right)(k) = \sigma_{p-1}\left(\underbrace{t_{p,i_p}(k)}_{=k}\right) = \sigma_{p-1}(k),$$ so that $\sigma_{p-1}(k) = \sigma_p(k)$, qed.

Now, the telescope principle yields

$$\sum_{p=1}^{n}\left(z\left(\sigma_{p-1}\right)-z\left(\sigma_{p}\right)\right)=z\left(\underbrace{\sigma_{0}}_{=\mathrm{id}}\right)-z\left(\underbrace{\sigma_{n}}_{=\sigma}\right)=\underbrace{z\left(\mathrm{id}\right)}_{\substack{=\sum_{k=1}^{n}x_{k}y_{\mathrm{id}(k)}\\ \text{(by the definition of }z(\mathrm{id}))}}-\underbrace{z\left(\sigma\right)}_{\substack{=\sum_{k=1}^{n}x_{k}y_{\sigma(k)}\\ \text{(by the definition of }z(\sigma))}}$$
$$=\sum_{k=1}^{n}x_{k}\underbrace{y_{\mathrm{id}(k)}}_{=y_{k}}-\sum_{k=1}^{n}x_{k}y_{\sigma(k)}=\sum_{k=1}^{n}x_{k}y_{k}-\sum_{k=1}^{n}x_{k}y_{\sigma(k)}.$$

Hence,

$$\sum_{k=1}^{n}x_{k}y_{k}-\sum_{k=1}^{n}x_{k}y_{\sigma(k)}$$
$$=\sum_{p=1}^{n}\underbrace{\left(z\left(\sigma_{p-1}\right)-z\left(\sigma_{p}\right)\right)}_{\substack{=\left(x_{i_p}-x_p\right)\left(y_{m_p}-y_p\right)\\ \text{(by (12))}}}=\sum_{p=1}^{n}\left(x_{i_p}-x_p\right)\left(y_{m_p}-y_p\right)=\sum_{k=1}^{n}\left(x_{i_k}-x_k\right)\left(y_{m_k}-y_k\right)$$

(here, we have renamed the summation index p as k). This solves part **(d)** of the exercise.

(e) Fix $k\in[n]$. Then, $i_k\in[k]$ (by (7)), so that $i_k\leq k$ and therefore $x_{i_k}\geq x_k$ (since $x_1\geq x_2\geq\cdots\geq x_n$). Hence, $x_{i_k}-x_k\geq 0$.

Also, $m_k\in[k]$ (by part **(a)** of the exercise), so that $m_k\leq k$ and thus $y_{m_k}\geq y_k$ (since $y_1\geq y_2\geq\cdots\geq y_n$). Hence, $y_{m_k}-y_k\geq 0$. Now,

$$\underbrace{\left(x_{i_k}-x_k\right)}_{\geq 0}\underbrace{\left(y_{m_k}-y_k\right)}_{\geq 0}\geq 0. \qquad (16)$$

Now, forget that we fixed k. We thus have proven (16) for each $k\in[n]$. Now, part **(d)** of the exercise yields

$$\sum_{k=1}^{n}x_{k}y_{k}-\sum_{k=1}^{n}x_{k}y_{\sigma(k)}=\sum_{k=1}^{n}\underbrace{\left(x_{i_k}-x_k\right)\left(y_{m_k}-y_k\right)}_{\substack{\geq 0\\ \text{(by (16))}}}\geq 0.$$

In other words,

$$\sum_{k=1}^{n}x_{k}y_{k}\geq\sum_{k=1}^{n}x_{k}y_{\sigma(k)}.$$

This solves part **(e)** of the exercise.

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6 Exercise 6

6.1 Problem

Prove the following:

(a) If $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are such that $m < n$, then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = 0.$$

(b) If $n \in \mathbb{N}$ and $r \in [n-1]$, then

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 0.$$

6.2 Solution sketch

(a) *First solution to part **(a)**:* Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $m < n$. We have $|[m]| = m < n = |[n]|$. Thus, there are no surjections from $[m]$ to $[n]$ (by the Pigeonhole Principle for Surjections). Recall that $\operatorname{sur}(m,n)$ denotes the number of all surjections from $[m]$ to $[n]$. Thus, $\operatorname{sur}(m,n) = 0$ (since there are no surjections from $[m]$ to $[n]$).

But Theorem 2.28 from class (2018-10-01) shows that

$$\operatorname{sur}(m,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$$

(here, we have renamed the summation index i as k). Comparing this with $\operatorname{sur}(m,n) = 0$, we obtain $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = 0$. This solves part **(a)** of the exercise.

*Second solution to part **(a)**:* Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $m < n$. Exercise 6 **(b)** on homework set #3 yields that if $A_1, A_2, \ldots, A_n$ are n numbers, then

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \left(\sum_{i \in I} A_i \right)^m = 0.$$

Applying this to $A_i = 1$, we obtain

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \left(\sum_{i \in I} 1 \right)^m = 0.$$

Thus,

$$0 = \sum_{I \subseteq [n]} (-1)^{n-|I|} \left(\underbrace{\sum_{i \in I} 1}_{=|I| \cdot 1 = |I|} \right)^m = \underbrace{\sum_{I \subseteq [n]}}_{= \sum_{i=0}^{n} \sum_{\substack{I \subseteq [n]; \\ |I| = i}}} (-1)^{n-|I|} |I|^m$$

$$= \sum_{i=0}^{n} \sum_{\substack{I \subseteq [n]; \\ |I| = i}} \underbrace{(-1)^{n-|I|} |I|^m}_{\substack{= (-1)^{n-i} i^m \\ (\text{since } |I| = i)}} = \sum_{i=0}^{n} \underbrace{\sum_{\substack{I \subseteq [n]; \\ |I| = i}} (-1)^{n-i} i^m}_{= (\text{the number of all } I \subseteq [n] \text{ satisfying } |I| = i) \cdot (-1)^{n-i} i^m}$$

$$= \sum_{i=0}^{n} \underbrace{(\text{the number of all } I \subseteq [n] \text{ satisfying } |I| = i)}_{\substack{= (\text{the number of all } i\text{-element subsets of } [n]) \\ = \binom{n}{i}}} \cdot (-1)^{n-i} i^m$$

$$= \sum_{i=0}^{n} \binom{n}{i} \cdot (-1)^{n-i} i^m = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^m = \sum_{k=0}^{n} (-1)^k \underbrace{\binom{n}{n-k}}_{\substack{= \binom{n}{k} \\ (\text{by the symmetry} \\ \text{of Pascal's triangle})}} (n-k)^m$$

(here, we have substituted $n-k$ for i in the sum)

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m .$$

This solves part **(a)** of the exercise again.

*Third solution to part **(a)**:* Part **(a)** of the exercise is a particular case of Theorem 6.1 further below (applied to $a = m$, $b = n$ and $c = n$).

(b) We need a generalization of part **(a)** of the exercise:

Theorem 6.1. *Let $a \in \mathbb{N}$, $b \in \mathbb{Q}$ and $c \in \mathbb{N}$ be such that $c > a$. Then,*

$$\sum_{k=0}^{c} (-1)^k \binom{c}{k} (b-k)^a = 0.$$

For the proof of Theorem 6.1, see [Grinbe18, Theorem 0.2].

Let $n \in \mathbb{N}$ and $r \in [n-1]$. We have $r \in [n-1]$, thus $r \leq n-1$ and therefore $2r \leq 2(n-1) < 2n$. Thus, $2n > 2r$. Hence, Theorem 6.1 (applied to $a = 2r$, $b = n$ and $c = 2n$) yields

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 0.$$

Thus,

$$0 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r}$$
$$= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} + \sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r} \tag{17}$$

(since $0 \leq n \leq 2n$). But

$$\sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r}$$
$$= \sum_{k=0}^{n-1} \underbrace{(-1)^{2n-k}}_{=(-1)^k} \underbrace{\binom{2n}{2n-k}}_{\substack{=\binom{2n}{k} \\ \text{(by the symmetry} \\ \text{of Pascal's triangle)}}} \Bigg(\underbrace{n-(2n-k)}_{=-(n-k)}\Bigg)^{2r}$$

(here, we have substituted $2n-k$ for k in the sum)

$$= \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \underbrace{(-(n-k))^{2r}}_{\substack{=(n-k)^{2r} \\ \text{(since } 2r \text{ is even)}}} = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (n-k)^{2r}$$
$$= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} - (-1)^n \binom{2n}{n} \underbrace{(n-n)^{2r}}_{\substack{=0^{2r}=0 \\ \text{(since } r>0)}}$$

(here, we have extended the range of the sum to include a new addend for $k = n$, and then subtracted that addend)

$$= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}.$$

Hence, (17) becomes

$$0 = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} + \underbrace{\sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r}}_{=\sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}}$$
$$= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} + \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 2 \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}.$$

Dividing this equality by 2, we find $0 = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}$. This solves part **(b)** of the exercise.

6.3 Remark

I have learnt part **(b)** of the exercise from MathOverflow question #312839, which also asks if the sum is $\neq 0$ when $2r$ is replaced by an **odd** integer between 1 and $2n-1$.

7 Exercise 7

7.1 Problem

Let $n \in \mathbb{N}$ and $d \in \mathbb{N}$. An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ is said to be *all-even* if each element of $[d]$ occurs an even number of times in this n-tuple (i.e., if for each $k \in [d]$, the number of all $i \in [n]$ satisfying $x_i = k$ is even). For example, the 4-tuple $(1, 4, 4, 1)$ and the 6-tuples $(1, 3, 3, 5, 1, 5)$ and $(2, 4, 2, 4, 3, 3)$ are all-even, while the 4-tuples $(1, 2, 2, 4)$ and $(2, 4, 6, 4)$ are not.

Prove that the number of all all-even n-tuples $(x_1, x_2, \ldots, x_n) \in [d]^n$ is

$$\frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n .$$

[**Hint:** Compute the sum $\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (e_1 + e_2 + \cdots + e_d)^n$ in two ways. One way is to split it according to the number of $i \in [d]$ satisfying $e_i = -1$; this is a number $k \in \{0, 1, \ldots, d\}$. Another way is by using the product rule:

$$(e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}$$

and then simplifying each sum $\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}$ using a form of destructive interference. This is not unlike the number of 1-even n-tuples, which we computed at the end of the 2018-10-10 class.]

7.2 Solution sketch

Recall the *product rule* (which we have already used when solving Exercise 6 on homework set #3):

Proposition 7.1 (Product rule)**.** *Let $m \in \mathbb{N}$. Let I be a finite set. Let $P_{u,v}$, for all $u \in [m]$ and $v \in I$, be numbers or polynomials or square matrices of the same size. Then,*

$$\left(\sum_{i\in I} P_{1,i}\right) \left(\sum_{i\in I} P_{2,i}\right) \cdots \left(\sum_{i\in I} P_{m,i}\right) = \sum_{(i_1,i_2,\ldots,i_m)\in I^m} P_{1,i_1} P_{2,i_2} \cdots P_{m,i_m}.$$

Fix a d-tuple $(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d$. We now apply Proposition 7.1 to $m = n$, $I = [d]$ and $P_{u,v} = e_v$. As a result, we obtain

$$\underbrace{\left(\sum_{i\in[d]} e_i\right) \left(\sum_{i\in[d]} e_i\right) \cdots \left(\sum_{i\in[d]} e_i\right)}_{n \text{ times}} = \sum_{(i_1,i_2,\ldots,i_n)\in[d]^n} e_{i_1} e_{i_2} \cdots e_{i_n} = \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}$$

(here, we have renamed the summation index $(i_1, i_2, \ldots, i_n)$ as $(x_1, x_2, \ldots, x_n)$). Thus,

$$\sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} e_{x_1} e_{x_2} \cdots e_{x_n} = \underbrace{\left(\sum_{i\in[d]} e_i\right)\left(\sum_{i\in[d]} e_i\right)\cdots\left(\sum_{i\in[d]} e_i\right)}_{n \text{ times}} = \left(\underbrace{\sum_{i\in[d]} e_i}_{=e_1+e_2+\cdots+e_d}\right)^n$$
$$= (e_1 + e_2 + \cdots + e_d)^n . \qquad (18)$$

Now, forget that we fixed $(e_1, e_2, \ldots, e_d)$. We thus have proven the equality (18) for each $(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d$.

Now,

$$\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} \underbrace{(e_1 + e_2 + \cdots + e_d)^n}_{\substack{=\sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} e_{x_1}e_{x_2}\cdots e_{x_n} \\ \text{(by (18))}}}$$
$$= \underbrace{\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n}}_{=\sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} \sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d}} e_{x_1} e_{x_2} \cdots e_{x_n}$$
$$= \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} \sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}. \qquad (19)$$

We shall now simplify the inner sum on the right hand side of this equality. Indeed, we claim the following:

Claim 1: Let $(x_1, x_2, \ldots, x_n) \in [d]^n$.

(a) If the n-tuple $(x_1, x_2, \ldots, x_n)$ is not all-even, then

$$\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = 0.$$

(b) If the n-tuple $(x_1, x_2, \ldots, x_n)$ is all-even, then

$$\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = 2^d.$$

[*Proof of Claim 1:* **(a)** Assume that the n-tuple $(x_1, x_2, \ldots, x_n)$ is not all-even. Thus, it is **not** true that for each $k \in [d]$, the number of all $i \in [n]$ satisfying $x_i = k$ is even (by the definition of "all-even"). In other words, there exists some $k \in [d]$ such that the number of all $i \in [n]$ satisfying $x_i = k$ is odd. Consider this k.

The number

$$\sum_{i\in[n]} [x_i = k] = \sum_{\substack{i\in[n];\\ x_i=k}} \underbrace{[x_i = k]}_{\substack{=1\\ (\text{since } x_i=k)}} + \sum_{\substack{i\in[n];\\ x_i\neq k}} \underbrace{[x_i = k]}_{\substack{=0\\ (\text{since } x_i\neq k)}} = \sum_{\substack{i\in[n];\\ x_i=k}} 1 + \underbrace{\sum_{\substack{i\in[n];\\ x_i\neq k}} 0}_{=0}$$
$$= \sum_{\substack{i\in[n];\\ x_i=k}} 1 = (\text{the number of all } i \in [n] \text{ satisfying } x_i = k) \cdot 1$$
$$= (\text{the number of all } i \in [n] \text{ satisfying } x_i = k)$$

is odd (by the definition of k). Now,

$$(-1)^{[x_1=k]}(-1)^{[x_2=k]}\cdots(-1)^{[x_n=k]}=\prod_{i\in[n]}(-1)^{[x_i=k]}=(-1)^{\sum_{i\in[n]}[x_i=k]}=-1$$

(since the number $\sum_{i\in[n]}[x_i=k]$ is odd).

Now, define the two subsets

$$N=\left\{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d\ \mid\ e_k=-1\right\}\qquad\text{and}$$
$$P=\left\{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d\ \mid\ e_k=1\right\}$$

of the set $\{-1,1\}^d$. Clearly, each element of $\{-1,1\}^d$ belongs to exactly one of these two subsets N and P (because for each $(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d$, we have either $e_k=-1$ or $e_k=1$ but not both).

Clearly, the map

$$N\to P,\qquad (e_1,e_2,\ldots,e_d)\mapsto(e_1,e_2,\ldots,e_{k-1},-e_k,e_{k+1},e_{k+2},\ldots,e_d)$$

(which replaces the k-th entry of a d-tuple by its negative, while leaving all other entries unchanged) is well-defined and bijective (indeed, its inverse map is defined by the same rule). We can rewrite this map (using the Iverson bracket notation) as the map

$$N\to P,\qquad (e_1,e_2,\ldots,e_d)\mapsto\left((-1)^{[1=k]}e_1,(-1)^{[2=k]}e_2,\ldots,(-1)^{[d=k]}e_d\right)$$

(because each $(e_1,e_2,\ldots,e_d)\in N$ satisfies

$$\left((-1)^{[1=k]}e_1,(-1)^{[2=k]}e_2,\ldots,(-1)^{[d=k]}e_d\right)=(e_1,e_2,\ldots,e_{k-1},-e_k,e_{k+1},e_{k+2},\ldots,e_d)$$

[7]). Hence, the map

$$N\to P,\qquad (e_1,e_2,\ldots,e_d)\mapsto\left((-1)^{[1=k]}e_1,(-1)^{[2=k]}e_2,\ldots,(-1)^{[d=k]}e_d\right)$$

is bijective, i.e., is a bijection from N to P.

[7] *Proof.* Let $(e_1,e_2,\ldots,e_d)\in N$. Then, each $i\in[d]$ satisfying $i\neq k$ satisfies $[i=k]=0$ and therefore $(-1)^{[i=k]}e_i=\underbrace{(-1)^0}_{=1}e_i=e_i$. Hence, the d-tuple $\left((-1)^{[1=k]}e_1,(-1)^{[2=k]}e_2,\ldots,(-1)^{[d=k]}e_d\right)$ differs from the d-tuple $(e_1,e_2,\ldots,e_d)$ only in its k-th entry. As for its k-th entry, it is $\underbrace{(-1)^{[k=k]}}_{=(-1)^1=-1}e_k=-e_k$. Thus, this d-tuple $\left((-1)^{[1=k]}e_1,(-1)^{[2=k]}e_2,\ldots,(-1)^{[d=k]}e_d\right)$ is obtained from the d-tuple $(e_1,e_2,\ldots,e_d)$ by replacing its k-th entry by $-e_k$. In other words,

$$\left((-1)^{[1=k]}e_1,(-1)^{[2=k]}e_2,\ldots,(-1)^{[d=k]}e_d\right)=(e_1,e_2,\ldots,e_{k-1},-e_k,e_{k+1},e_{k+2},\ldots,e_d).$$

Recall that each element of $\{-1,1\}^d$ belongs to exactly one of the two subsets N and P. Hence, we can split the sum $\sum\limits_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n}$ as follows:

$$\begin{aligned}
&\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n} \\
&= \sum_{(e_1,e_2,\ldots,e_d)\in N} e_{x_1}e_{x_2}\cdots e_{x_n} + \sum_{(e_1,e_2,\ldots,e_d)\in P} e_{x_1}e_{x_2}\cdots e_{x_n} \\
&= \sum_{(e_1,e_2,\ldots,e_d)\in N} e_{x_1}e_{x_2}\cdots e_{x_n} + \sum_{(e_1,e_2,\ldots,e_d)\in N} \underbrace{\left((-1)^{[x_1=k]} e_{x_1}\right)\left((-1)^{[x_2=k]} e_{x_2}\right)\cdots\left((-1)^{[x_n=k]} e_{x_n}\right)}_{=\left((-1)^{[x_1=k]}(-1)^{[x_2=k]}\cdots(-1)^{[x_n=k]}\right)\left(e_{x_1}e_{x_2}\cdots e_{x_n}\right)} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \left((-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d\right) \\ \text{for } (e_1,e_2,\ldots,e_d) \text{ in the second sum, since} \\ \text{the map } N\to P, \qquad (e_1,e_2,\ldots,e_d)\mapsto\left((-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d\right) \\ \text{is a bijection}\end{array}\right) \\
&= \sum_{(e_1,e_2,\ldots,e_d)\in N} e_{x_1}e_{x_2}\cdots e_{x_n} + \sum_{(e_1,e_2,\ldots,e_d)\in N} \underbrace{\left((-1)^{[x_1=k]}(-1)^{[x_2=k]}\cdots(-1)^{[x_n=k]}\right)}_{=-1}\left(e_{x_1}e_{x_2}\cdots e_{x_n}\right) \\
&= \sum_{(e_1,e_2,\ldots,e_d)\in N} e_{x_1}e_{x_2}\cdots e_{x_n} - \sum_{(e_1,e_2,\ldots,e_d)\in N} e_{x_1}e_{x_2}\cdots e_{x_n} = 0.
\end{aligned}$$

This proves Claim 1 **(a)**.

(b) Assume that the n-tuple $(x_1,x_2,\ldots,x_n)$ is all-even. Thus, for each $k\in[d]$, the number of all $i\in[n]$ satisfying $x_i=k$ is even (by the definition of "all-even").

Let $k\in[d]$. As we have just seen, the number of all $i\in[n]$ satisfying $x_i=k$ is even. In other words, there exists some $h\in\mathbb{Z}$ such that

$$(\text{the number of all } i\in[n] \text{ satisfying } x_i=k) = 2h. \tag{20}$$

Consider this h.

Now, let $(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d$ be arbitrary. Thus, $e_k\in\{-1,1\}$, so that $e_k^2=1$. Now,

$$\begin{aligned}
\prod_{\substack{i\in[n];\\ x_i=k}} \underbrace{e_{x_i}}_{\substack{=e_k\\ (\text{since } x_i=k)}} &= \prod_{\substack{i\in[n];\\ x_i=k}} e_k = e_k^{(\text{the number of all } i\in[n] \text{ satisfying } x_i=k)} = e_k^{2h} \qquad (\text{by (20)}) \\
&= \Big(\underbrace{e_k^2}_{=1}\Big)^h = 1^h = 1. \qquad (21)
\end{aligned}$$

Now, forget that we fixed $(e_1,e_2,\ldots,e_d)$ and k. We thus have proven (21) for each $(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d$ and $k\in[d]$.

Now, each $(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d$ satisfies

$$e_{x_1}e_{x_2}\cdots e_{x_n} = \underbrace{\prod_{i\in[n]}}_{=\prod_{k\in[d]}\prod_{\substack{i\in[n];\\ x_i=k}}} e_{x_i} = \prod_{k\in[d]} \underbrace{\prod_{\substack{i\in[n];\\ x_i=k}} e_{x_i}}_{\substack{=1\\ (\text{by (21)})}} = \prod_{k\in[d]} 1 = 1.$$

Hence,

$$\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} \underbrace{e_{x_1}e_{x_2}\cdots e_{x_n}}_{=1} = \sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} 1 = \left|\{-1,1\}^d\right| \cdot 1$$
$$= \left|\{-1,1\}^d\right| = \left|\{-1,1\}\right|^d = 2^d.$$

This proves Claim 1 **(b)**.]

Now, (19) becomes

$$\begin{aligned}
&\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (e_1+e_2+\cdots+e_d)^n \\
&= \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} \sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n} \\
&= \sum_{\substack{(x_1,x_2,\ldots,x_n)\in[d]^n;\\ (x_1,x_2,\ldots,x_n)\text{ is all-even}}} \underbrace{\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n}}_{\substack{=2^d\\ \text{(by Claim 1 (b))}}} \\
&\qquad + \sum_{\substack{(x_1,x_2,\ldots,x_n)\in[d]^n;\\ (x_1,x_2,\ldots,x_n)\text{ is not all-even}}} \underbrace{\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n}}_{\substack{=0\\ \text{(by Claim 1 (a))}}} \\
&= \sum_{\substack{(x_1,x_2,\ldots,x_n)\in[d]^n;\\ (x_1,x_2,\ldots,x_n)\text{ is all-even}}} 2^d + \underbrace{\sum_{\substack{(x_1,x_2,\ldots,x_n)\in[d]^n;\\ (x_1,x_2,\ldots,x_n)\text{ is not all-even}}} 0}_{=0} = \sum_{\substack{(x_1,x_2,\ldots,x_n)\in[d]^n;\\ (x_1,x_2,\ldots,x_n)\text{ is all-even}}} 2^d \\
&= \left(\text{the number of all all-even } (x_1,x_2,\ldots,x_n)\in[d]^n\right)\cdot 2^d. \qquad (22)
\end{aligned}$$

For each d-tuple $(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d$, we have

$$\begin{aligned}
d-(e_1+e_2+\cdots+e_d) &= \underbrace{d}_{=\sum_{i\in[d]} 1} - \sum_{i\in[d]} e_i = \sum_{i\in[d]} 1 - \sum_{i\in[d]} e_i = \sum_{i\in[d]} (1-e_i) \\
&= \sum_{\substack{i\in[d];\\ e_i=-1}} \left(1-\underbrace{e_i}_{=-1}\right) + \sum_{\substack{i\in[d];\\ e_i=1}} \left(1-\underbrace{e_i}_{=1}\right) \\
&\qquad \left(\begin{array}{c}\text{since each } i\in[d] \text{ satisfies either } e_i=-1 \text{ or } e_i=1\\ \text{(but not both) (because } (e_1,e_2,\ldots,e_d)\in\{-1,1\}^d\text{)}\end{array}\right) \\
&= \sum_{\substack{i\in[d];\\ e_i=-1}} \underbrace{(1-(-1))}_{=2} + \sum_{\substack{i\in[d];\\ e_i=1}} \underbrace{(1-1)}_{=0} = \sum_{\substack{i\in[d];\\ e_i=-1}} 2 + \underbrace{\sum_{\substack{i\in[d];\\ e_i=1}} 0}_{=0} \\
&= \sum_{\substack{i\in[d];\\ e_i=-1}} 2 = \left|\{i\in[d] \mid e_i=-1\}\right|\cdot 2 = 2\cdot\left|\{i\in[d] \mid e_i=-1\}\right|
\end{aligned}$$

and thus

$$e_1+e_2+\cdots+e_d = d-2\cdot\left|\{i\in[d] \mid e_i=-1\}\right|.$$

Hence,

$$\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} \left(\underbrace{e_1+e_2+\cdots+e_d}_{=d-2\cdot|\{i\in[d]\ \mid\ e_i=-1\}|}\right)^n$$
$$= \sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (d-2\cdot|\{i\in[d]\ \mid\ e_i=-1\}|)^n . \qquad (23)$$

On the other hand, a d-tuple $(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d$ is uniquely determined by the set $\{i\in[d]\ \mid\ e_i=-1\}$ of all positions at which it contains a -1 (and conversely, for every subset S of $[d]$, there exists such a d-tuple whose set $\{i\in[d]\ \mid\ e_i=-1\}$ is S). Thus, the map

$$\{-1,1\}^d \to \{S\subseteq[d]\}, \qquad (e_1,e_2,\ldots,e_d)\mapsto\{i\in[d]\ \mid\ e_i=-1\}$$

is a bijection. Hence, we can substitute S for $\{i\in[d]\ \mid\ e_i=-1\}$ in the sum on the right hand side of (23). We thus obtain

$$\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (d-2\cdot|\{i\in[d]\ \mid\ e_i=-1\}|)^n$$
$$= \underbrace{\sum_{S\subseteq[d]}}_{=\sum_{k=0}^{d}\sum_{S\subseteq[d];\ |S|=k}} (d-2\cdot|S|)^n = \sum_{k=0}^{d}\sum_{\substack{S\subseteq[d];\\ |S|=k}} \Bigl(d-2\cdot\underbrace{|S|}_{=k}\Bigr)^n$$
$$= \sum_{k=0}^{d} \underbrace{\sum_{\substack{S\subseteq[d];\\ |S|=k}} (d-2k)^n}_{=(\text{the number of all } S\subseteq[d] \text{ satisfying } |S|=k)\cdot(d-2k)^n}$$
$$= \sum_{k=0}^{d} \underbrace{(\text{the number of all } S\subseteq[d] \text{ satisfying } |S|=k)}_{=(\text{the number of all } k\text{-element subsets of } [d])=\binom{d}{k}} \cdot (d-2k)^n$$
$$= \sum_{k=0}^{d} \binom{d}{k} (d-2k)^n .$$

Hence, (23) becomes

$$\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (e_1+e_2+\cdots+e_d)^n$$
$$= \sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (d-2\cdot|\{i\in[d]\ \mid\ e_i=-1\}|)^n = \sum_{k=0}^{d}\binom{d}{k}(d-2k)^n .$$

Comparing this with (22), we obtain

$$(\text{the number of all all-even } (x_1,x_2,\ldots,x_n)\in[d]^n)\cdot 2^d = \sum_{k=0}^{d}\binom{d}{k}(d-2k)^n .$$

Solving this for (the number of all all-even $(x_1, x_2, \ldots, x_n) \in [d]^n$), we obtain

$$\text{(the number of all all-even } (x_1, x_2, \ldots, x_n) \in [d]^n) = \frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d-2k)^n .$$

This solves the exercise.

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The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

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