University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 3

Darij Grinberg

October 15, 2018

due date: **Wednesday, 10 October 2018** at the beginning of class, or before that by email or canvas.
Please solve **at most 4 of the 6 exercises**!

---

1 Exercise 1

1.1 Problem

Let A and B be two sets, and let $f : A \to B$ be a map. A *left inverse* of f shall mean a map $g : B \to A$ such that $g \circ f = \operatorname{id}_A$. We say that f is *left-invertible* if and only if a left inverse of f exists. (It is usually not unique.)

Assume that the sets A and B are finite.

(a) If the set A is nonempty, then prove that f is left-invertible if and only if f is injective.[1]

(b) Assume that f is injective. Prove that the number of left inverses of f is $|A|^{|B|-|A|}$.

1.2 Solution

[...]

---

[1] This holds even when A and B are infinite. Feel free to prove this if you wish.

2 Exercise 2

2.1 Problem

Let A and B be two sets, and let $f : A \to B$ be a map. A *right inverse* of f shall mean a map $h : B \to A$ such that $f \circ h = \operatorname{id}_B$. We say that f is *right-invertible* if and only if a right inverse of f exists. (It is usually not unique.)

Assume that the sets A and B are finite.

(a) Prove that f is right-invertible if and only if f is surjective.[2]

(b) Prove that the number of right inverses of f is $\prod_{b \in B} \left|f^{-1}(b)\right|$. Here, $f^{-1}(b)$ denotes the **set** of all $a \in A$ satisfying $f(a) = b$.

2.2 Solution

[...]

---

3 Exercise 3

3.1 Problem

(a) Prove that

$$\binom{-1/2}{n} = \left(\frac{-1}{4}\right)^n \binom{2n}{n} \qquad \text{for each } n \in \mathbb{N}.$$

(b) Prove that

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n \qquad \text{for each } n \in \mathbb{N}.$$

[**Hint:** Part **(b)** is highly difficult to prove combinatorially. Try using part **(a)** instead.]

3.2 Solution

[...]

---

[2] This holds even when A and B are infinite, if you assume the axiom of choice. But this is not the subject of our class.

4 Exercise 4

4.1 Problem

Recall once again the *Fibonacci sequence* $(f_0, f_1, f_2, \ldots)$, which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 2. \tag{1}$$

It is easy to see that $f_1, f_2, f_3, \ldots$ are positive integers (which will allow us to divide by them soon).

For any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, define the rational number $\dbinom{n}{k}_F$ (a slight variation on the corresponding binomial coefficient) by

$$\binom{n}{k}_F = \begin{cases} \dfrac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k f_{k-1} \cdots f_1}, & \text{if } n \geq k \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Let n be a positive integer, and let $k \in \mathbb{N}$ be such that $n \geq k$. Prove that

$$\binom{n}{k}_F = f_{k+1} \binom{n-1}{k}_F + f_{n-k-1} \binom{n-1}{k-1}_F,$$

where we set $f_{-1} = 1$.

(b) Prove that $\dbinom{n}{k}_F \in \mathbb{N}$ for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

4.2 Solution

[...]

---

5 Exercise 5

5.1 Problem

Let $j \in \mathbb{N}$, $r \in \mathbb{R}$ and $s \in \mathbb{R}$. Prove that

$$\sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{r - sk}{j} = s^j.$$

[**Hint:** First, argue that it suffices to prove this only for $s \in \mathbb{N}$ and $r \in \mathbb{Z}$ satisfying $r \geq sj$. Next, consider r distinct stones, sj of which are arranged in j piles containing s stones each, while the remaining $r - sj$ stones are forming a separate heap. How many ways are there to pick j of these r stones such that each of the j piles loses at least one stone?]

5.2 Solution

[...]

---

6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. The summation sign $\sum\limits_{I \subseteq [n]}$ shall always stand for a sum over all subsets I of $[n]$. (This sum has 2^n addends.)

Let $A_1, A_2, \ldots, A_n$ be n numbers or polynomials or square matrices of the same size. (Allowing matrices means that $A_i A_j$ is not necessarily equal to $A_j A_i$, so beware of using the binomial formula or similar identities!)

(a) Show that

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \left(\sum_{i \in I} A_i \right)^m = \sum_{\substack{(i_1, i_2, \ldots, i_m) \in [n]^m; \\ \{i_1, i_2, \ldots, i_m\} = [n]}} A_{i_1} A_{i_2} \cdots A_{i_m} \qquad \text{for all } m \in \mathbb{N}.$$

(Example: If $n = 2$ and $m = 3$, then this is saying

$$(A+B)^3 - A^3 - B^3 + 0^3 = AAB + ABA + ABB + BAA + BAB + BBA,$$

where we have renamed A_1 and A_2 as A and B.)

(b) Show that

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \left(\sum_{i \in I} A_i \right)^m = 0 \qquad \text{for all } m \in \mathbb{N} \text{ satisfying } m < n.$$

(Example: If $n = 3$ and $m = 2$, then this is saying

$$(A+B+C)^2 - (A+B)^2 - (A+C)^2 - (B+C)^2 + A^2 + B^2 + C^2 - 0^2 = 0,$$

where we have renamed A_1, A_2, A_3 as A, B, C.)

(c) Show that

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \left(\sum_{i \in I} A_i \right)^n = \sum_{\sigma \in S_n} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)},$$

where S_n stands for the set of all ($n!$) permutations of $[n]$.

(Example: If $n = 3$, then this is saying

$$\begin{aligned} &(A+B+C)^3 - (A+B)^3 - (A+C)^3 - (B+C)^3 + A^3 + B^3 + C^3 - 0^3 \\ &= ABC + ACB + BAC + BCA + CAB + CBA, \end{aligned}$$

where we have renamed A_1, A_2, A_3 as A, B, C.)

[**Hint:** You can use the *product rule*, which says the following:

Proposition 6.1 (Product rule)**.** *Let m and n be two nonnegative integers. Let $P_{u,v}$, for all $u \in [m]$ and $v \in [n]$, be numbers or polynomials or square matrices of the same size. Then,*

$$\left(P_{1,1} + P_{1,2} + \cdots + P_{1,n}\right)\left(P_{2,1} + P_{2,2} + \cdots + P_{2,n}\right) \cdots \left(P_{m,1} + P_{m,2} + \cdots + P_{m,n}\right) = \sum_{(i_1,i_2,\ldots,i_m)\in[n]^m} P_{1,i_1} P_{2,i_2} \cdots P_{m,i_m}.$$

(This frightening formula merely says that a product of sums can be expanded, and the result will be a sum of products, with each of the latter products being obtained by multiplying together one addend from each sum. You have probably used this sometime already.)

]

6.2 Solution

[...]