University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 4

Jacob Elafandi (edited by Darij Grinberg)

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Exercise 1

Problem

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Let i and j be two elements of $[n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$. Let Q be the set of all $k \in \{i+1, i+2, \ldots, j-1\}$ satisfying $\sigma(i) > \sigma(k) > \sigma(j)$. Prove that

$$\ell(\sigma \circ t_{i,j}) = \ell(\sigma) - 2|Q| - 1.$$

Remark

This exercise implies that, in particular, $\ell(\sigma \circ t_{i,j}) < \ell(\sigma)$; this answers the question on page 213 of the notes from class (2018-10-22).

Solution

For fixed $i, j \in [n]$ such that $i < j$, the inversions of $\pi \in S_n$ can be divided into ten disjoint categories (whose sizes sum to $\ell(\pi)$):

- $A(\pi) = \{(k, y) \in [n]^2 : k < y \text{ and } \{k, y\} \cap \{i, j\} = \emptyset\}$.
- $B(\pi) = \{(k, i) \in [n]^2 : k < i \text{ and } \pi(k) > \pi(i)\}$.
- $C(\pi) = \{(i, k) \in [n]^2 : i < k < j \text{ and } \pi(i) > \pi(k) > \pi(j)\}$.

- $D(\pi) = \{(i,k) \in [n]^2 : i < k < j \text{ and } \pi(i) > \pi(k) \text{ and } \pi(j) > \pi(k)\}$.
- $E(\pi) = \{(i,k) \in [n]^2 : k > j \text{ and } \pi(i) > \pi(k)\}$.
- $F(\pi) = \{(k,j) \in [n]^2 : k < i \text{ and } \pi(k) > \pi(j)\}$.
- $G(\pi) = \{(k,j) \in [n]^2 : i < k < j \text{ and } \pi(k) > \pi(i) \text{ and } \pi(k) > \pi(j)\}$.
- $H(\pi) = \{(k,j) \in [n]^2 : i < k < j \text{ and } \pi(i) > \pi(k) > \pi(j)\}$.
- $I(\pi) = \{(j,k) \in [n]^2 : k > j \text{ and } \pi(j) > \pi(k)\}$.
- $J(\pi) = \{(i,j) : \pi(i) > \pi(j)\}$.

Note that $(\sigma \circ t_{i,j})(i) = \sigma(j)$ and $(\sigma \circ t_{i,j})(j) = \sigma(i)$. Therefore, since $i < j$ and $\sigma(i) > \sigma(j)$, we see that the inversions of $\sigma \circ t_{i,j}$ can be divided into the following disjoint categories (whose sizes sum to $\ell(\pi \circ t_{i,j})$):

- $A(\sigma \circ t_{i,j}) = A(\sigma)$.
- $|B(\sigma \circ t_{i,j})| = |F(\sigma)|$ (via the bijection $B(\sigma \circ t_{i,j}) \to F(\sigma)$ sending each (k,i) to (k,j)); $|F(\sigma \circ t_{i,j})| = |B(\sigma)|$ (via the bijection $B(\sigma \circ t_{i,j}) \to F(\sigma)$ sending each (k,j) to (k,i)).
- $D(\sigma \circ t_{i,j}) = D(\sigma)$.
- $|E(\sigma \circ t_{i,j})| = |I(\sigma)|$ (via the bijection $E(\sigma \circ t_{i,j}) \to I(\sigma)$ sending each (i,k) to (j,k)); $|I(\sigma \circ t_{i,j})| = |E(\sigma)|$ (via the bijection $I(\sigma \circ t_{i,j}) \to E(\sigma)$ sending each (j,k) to (i,k)).
- $G(\sigma \circ t_{i,j}) = G(\sigma)$.
- $C(\sigma \circ t_{i,j}) = H(\sigma \circ t_{i,j}) = J(\sigma \circ t_{i,j}) = \emptyset$.

Therefore $\ell(\sigma \circ t_{i,j}) = \ell(\sigma) - |C(\sigma)| - |H(\sigma)| - |J(\sigma)|$. By the definitions of C, H, J, and Q, one has that $|C(\sigma)| = |H(\sigma)| = |Q|$ and $|J(\sigma)| = 1$. The desired equality follows.

Exercise 2

Problem

Let $n \in \mathbb{N}$ and $\pi \in S_n$.

(a) Prove that

$$\sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) > \pi(j)}} (\pi(j) - \pi(i)) = \sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) > \pi(j)}} (i - j).$$

(b) Prove that

$$\sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) < \pi(j)}} (\pi(j) - \pi(i)) = \sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) < \pi(j)}} (j - i).$$

[**Hint:** Exercise 5.23 in [Grinbe16] says something about sums of the form appearing in part **(a)**. (See also Nathaniel Gorski's solution of the same exercise in Spring 2018 Math 4707 homework set #4.) You may want to use the result or the ideas.]

Solution

The following solution was inspired by that of Nathaniel Gorski, as suggested in the hint. For $i \in [n]$, define the following subsets of $[n]$:

- $U(i) = \{j \in [n] : j < i \text{ and } \pi(j) > \pi(i)\}$.
- $M(i) = \{j \in [n] : j < i \text{ and } \pi(j) < \pi(i)\}$.
- $L(i) = \{j \in [n] : j > i \text{ and } \pi(j) < \pi(i)\}$.
- $T(i) = \{j \in [n] : j > i \text{ and } \pi(j) > \pi(i)\}$.

Part (a)

Note first that $U(i) \cup M(i) = \{j \in [n] : j < i\}$. Because $U(i)$ and $M(i)$ are disjoint, $|U(i)| + |M(i)| = |U(i) \cup M(i)| = i - 1$.

Note also that $M(i) \cup L(i) = \{j \in [n] : \pi(j) < \pi(i)\}$. Because π is a permutation of $[n]$, this union has size $\pi(i) - 1$. Because $M(i)$ and $L(i)$ are disjoint, $|M(i)| + |L(i)| = |M(i) \cup L(i)| = \pi(i) - 1$.

Subtracting the second equality from the first yields $|U(i)| - |L(i)| = i - \pi(i)$. This enables the following string of computations:

$$\begin{aligned}
&\sum_{\substack{1\le i<j\le n;\\ \pi(i)>\pi(j)}} (\pi(j) - \pi(i)) - \sum_{\substack{1\le i<j\le n;\\ \pi(i)>\pi(j)}} (i - j) \\
&= \sum_{\substack{1\le i<j\le n;\\ \pi(i)>\pi(j)}} (\pi(j) + j) - \sum_{\substack{1\le i<j\le n;\\ \pi(i)>\pi(j)}} (\pi(i) + i) \\
&= \sum_{j=1}^{n} \sum_{i \in U(j)} (\pi(j) + j) - \sum_{i=1}^{n} \sum_{j \in L(i)} (\pi(i) + i) \\
&= \sum_{j=1}^{n} |U(j)| (\pi(j) + j) - \sum_{i=1}^{n} |L(i)| (\pi(i) + i) \\
&= \sum_{i=1}^{n} (|U(i)| - |L(i)|) (\pi(i) + i) \\
&= \sum_{i=1}^{n} (i - \pi(i)) (\pi(i) + i) \\
&= \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} (\pi(i))^2 \\
&= (1^2 + 2^2 + \cdots + n^2) - (1^2 + 2^2 + \cdots + n^2) \\
&= 0.
\end{aligned}$$

The desired equality follows.

Part (b)

Note that $L(i) \cup T(i) = \{j \in [n] : j > i\}$. Because $L(i)$ and $T(i)$ are disjoint, $|L(i)| + |T(i)| = |L(i) \cup T(i)| = n - i$.

We know from part **(a)** that $|M(i)| + |L(i)| = \pi(i) - 1$. Subtracting the above equation from this one gives $|M(i)| - |T(i)| = \pi(i) + i - n - 1$. Therefore:

$$\begin{aligned}
&\sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} (\pi(j) - \pi(i)) - \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} (j-i) \\
&= \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} (\pi(j) - j) - \sum_{\substack{1\leq i<j\leq n;\\ \pi(i)<\pi(j)}} (\pi(i) - i) \\
&= \sum_{j=1}^{n} \sum_{i\in M(j)} (\pi(j) - j) - \sum_{i=1}^{n} \sum_{j\in T(i)} (\pi(i) - i) \\
&= \sum_{j=1}^{n} |M(j)|(\pi(j) - j) - \sum_{i=1}^{n} |T(i)|(\pi(i) - i) \\
&= \sum_{i=1}^{n} (|M(i) - |T(i)|)(\pi(i) - i) \\
&= \sum_{i=1}^{n} (\pi(i) + i - n - 1)(\pi(i) - i) \\
&= \sum_{i=1}^{n} (\pi(i))^2 - \sum_{i=1}^{n} i^2 - (n-1)\left(\sum_{i=1}^{n} \pi(i) - \sum_{i=1}^{n} i\right) \\
&= (1^2 + \cdots + n^2) - (1^2 + \cdots + n^2) - (n-1)\left((1 + \cdots + n) - (1 + \cdots + n)\right) \\
&= 0.
\end{aligned}$$

The desired equality follows.

Exercise 3

Problem

Let $n \in \mathbb{N}$. For each $p \in \mathbb{Z}$, we let

$$D_{n,p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents}\}.$$

(Recall that a *descent* of a permutation $\sigma \in S_n$ denotes an element $k \in [n-1]$ satisfying $\sigma(k) > \sigma(k+1)$.)

Let $p \in \mathbb{Z}$. Prove that $|D_{n,p}| = |D_{n,n-1-p}|$.

Solution

Let $\sigma \in D_{n,p}$ and let $H = \{k \in [n-1] \mid \sigma(k) > \sigma(k+1)\}$ be the set of all descents of σ. For all $k \in H$, we have $(w_0 \circ \sigma)(k) = n+1-\sigma(k) < n+1-\sigma(k+1) = (w_0 \circ \sigma)(k+1)$, so k is not a descent of $w_0 \circ \sigma$. For all $k \in [n-1] \setminus H$, we have $(w_0 \circ \sigma)(k) = n+1-\sigma(k) > n+1-\sigma(k+1) = (w_0 \circ \sigma)(k+1)$, so k is a descent of $w_0 \circ \sigma$. Thus $w_0 \circ \sigma$ has $n-1-|H| = n-1-p$ descents, so the map $f : D_{n,p} \to D_{n,n-1-p}$ defined by $f(\sigma) = w_0 \circ \sigma$ is well-defined.

For any $\sigma \in D_{n,p}$, we have $w_0 \circ f(\sigma) = w_0 \circ w_0 \circ \sigma = \sigma$. This implies that f is injective, so $|D_{n,p}| \leq |D_{n,n-1-p}|$.

The same chain of reasoning, applied to $D_{n,p}$ from $D_{n,n-1-p}$, can be used to show that $|D_{n,n-1-p}| \leq |D_{n,p}|$. Therefore $|D_{n,p}| = |D_{n-1-p}|$.

Exercise 7

Problem

Let $n \in \mathbb{N}$ and $d \in \mathbb{N}$. An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ is said to be *all-even* if each element of $[d]$ occurs an even number of times in this n-tuple (i.e., if for each $k \in [d]$, the number of all $i \in [n]$ satisfying $x_i = k$ is even). For example, the 4-tuple $(1, 4, 4, 1)$ and the 6-tuples $(1, 3, 3, 5, 1, 5)$ and $(2, 4, 2, 4, 3, 3)$ are all-even, while the 4-tuples $(1, 2, 2, 4)$ and $(2, 4, 6, 4)$ are not.

Prove that the number of all all-even n-tuples $(x_1, x_2, \ldots, x_n) \in [d]^n$ is

$$\frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d-2k)^n .$$

[**Hint:** Compute the sum $\sum\limits_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (e_1 + e_2 + \cdots + e_d)^n$ in two ways. One way is to split it according to the number of $i \in [d]$ satisfying $e_i = -1$; this is a number $k \in \{0, 1, \ldots, d\}$. Another way is by using the product rule:

$$(e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}$$

and then simplifying each sum $\sum\limits_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}$ using a form of destructive interference. This is not unlike the number of 1-even n-tuples, which we computed at the end of the 2018-10-10 class.]

Solution

The following solution will use the shorthand symbols $\vec{x}$ and $\vec{e}$ to represent $(x_1, x_2, \ldots, x_n)$ and $(e_1, e_2, \ldots, e_d)$, respectively.

Let $f : \{-1, 1\}^d \to \{0, 1, \ldots, d\}$ send each d-tuple $\vec{e}$ to the number of $i \in [d]$ such that $e_i = -1$. For each $k \in \mathbb{Z}$, there are exactly $\binom{d}{k}$ such d-tuples $\vec{e}$ satisfying $f(\vec{e}) = k$. Thus:

$$\begin{aligned}
\sum_{\vec{e}\in\{-1,1\}^d} (e_1 + \cdots + e_d)^n &= \sum_{k=0}^{d} \sum_{\substack{\vec{e}\in\{-1,1\}^d;\\ f(\vec{e})=k}} (e_1 + \cdots + e_d)^n \\
&= \sum_{k=0}^{d} \sum_{\substack{\vec{e}\in\{-1,1\}^d;\\ f(\vec{e})=k}} (k(-1) + (d-k)(1))^n \\
&= \sum_{k=0}^{d} \binom{d}{k} (d-2k)^n.
\end{aligned}$$

Now use the product rule to expand the left-hand sum, and flip the order of summation.

$$\begin{aligned}\sum_{\vec{e}\in\{-1,1\}^d} (e_1+\cdots+e_d)^n &= \sum_{\vec{e}\in\{-1,1\}^d} \sum_{\vec{x}\in[d]^n} e_{x_1}e_{x_2}\cdots e_{x_n} \\ &= \sum_{\vec{x}\in[d]^n} \sum_{\vec{e}\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n}.\end{aligned}$$

Let $g : [d]^n \times [d] \to \{0,1,\ldots,n\}$ send each $(\vec{x},j)$ to the number of $i \in [n]$ such that $x_i = j$. Then:

$$\sum_{\vec{x}\in[d]^n} \sum_{\vec{e}\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n} = \sum_{\vec{x}\in[d]^n} \sum_{\vec{e}\in\{-1,1\}^d} \prod_{j\in[d]} e_j^{g(\vec{x},j)}.$$

For every n-tuple $\vec{x} \in [d]^n$ which is not all-even, there exists $h(\vec{x}) \in [d]$ such that $g(\vec{x},h(\vec{x}))$ is odd. For convenience's sake, let $\vec{e'}$ represent the $(d-1)$-tuple $(e_1,\ldots,e_{h(\vec{x})-1},e_{h(\vec{x})+1},\ldots,e_d)$. Then:

$$\begin{aligned}\sum_{\substack{\vec{x}\in[d]^n;\\ \vec{x}\text{ not all-even}}} \sum_{\vec{e}\in\{-1,1\}^d} \prod_{j\in[d]} e_j^{g(\vec{x},j)} &= \sum_{\substack{\vec{x}\in[d]^n;\\ \vec{x}\text{ not all-even}}} \sum_{e_{h(\vec{x})}\in\{-1,1\}} \sum_{\vec{e'}\in\{-1,1\}^{d-1}} \prod_{j\in[d]} e_j^{g(\vec{x},j)} \\ &= \sum_{\substack{\vec{x}\in[d]^n;\\ \vec{x}\text{ not all-even}}} \sum_{e_{h(\vec{x})}\in\{-1,1\}} e_{h(\vec{x})}^{g(\vec{x},h(\vec{x}))} \sum_{\vec{e'}\in\{-1,1\}^{d-1}} \prod_{j\in[d]\setminus h(\vec{x})} e_j^{g(\vec{x},j)} \\ &= \sum_{\substack{\vec{x}\in[d]^n;\\ \vec{x}\text{ not all-even}}} (-1+1) \sum_{\vec{e'}\in\{-1,1\}^{d-1}} \prod_{j\in[d]\setminus h(\vec{x})} e_j^{g(\vec{x},j)} \\ &= \sum_{\substack{\vec{x}\in[d]^n;\\ \vec{x}\text{ not all-even}}} 0 \\ &= 0.\end{aligned}$$

Therefore the only n-tuples $\vec{x}$ which contribute to the overall sum are those which are all-even. For these, $g(\vec{x},j)$ is even for all $j \in [d]$, so $e_j^{g(\vec{x},j)} = 1$ for all $j \in [d]$. Thus:

$$\begin{aligned}\sum_{\vec{x}\in[d]^n} \sum_{\vec{e}\in\{-1,1\}^d} \prod_{j\in[d]} e_j^{g(\vec{x},j)} &= \sum_{\substack{\vec{x}\in[d]^n;\\ \vec{x}\text{ all-even}}} \sum_{\vec{e}\in\{-1,1\}^d} 1 \\ &= \sum_{\substack{\vec{x}\in[d]^n;\\ \vec{x}\text{ all-even}}} 2^d \\ &= 2^d \cdot (\text{number of all-even } n\text{-tuples}).\end{aligned}$$

Chaining these equalities together yields

$$\sum_{k=0}^{d} \binom{d}{k} (d-2k)^n = 2^d \cdot (\text{number of all-even } n\text{-tuples}),$$

and dividing by 2^d gives the desired result.

References

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.

http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10 .