University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 2

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due date: **Wednesday, 26 September 2018** at the beginning of class,
or before that by email or canvas.
Please solve **at most 3 of the 5 exercises**!

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1 Exercise 1

1.1 Problem

For any nonnegative integers a and b and any real x, prove that

$$\binom{x}{a}\binom{x}{b} = \sum_{r=\max\{a,b\}}^{a+b} \binom{a}{a+b-r}\binom{r}{a}\binom{x}{r}. \tag{1}$$

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{k-2i} 2^{k-2i} = \binom{2n}{k}. \tag{2}$$

[**Hint:** You have n pairs of shoes $(L_1, R_1), (L_2, R_2), \ldots, (L_n, R_n)$, where the $2n$ shoes $L_1, R_1, L_2, R_2, \ldots, L_n, R_n$ are all distinguishable. You grab k of these $2n$ shoes at random (i.e., pick a k-element subset of the set of all $2n$ shoes). For a given $i \in \{0, 1, \ldots, n\}$, what is the probability that among your k shoes are exactly i pairs?]

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. For each $i \in \{0, 1, 2\}$, we let $g_{n,i}$ denote the number of all subsets S of $[n]$ satisfying $|S| \equiv i \mod 3$.

(a) Show that if $n > 0$, then

$$g_{n,0} = g_{n-1,0} + g_{n-1,2}; \qquad g_{n,1} = g_{n-1,1} + g_{n-1,0}; \qquad g_{n,2} = g_{n-1,2} + g_{n-1,1}.$$

(b) Find closed-form expressions (with no summation signs) for $g_{n,0}, g_{n,1}, g_{n,2}$ depending on the remainder of n upon division by 3.

3.2 Remark

Remark 3.1. The combinatorial interpretation of binomial coefficients shows that

$$g_{n,i} = \sum_{\substack{k \in \mathbb{Z}; \\ k \equiv i \mod 3}} \binom{n}{k} \qquad \text{for each } i.$$

This is not what the problem is asking for – find formulas with no summation signs.

3.3 Solution

[...]

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$ be positive. Let $m \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}. \qquad (3)$$

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. If $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ and $k \in [n]$, then

- we say that k is a *1-position* of $\mathbf{i}$ if $i_k = 1$;
- we say that k is a *10-position* of $\mathbf{i}$ if $k < n$, $i_k = 1$ and $i_{k+1} = 0$.

For example, the 7-tuple $(0,1,1,0,1,0,1)$ has 1-positions $2,3,5,7$ and 10-positions $3,5$.

It is easy to see that for each $k \in \mathbb{Z}$, the number of all n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly k 1-positions is $\binom{n}{k}$. (In fact, these n-tuples are in bijection with the k-element subsets of $[n]$.) In this problem, we shall count the n-tuples having exactly k 10-positions.

We use the notation $a\%b$ for the remainder of an integer a upon division by a positive integer b. For example, $5\%3 = 2$. Also, $\lfloor x \rfloor$ denotes the integer part (i.e., floor) of a real number x (that is, the largest integer that is smaller or equal to x).

Let $A : \{0,1\}^n \to \{0,1\}^n$ be the map that sends any n-tuple $(i_1, i_2, \ldots, i_n)$ to the n-tuple $(j_1, j_2, \ldots, j_n)$, where

$$j_k = (i_1 + i_2 + \cdots + i_k) \%2 \qquad \text{for all } k.$$

For example, $A((0,1,1,0,0,1,0)) = (0,1,0,0,0,1,1)$.

Prove the following:

(a) The map A is bijective.

(b) If the number of 1-positions of some n-tuple $\mathbf{i} \in \{0,1\}^n$ is p, then the number of 10-positions of the n-tuple $A(\mathbf{i})$ is $\lfloor p/2 \rfloor$.

(c) Let $k \in \mathbb{Z}$. Then, the number of n-tuples $\mathbf{i} \in \{0,1\}^n$ having exactly k 10-positions is $\binom{n+1}{2k+1}$.

5.2 Solution

[...]