University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 4

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due date: **Wednesday, 31 October 2018** at the beginning of class, or before that by email or canvas.
Please solve **at most 4 of the 7 exercises**!

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Let i and j be two elements of $[n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$. Let Q be the set of all $k \in \{i+1, i+2, \ldots, j-1\}$ satisfying $\sigma(i) > \sigma(k) > \sigma(j)$. Prove that

$$\ell(\sigma \circ t_{i,j}) = \ell(\sigma) - 2|Q| - 1.$$

1.2 Remark

This exercise implies that, in particular, $\ell(\sigma \circ t_{i,j}) < \ell(\sigma)$; this answers the question on page 213 of the notes from class (2018-10-22).

1.3 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $\pi \in S_n$.

(a) Prove that

$$\sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) > \pi(j)}} (\pi(j) - \pi(i)) = \sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) > \pi(j)}} (i - j).$$

(b) Prove that

$$\sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) < \pi(j)}} (\pi(j) - \pi(i)) = \sum_{\substack{1 \leq i < j \leq n; \\ \pi(i) < \pi(j)}} (j - i).$$

[**Hint:** Exercise 5.23 in [Grinbe16] says something about sums of the form appearing in part **(a)**. (See also Nathaniel Gorski's solution of the same exercise in Spring 2018 Math 4707 homework set #4.) You may want to use the result or the ideas.]

2.2 Solution

[...]

3 Exercise 3

3.1 Problem

Let n be a positive integer. For each $p \in \mathbb{Z}$, we let

$$D_{n,p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents}\}.$$

(Recall that a *descent* of a permutation $\sigma \in S_n$ denotes an element $k \in [n-1]$ satisfying $\sigma(k) > \sigma(k+1)$.)

Let $p \in \mathbb{Z}$. Prove that $|D_{n,p}| = |D_{n,n-1-p}|$.

3.2 Solution

[...]

4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Let $S = \{s_1 < s_2 < \cdots < s_k\}$ be a subset of $[n-1]$. Set $s_0 = 0$ and $s_{k+1} = n$. For each $i \in [k+1]$, set $d_i = s_i - s_{i-1}$. (You might remember this construction from the definition of the map D in the solution to Exercise 1 on homework set #0.)

(a) Prove that

$$\left|\{\sigma \in S_n \mid \operatorname{Des}\sigma \subseteq S\}\right| = \binom{n}{d_1, d_2, \ldots, d_{k+1}}.$$

(The term on the right hand side is a multinomial coefficient. The $\operatorname{Des}\sigma$ on the left hand side denotes the descent set of σ, that is, the set of all descents of σ.)

(b) Prove that

$$\left|\{\sigma \in S_n \mid \operatorname{Des}\sigma = S\}\right| = \sum_{T \subseteq S} (-1)^{|S|-|T|} \left|\{\sigma \in S_n \mid \operatorname{Des}\sigma \subseteq T\}\right|.$$

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. We shall follow the convention that $t_{i,i}$ denotes the identity permutation $\operatorname{id} \in S_n$ for each $i \in [n]$.

Let $\sigma \in S_n$.

It is known that there is a unique n-tuple $(i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n]$ satisfying $\sigma = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n}$. (See [Grinbe16, Exercise 5.9] for the proof of this fact, or – easier – do it on your own.) Consider this n-tuple. (It is sometimes called the *transposition code* of σ.)

For each $k \in \{0, 1, \ldots, n\}$, we define a permutation $\sigma_k \in S_n$ by $\sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k}$. Note that this permutation σ_k leaves each of the numbers $k+1, k+2, \ldots, n$ unchanged (since all of $i_1, i_2, \ldots, i_k$, as well as $1, 2, \ldots, k$, are $\leq k$).

For each $k \in [n]$, let $m_k = \sigma_k(k)$.

(a) Show that $m_k \in [k]$ for all $k \in [n]$.

(b) Show that $\sigma_k(i_k) = k$ for all $k \in [n]$.

(c) Show that $\sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}$.

(d) Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be any $2n$ numbers. Prove that

$$\sum_{k=1}^n x_k y_k - \sum_{k=1}^n x_k y_{\sigma(k)} = \sum_{k=1}^n (x_{i_k} - x_k)(y_{m_k} - y_k).$$

(e) Now assume that the numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real and satisfy $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. Conclude that

$$\sum_{k=1}^n x_k y_k \geq \sum_{k=1}^n x_k y_{\sigma(k)}.$$

5.2 Remark

Parts **(a)** and **(c)**, combined, show that $(m_1, m_2, \ldots, m_n)$ is the transposition code of σ^{-1}.

Part **(e)** of the exercise is known as the *rearrangement inequality.* The proof in this exercise is far from its easiest proof, but has the advantage of "manifest positivity" – i.e., it gives an explicit formula for the difference between the two sides as a sum of products of nonnegative numbers.

5.3 Solution

[...]

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6 Exercise 6

6.1 Problem

Prove the following:

(a) If $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are such that $m < n$, then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = 0.$$

(b) If $n \in \mathbb{N}$ and $r \in [n-1]$, then

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 0.$$

6.2 Solution

[...]

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7 Exercise 7

7.1 Problem

Let $n \in \mathbb{N}$ and $d \in \mathbb{N}$. An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ is said to be *all-even* if each element of $[d]$ occurs an even number of times in this n-tuple (i.e., if for each $k \in [d]$, the number of all $i \in [n]$ satisfying $x_i = k$ is even). For example, the 4-tuple $(1, 4, 4, 1)$ and the 6-tuples $(1, 3, 3, 5, 1, 5)$ and $(2, 4, 2, 4, 3, 3)$ are all-even, while the 4-tuples $(1, 2, 2, 4)$ and $(2, 4, 6, 4)$ are not.

Prove that the number of all all-even n-tuples $(x_1, x_2, \ldots, x_n) \in [d]^n$ is

$$\frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d-2k)^n .$$

[**Hint:** Compute the sum $\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (e_1+e_2+\cdots+e_d)^n$ in two ways. One way is to split it according to the number of $i \in [d]$ satisfying $e_i = -1$; this is a number $k \in \{0,1,\ldots,d\}$. Another way is by using the product rule:

$$(e_1+e_2+\cdots+e_d)^n = \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} e_{x_1}e_{x_2}\cdots e_{x_n}$$

and then simplifying each sum $\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n}$ using a form of destructive interference. This is not unlike the number of 1-even n-tuples, which we computed at the end of the 2018-10-10 class.]

7.2 Solution

[...]

References

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .