University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 1

Darij Grinberg

October 6, 2019

1 Exercise 1

1.1 Problem

Let n be a positive integer.

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0,1,2,3\}^n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(2,3,1,2)$ is even, whereas $(1,2,3,1)$ is not.)

Compute the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0,1,2,3\}^n$.

(Here and in all future exercises, all answers need to be proven.)

[**Hint:** Compare with Exercise 3 on Homework set #0.]

1.2 Solution sketch

The number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0,1,2,3\}^n$ is 2^{2n-1}.

There are several possible proofs of this fact. Here are two:

First proof (outline).

We proceed similarly to the solution of Exercise 3 on homework set #0:

Let E_n be the set of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0,1,2,3\}^n$. Then, we must show that $|E_n| = 2^{2n-1}$.

We shall achieve this by finding a bijection from E_n to $\{0,1,2,3\}^{n-1} \times \{0,1\}$.

If a is an integer and b is a positive integer, then $a \mathbin{/\!/} b$ shall denote the quotient obtained when dividing a by b (with remainder), whereas $a\%b$ shall denote the remainder. For example, $7 \mathbin{/\!/} 3 = 2$ and $7\%3 = 1$. For any integer a and positive integer b, we have $a = (a \mathbin{/\!/} b)\, b + (a\%b)$.

Note that if a is a nonnegative integer, then $a\%2$ is the last bit[1] of the binary representation of a, whereas $a \mathbin{/\!/} 2$ is the number obtained by throwing away this last bit.

If $a \in \{0,1,2,3\}$, then the binary representation of a consists of at most two bits. If we write it as a two-bit string (possibly with a leading zero), then its first bit is $a \mathbin{/\!/} 2$, whereas the last bit is $a\%2$. This should be kept in mind when reading what follows.

Define a map $A : E_n \to \{0,1,2,3\}^{n-1} \times \{0,1\}$ by setting

$$A\left(\left(i_1, i_2, \ldots, i_n\right)\right) = \left(\left(i_1, i_2, \ldots, i_{n-1}\right), i_n \mathbin{/\!/} 2\right) \qquad \text{for each } \left(i_1, i_2, \ldots, i_n\right) \in E_n.$$

Thus, the map A essentially throws away the last bit of the last entry of the even n-tuple.

It is not hard to check that A is injective and surjective. (For example, to prove the surjectivity of A, we just need to check that for any $\left(\left(j_1, j_2, \ldots, j_{n-1}\right), k\right) \in \{0,1,2,3\}^{n-1} \times \{0,1\}$, there exists some even n-tuple $\left(i_1, i_2, \ldots, i_n\right) \in E_n$ such that $\left(\left(i_1, i_2, \ldots, i_{n-1}\right), i_n \mathbin{/\!/} 2\right) = \left(\left(j_1, j_2, \ldots, j_{n-1}\right), k\right)$. This is easily done: Set $i_p = j_p$ for all $p \in [n-1]$, and set $i_n = 2k + \begin{cases} 0, & \text{if } i_1 + i_2 + \cdots + i_{n-1} \text{ is even;} \\ 1, & \text{if } i_1 + i_2 + \cdots + i_{n-1} \text{ is odd.} \end{cases}$)

In other words, $A : E_n \to \{0,1,2,3\}^{n-1} \times \{0,1\}$ is a bijection. Hence,

$$\left|E_n\right| = \left|\{0,1,2,3\}^{n-1} \times \{0,1\}\right| = 4^{n-1} \cdot 2 = 2^{2n-1}.$$

This solves the exercise.

Second proof (outline).

Exercise 3 on homework set #0 says that the number of even n-tuples $\left(i_1, i_2, \ldots, i_n\right) \in \{0,1\}^n$ is 2^{n-1}. Applying this to $2n$ instead of n, we conclude that the number of even $2n$-tuples $\left(i_1, i_2, \ldots, i_{2n}\right) \in \{0,1\}^{2n}$ is 2^{2n-1}. In other words,

$$\left|\left\{\text{even } 2n\text{-tuples } \left(i_1, i_2, \ldots, i_{2n}\right) \in \{0,1\}^{2n}\right\}\right| = 2^{2n-1}. \tag{1}$$

Consider the map $D : \{0,1\}^2 \to \{0,1,2,3\}$ given by

$$D(0,0) = 0, \qquad D(0,1) = 1, \qquad D(1,0) = 3, \qquad D(1,1) = 2.$$

(Strictly speaking, we should be writing $D((0,0))$ instead of $D(0,0)$, and so on. But that's a common shorthand.)

The map D is a bijection and has the property that

$$D(a,b) \equiv a + b \mod 2 \qquad \text{for all } (a,b) \in \{0,1\}^2. \tag{2}$$

Hence, the map

$$\left\{\text{even } 2n\text{-tuples } \left(i_1, i_2, \ldots, i_{2n}\right) \in \{0,1\}^{2n}\right\} \to \left\{\text{even } n\text{-tuples } \left(i_1, i_2, \ldots, i_n\right) \in \{0,1,2,3\}^n\right\},$$
$$\left(a_1, b_1, a_2, b_2, \ldots, a_n, b_n\right) \mapsto \left(D\left(a_1, b_1\right), D\left(a_2, b_2\right), \ldots, D\left(a_n, b_n\right)\right)$$

is well-defined. It is not hard to see that this map is a bijection (indeed, its inverse sends each even n-tuple $\left(i_1, i_2, \ldots, i_n\right) \in \{0,1,2,3\}^n$ to $\left(a_1, b_1, a_2, b_2, \ldots, a_n, b_n\right) \in \{0,1\}^{2n}$, where the a_k and b_k are defined by letting $\left(a_k, b_k\right) = D^{-1}\left(i_k\right)$ for each $k \in [n]$). Thus,

$$\left|\left\{\text{even } n\text{-tuples } \left(i_1, i_2, \ldots, i_n\right) \in \{0,1,2,3\}^n\right\}\right| = \left|\left\{\text{even } 2n\text{-tuples } \left(i_1, i_2, \ldots, i_{2n}\right) \in \{0,1\}^{2n}\right\}\right| = 2^{2n-1}$$

(by (1)). This solves the exercise again.

---

[1] A *bit* means an element of $\{0,1\}$. So the "digits" in the binary representation of an integer are bits.

2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$.

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0, 1, 2\}^n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(2, 1, 1, 2)$ is even, whereas $(1, 2, 2, 2)$ is not.)

Let e_n be the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, 2\}^n$.

Prove that $e_n = \dfrac{3^n + 1}{2}$.

2.2 Solution sketch

We proceed by induction on n:

Induction base: There is only one 0-tuple $(i_1, i_2, \ldots, i_0) \in \{0, 1, 2\}^0$, namely the empty list $()$. This empty list is even (since the sum $i_1 + i_2 + \cdots + i_0 = (\text{empty sum}) = 0$ is even). Thus, $e_0 = 1$. Comparing this with $\dfrac{3^0 + 1}{2} = \dfrac{1 + 1}{2} = 1$, we conclude that $e_0 = \dfrac{3^0 + 1}{2}$. Hence, the exercise is solved for $n = 0$. This completes the induction base.

Induction step: Let N be a positive integer. Assume that the exercise holds for $n = N - 1$. We must prove that the exercise holds for $n = N$.

In the following, the word "k-tuple" (for k being a nonnegative integer) shall always mean "k-tuple in $\{0, 1, 2\}^k$". Thus, for each $n \in \mathbb{N}$, the number e_n is simply the number of all even n-tuples.

We have assumed that the exercise holds for $n = N - 1$. In other words, $e_{N-1} = \dfrac{3^{N-1} + 1}{2}$.

Recall that an $(N - 1)$-tuple $(i_1, i_2, \ldots, i_{N-1}) \in \{0, 1, 2\}^{N-1}$ is even if and only if the sum $i_1 + i_2 + \cdots + i_{N-1}$ is even. Let us introduce the natural counterpart to this notion: An $(N - 1)$-tuple $(i_1, i_2, \ldots, i_{N-1}) \in \{0, 1, 2\}^{N-1}$ is said to be *odd* if the sum $i_1 + i_2 + \cdots + i_{N-1}$ is odd.

Thus, each $(N - 1)$-tuple is either even or odd, but not both at the same time. Hence,

$$\begin{aligned}
&(\text{the number of all } (N-1)\text{-tuples}) \\
&= \underbrace{(\text{the number of all even } (N-1)\text{-tuples})}_{\substack{=e_{N-1} \\ (\text{by the definition of } e_{N-1})}} + (\text{the number of all odd } (N-1)\text{-tuples}) \\
&= e_{N-1} + (\text{the number of all odd } (N-1)\text{-tuples}).
\end{aligned}$$

Thus,

$$\begin{aligned}
(\text{the number of all odd } (N-1)\text{-tuples}) &= \underbrace{(\text{the number of all } (N-1)\text{-tuples})}_{=3^{N-1}} - e_{N-1} \\
&= 3^{N-1} - e_{N-1}.
\end{aligned}$$

Now, we want to count the even N-tuples $(i_1, i_2, \ldots, i_N)$. Let us first count those among these N-tuples whose last entry i_N is 0; then, those whose last entry i_N is 1; then, those whose last entry i_N is 2:

- There is a bijection

$$\begin{aligned}
\{\text{even } (N-1)\text{-tuples}\} &\to \{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 0\}, \\
(i_1, i_2, \ldots, i_{N-1}) &\mapsto (i_1, i_2, \ldots, i_{N-1}, 0).
\end{aligned}$$

[2] Hence,

$$|\{\text{even } (N-1)\text{-tuples}\}| = |\{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 0\}|.$$

In other words,

$$\begin{aligned}&(\text{the number of all even } (N-1)\text{-tuples})\\&= (\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 0).\end{aligned}$$

Hence,

$$\begin{aligned}&(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 0)\\&= (\text{the number of all even } (N-1)\text{-tuples})\\&= e_{N-1}. \qquad (3)\end{aligned}$$

- There is a bijection

$$\begin{aligned}\{\text{odd } (N-1)\text{-tuples}\} &\to \{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 1\},\\(i_1, i_2, \ldots, i_{N-1}) &\mapsto (i_1, i_2, \ldots, i_{N-1}, 1).\end{aligned}$$

[3] Hence,

$$|\{\text{odd } (N-1)\text{-tuples}\}| = |\{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 1\}|.$$

In other words,

$$\begin{aligned}&(\text{the number of all odd } (N-1)\text{-tuples})\\&= (\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 1).\end{aligned}$$

Hence,

$$\begin{aligned}&(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 1)\\&= (\text{the number of all odd } (N-1)\text{-tuples})\\&= 3^{N-1} - e_{N-1}. \qquad (4)\end{aligned}$$

- There is a bijection

$$\begin{aligned}\{\text{even } (N-1)\text{-tuples}\} &\to \{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 2\},\\(i_1, i_2, \ldots, i_{N-1}) &\mapsto (i_1, i_2, \ldots, i_{N-1}, 2).\end{aligned}$$

[4] Hence,

$$|\{\text{even } (N-1)\text{-tuples}\}| = |\{\text{even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 2\}|.$$

In other words,

$$\begin{aligned}&(\text{the number of all even } (N-1)\text{-tuples})\\&= (\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 2).\end{aligned}$$

Hence,

$$\begin{aligned}&(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 2)\\&= (\text{the number of all even } (N-1)\text{-tuples})\\&= e_{N-1}. \qquad (5)\end{aligned}$$

[2] We leave it to the reader to verify that this map is well-defined and is actually a bijection.
[3] We leave it to the reader to verify that this map is well-defined and is actually a bijection.
[4] We leave it to the reader to verify that this map is well-defined and is actually a bijection.

But e_N is the number of all even N-tuples. Thus,

$$\begin{aligned} e_N &= \left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N)\right) \\ &= \underbrace{\left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 0\right)}_{\substack{=e_{N-1} \\ \text{(by (3))}}} \\ &\quad + \underbrace{\left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 1\right)}_{\substack{=3^{N-1}-e_{N-1} \\ \text{(by (4))}}} \\ &\quad + \underbrace{\left(\text{the number of all even } N\text{-tuples } (i_1, i_2, \ldots, i_N) \text{ with } i_N = 2\right)}_{\substack{=e_{N-1} \\ \text{(by (5))}}} \\ &\qquad \left(\begin{array}{c} \text{since each } N\text{-tuple } (i_1, i_2, \ldots, i_N) \text{ satisfies exactly one} \\ \text{of the equations } i_N = 0 \text{ and } i_N = 1 \text{ and } i_N = 2 \end{array}\right) \\ &= e_{N-1} + \left(3^{N-1} - e_{N-1}\right) + e_{N-1} = 3^{N-1} + \underbrace{e_{N-1}}_{=\frac{3^{N-1}+1}{2}} \\ &= 3^{N-1} + \frac{3^{N-1}+1}{2} = \frac{3 \cdot 3^{N-1}+1}{2} = \frac{3^N+1}{2} \end{aligned}$$

(since $3 \cdot 3^{N-1} = 3^N$). In other words, the exercise holds for $n = N$. This completes the induction step. Hence, the exercise is solved.

---

3 Exercise 3

3.1 Problem

For any real number x and any $k \in \mathbb{N}$, we define the lower factorial $x^{\underline{k}}$ as in Exercise 2 of Homework set #0. (Thus, $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1) = \prod_{i=0}^{k-1}(x-i)$. This boils down to $x^{\underline{0}} = 1$ when $k = 0$, since empty products are defined to be 1.)

Let k, a and b be three positive integers such that $k \leq a \leq b$. Prove that

$$(k-1)\sum_{i=a}^{b}\frac{1}{i^{\underline{k}}} = \frac{1}{(a-1)^{\underline{k-1}}} - \frac{1}{b^{\underline{k-1}}}. \qquad (6)$$

3.2 Remark

Remark 3.1. This is similar to Exercise 2 of Homework set #0, but here the lower factorials are in the denominators. The analogous fact from calculus is

$$(k-1)\int_a^b \frac{1}{x^k}dx = \frac{1}{a^{k-1}} - \frac{1}{b^{k-1}}.$$

3.3 Solution sketch

First of all, it is easy to see that all the fractions occurring in the exercise are well-defined:

- For each $i \in \{a, a+1, \ldots, b\}$, the number $i^{\underline{k}} = i(i-1)\cdots(i-k+1)$ is a product of positive integers (since $i \geq a \geq k$ and thus $i-k+1 \geq k-k+1 = 1 > 0$), and thus nonzero; hence, $\dfrac{1}{i^{\underline{k}}}$ is well-defined.
- The number $(a-1)^{\underline{k-1}} = (a-1)(a-2)\cdots(a-k+1)$ is a product of positive integers (since $a \geq k$ and thus $a-k+1 \geq k-k+1 = 1 > 0$), and thus nonzero; hence, $\dfrac{1}{(a-1)^{\underline{k-1}}}$ is well-defined.
- The number $b^{\underline{k-1}} = b(b-1)\cdots(b-k+2)$ is a product of positive integers (since $b \geq a \geq k$ and thus $b-k+2 \geq k-k+2 = 2 > 0$), and thus nonzero; hence, $\dfrac{1}{b^{\underline{k-1}}}$ is well-defined.

Now, we can begin the actual solution. Recall the telescope principle (stated and proven in the solution to Exercise 2 of Homework set #0):

Proposition 3.2. *Let $m \in \mathbb{N}$. Let $a_0, a_1, \ldots, a_m$ be $m+1$ real numbers. Then,*

$$\sum_{i=1}^{m} (a_i - a_{i-1}) = a_m - a_0.$$

We can easily turn this formula around, obtaining a "reverse telescope principle":

Proposition 3.3. *Let $m \in \mathbb{N}$. Let $a_0, a_1, \ldots, a_m$ be $m+1$ real numbers. Then,*

$$\sum_{i=1}^{m} (a_{i-1} - a_i) = a_0 - a_m.$$

Proof of Proposition 3.3. We have

$$\begin{aligned}
\sum_{i=1}^{m} \underbrace{(a_{i-1} - a_i)}_{=(-a_i)-(-a_{i-1})} &= \sum_{i=1}^{m} ((-a_i) - (-a_{i-1})) = (-a_m) - (-a_0) \\
&\qquad \text{(by Proposition 3.2, applied to } -a_i \text{ instead of } a_i) \\
&= a_0 - a_m.
\end{aligned}$$

This proves Proposition 3.3. □

Next, we observe the following:

Lemma 3.4. *Let $k \in \mathbb{N}$. Let $i \geq k$ be a real number. Then,*

$$\frac{k-1}{i^{\underline{k}}} = \frac{1}{(i-1)^{\underline{k-1}}} - \frac{1}{i^{\underline{k-1}}}.$$

Proof of Lemma 3.4. We have $i \geq k$, thus $i-k \geq 0$ and therefore $i-k+1 > 0$.

The definition of $(i-1)^{\underline{k-1}}$ yields

$$\begin{aligned}
(i-1)^{\underline{k-1}} &= (i-1)((i-1)-1)((i-1)-2)\cdots((i-1)-(k-1)+1) \\
&= (i-1)(i-2)\cdots(i-k+1).
\end{aligned}$$

Hence, $(i-1)^{\underline{k-1}}$ is a product of positive integers (since $i-k+1 > 0$), and therefore nonzero. Thus, the fraction $\dfrac{1}{(i-1)^{\underline{k-1}}}$ is well-defined.

The definition of $i^{\underline{k-1}}$ yields

$$i^{\underline{k-1}} = i(i-1)\cdots(i-(k-1)+1) = i(i-1)\cdots(i-k+2).$$

Hence, $i^{\underline{k-1}}$ is a product of positive integers (since $i-k+2 > i-k+1 > 0$), and therefore nonzero. Thus, the fraction $\dfrac{1}{i^{\underline{k-1}}}$ is well-defined.

The definition of $i^{\underline{k}}$ yields

$$i^{\underline{k}} = i(i-1)\cdots(i-k+1).$$

Hence, $i^{\underline{k}}$ is a product of positive integers (since $i-k+1>0$), and therefore nonzero. Thus, the fraction $\dfrac{1}{i^{\underline{k}}}$ is well-defined.

We have

$$i^{\underline{k}} = i(i-1)\cdots(i-k+1) = i\cdot\underbrace{((i-1)(i-2)\cdots(i-k+1))}_{=(i-1)^{\underline{k-1}}} = i\cdot(i-1)^{\underline{k-1}}.$$

Hence,

$$\frac{1}{i^{\underline{k}}} = \frac{1}{i\cdot(i-1)^{\underline{k-1}}}.$$

Multiplying this equality by i, we obtain

$$\frac{i}{i^{\underline{k}}} = \frac{1}{(i-1)^{\underline{k-1}}}. \tag{7}$$

We have

$$i^{\underline{k}} = i(i-1)\cdots(i-k+1) = \underbrace{i(i-1)\cdots(i-k+2)}_{=i^{\underline{k-1}}}\cdot(i-k+1) = i^{\underline{k-1}}\cdot(i-k+1).$$

Hence,

$$\frac{1}{i^{\underline{k}}} = \frac{1}{i^{\underline{k-1}}\cdot(i-k+1)}.$$

Multiplying this equality by $i-k+1$, we obtain

$$\frac{i-k+1}{i^{\underline{k}}} = \frac{1}{i^{\underline{k-1}}}. \tag{8}$$

Subtracting this equality from (7), we obtain

$$\frac{i}{i^{\underline{k}}} - \frac{i-k+1}{i^{\underline{k}}} = \frac{1}{(i-1)^{\underline{k-1}}} - \frac{1}{i^{\underline{k-1}}}.$$

Hence,

$$\frac{1}{(i-1)^{\underline{k-1}}} - \frac{1}{i^{\underline{k-1}}} = \frac{i}{i^{\underline{k}}} - \frac{i-k+1}{i^{\underline{k}}} = \frac{i-(i-k+1)}{i^{\underline{k}}} = \frac{k-1}{i^{\underline{k}}}.$$

This proves Lemma 3.4. □

For each $i \in \{a, a+1, \ldots, b\}$, we have $i \geq a \geq k$. Thus, for each $i \in \{a, a+1, \ldots, b\}$, we have

$$\frac{k-1}{i^{\underline{k}}} = \frac{1}{(i-1)^{\underline{k-1}}} - \frac{1}{i^{\underline{k-1}}} \qquad \text{(by Lemma 3.4)}.$$

Summing these equalities over all $i \in \{a, a+1, \ldots, b\}$, we obtain

$$\begin{aligned}
\sum_{i=a}^{b} \frac{k-1}{i^{\underline{k}}} &= \sum_{i=a}^{b} \left(\frac{1}{(i-1)^{\underline{k-1}}} - \frac{1}{i^{\underline{k-1}}} \right) \\
&= \sum_{i=1}^{b-a+1} \left(\frac{1}{(i+a-1-1)^{\underline{k-1}}} - \frac{1}{(i+a-1)^{\underline{k-1}}} \right) \\
&\qquad \text{(here, we have substituted } i+a-1 \text{ for } i \text{ in the sum)} \\
&= \sum_{i=1}^{b-a+1} \left(\frac{1}{((i-1)+a-1)^{\underline{k-1}}} - \frac{1}{(i+a-1)^{\underline{k-1}}} \right) \\
&\qquad \text{(since } i+a-1-1 = (i-1)+a-1 \text{ for all } i\text{)} \\
&= \frac{1}{(0+a-1)^{\underline{k-1}}} - \frac{1}{((b-a+1)+a-1)^{\underline{k-1}}} \\
&\qquad \left(\text{by Proposition 3.3, applied to } m = b-a+1 \text{ and } a_i = \frac{1}{(i+a-1)^{\underline{k-1}}} \right) \\
&= \frac{1}{(a-1)^{\underline{k-1}}} - \frac{1}{b^{\underline{k-1}}} \\
&\qquad \text{(since } (b-a+1)+a-1 = b \text{ and } 0+a-1 = a-1\text{)}.
\end{aligned}$$

Hence,

$$\frac{1}{(a-1)^{\underline{k-1}}} - \frac{1}{b^{\underline{k-1}}} = \sum_{i=a}^{b} \frac{k-1}{i^{\underline{k}}} = (k-1) \sum_{i=a}^{b} \frac{1}{i^{\underline{k}}}.$$

This solves the exercise.

---

4 Exercise 4

4.1 Problem

Definition 4.1. The *Fibonacci sequence* is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 2. \tag{9}$$

Here is a table of some of its first terms:

n	0	1	2	3	4	5	6	7	8	9
f_n	0	1	1	2	3	5	8	13	21	34

Let $n \in \mathbb{N}$. Recall some definitions from class:

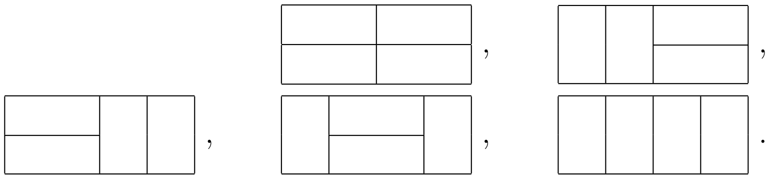
Let $R_{n,2}$ denote the set $[n] \times [2]$, which we regard as a rectangle of width n and height 2 (by identifying the squares with pairs of coordinates).

A *vertical domino* is a set of the form $\{(i,j), (i,j+1)\}$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

A *horizontal domino* is a set of the form $\{(i,j), (i+1,j)\}$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

A *domino tiling* of $R_{n,2}$ means a set of disjoint dominos (i.e., vertical dominos and horizontal dominos) whose union is $R_{n,2}$.

For example, there are 5 domino tilings of $R_{4,2}$, namely



Written as a set of dominos, the second of these tilings is

$$\{\{(1,1), (1,2)\}, \{(2,1), (2,2)\}, \{(3,1), (4,1)\}, \{(3,2), (4,2)\}\}.$$

We have seen in class (September 5) that

$$\text{the number of domino tilings of } R_{n,2} \text{ is } f_{n+1}. \tag{10}$$

We have also counted "axisymmetric" domino tilings.

Let us now define a different kind of symmetry: A domino tiling S of $R_{n,2}$ is said to be *centrosymmetric* if reflecting it across the center of the rectangle $R_{n,2}$ leaves it unchanged. (Formally, if S is regarded as a set, it means that for every domino $\{(i,j), (i',j')\} \in S$, its "opposite domino" $\{(n+1-i, 3-j), (n+1-i', 3-j')\}$ is also in S.) For example, among the 5 domino tilings of $R_{4,2}$ listed above, exactly 3 are centrosymmetric (namely, the first, the fourth and the fifth).

Let s_n be the number of centrosymmetric domino tilings of $R_{n,2}$.

(a) Prove that $s_n = f_{(n+1)/2}$ if n is odd.

(b) Prove that $s_n = f_{n/2+2}$ if n is even.

(Note that these are the same numbers as for axisymmetric domino tilings!)

[**Hint:** This is a bit of a trick problem.]

4.2 Solution sketch

Let us first recall the definition of axisymmetric domino tilings (as given in the September 5 lecture and in Exercise 5 of UMN Spring 2018 Math 4707 Exercise 5):

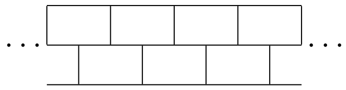
A domino tiling S of $R_{n,2}$ is said to be *axisymmetric* if reflecting it across the vertical axis of the rectangle $R_{n,2}$ leaves it unchanged. (Formally, if S is regarded as a set, it means that for every domino $\{(i,j), (i',j')\} \in S$, its "mirror domino" $\{(n+1-i, j), (n+1-i', j')\}$ is also in S.) For example, among the 5 domino tilings of $R_{4,2}$ listed in the exercise, exactly 3 are axisymmetric (namely, the first, the fourth and the fifth).

Now, the claim of our exercise is identical with the claim of UMN Spring 2018 Math 4707 Exercise 5, except that the word "axisymmetric" has been replaced by "centrosymmetric". This suggests that there might be a bijection between the axisymmetric domino tilings and the centrosymmetric domino tilings. It turns out that this is the case, and this bijection is as simple as one could hope for: It is the identity map! That is, the axisymmetric domino tilings are exactly the centrosymmetric domino tilings. We shall prove this below (in Claim 2).

First, we introduce one more concept: A domino tiling S of $R_{n,2}$ is said to be *horisymmetric* if reflecting it across the horizontal axis of the rectangle $R_{n,2}$ leaves it unchanged. (Formally, if S is regarded as a set, it means that for every domino $\{(i,j),(i',j')\} \in S$, its "up-down mirror domino" $\{(i,3-j),(i',3-j')\}$ is also in S.) Now, we claim the following:

Claim 1: Every domino tiling of $R_{n,2}$ is horisymmetric.

Note that this isn't completely obvious! For example, a rectangular strip of height 2 that is infinite in both directions has non-horisymmetric domino tilings:



Before we prove Claim 1, let us introduce some notations:

- We shall refer to the elements of $R_{n,2}$ as "squares", even though they are just pairs of integers.
- The words "domino tiling" will always mean "domino tiling of the rectangle $R_{n,2}$" (in this exercise).
- The words "horizontal axis" will always mean the horizontal axis of symmetry of the rectangle $R_{n,2}$. Likewise, the words "vertical axis" will always mean the vertical axis of symmetry of the rectangle $R_{n,2}$. Finally, the word "center" will always mean the center of the rectangle $R_{n,2}$.
- Let $\mathbf{H} : R_{n,2} \to R_{n,2}$ be the map that sends each square (i,j) to $(i,3-j)$. Visually speaking, $\mathbf{H}$ is just reflection in the horizontal axis.
- For each domino d, we let $\boxed{\mathbf{H}}(d)$ denote the domino $\{\mathbf{H}(c) \mid c \in d\}$. Visually speaking, $\boxed{\mathbf{H}}(d)$ is just the reflection of d in the horizontal axis.
- For each domino tiling D, we let $\boxed{\boxed{\mathbf{H}}}(D)$ denote the domino tiling $\left\{\boxed{\mathbf{H}}(d) \mid d \in D\right\}$. Visually speaking, $\boxed{\boxed{\mathbf{H}}}(D)$ is just the reflection of D in the horizontal axis.
- Let $\mathbf{V} : R_{n,2} \to R_{n,2}$ be the map that sends each square (i,j) to $(n+1-i,j)$. Visually speaking, $\mathbf{V}$ is just reflection in the vertical axis.
- For each domino d, we let $\boxed{\mathbf{V}}(d)$ denote the domino $\{\mathbf{V}(c) \mid c \in d\}$. Visually speaking, $\boxed{\mathbf{V}}(d)$ is just the reflection of d in the vertical axis.
- For each domino tiling D, we let $\boxed{\boxed{\mathbf{V}}}(D)$ denote the domino tiling $\left\{\boxed{\mathbf{V}}(d) \mid d \in D\right\}$. Visually speaking, $\boxed{\boxed{\mathbf{V}}}(D)$ is just the reflection of D in the vertical axis.
- Let $\mathbf{C} : R_{n,2} \to R_{n,2}$ be the map that sends each square (i,j) to $(n+1-i,3-j)$. Visually speaking, $\mathbf{C}$ is just reflection across the center.
- For each domino d, we let $\boxed{\mathbf{C}}(d)$ denote the domino $\{\mathbf{C}(c) \mid c \in d\}$. Visually speaking, $\boxed{\mathbf{C}}(d)$ is just the reflection of d across the center.

- For each domino tiling D, we let $\boxed{\boxed{\mathbf{C}}}(D)$ denote the domino tiling $\left\{\boxed{\mathbf{C}}(d) \mid d \in D\right\}$. Visually speaking, $\boxed{\boxed{\mathbf{C}}}(D)$ is just the reflection of D across the center.

Thus:

- A domino tiling D is horisymmetric if and only if $\boxed{\boxed{\mathbf{H}}}(D) = D$.
- A domino tiling D is axisymmetric if and only if $\boxed{\boxed{\mathbf{V}}}(D) = D$.
- A domino tiling D is centrosymmetric if and only if $\boxed{\boxed{\mathbf{C}}}(D) = D$.

Note that $\mathbf{C} = \mathbf{V} \circ \mathbf{H}$ (this is easy to check by hand[5]). Thus, every domino d satisfies $\boxed{\mathbf{C}}(d) = \boxed{\mathbf{V}}\left(\boxed{\mathbf{H}}(d)\right)$. Hence, every domino tiling D satisfies

$$\boxed{\boxed{\mathbf{C}}}(D) = \boxed{\boxed{\mathbf{V}}}\left(\boxed{\boxed{\mathbf{H}}}(D)\right). \tag{11}$$

We are now ready to prove Claim 1:

[*Proof of Claim 1:* Let D be a domino tiling. We must prove that D is horisymmetric.

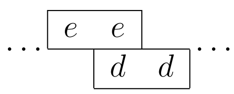
Indeed, assume the contrary. Thus, there exists some domino $d \in D$ such that $\boxed{\mathbf{H}}(d) \notin D$. We call such a domino *asymmetric*. Note that every asymmetric domino must be horizontal (since a vertical domino d always satisfies $\boxed{\mathbf{H}}(d) = d$).

If $d = \{(i, j), (i+1, j)\}$ is a horizontal domino, then we define the *rightness* of d to be the number i. (Thus, visually speaking, the further to the right a horizontal domino lies, the larger its rightness.)

We know that there exists an asymmetric domino $d \in D$. Each such domino is horizontal, and thus has a well-defined rightness. Consider an asymmetric domino $d \in D$ with **minimum rightness**. This domino d is asymmetric and therefore horizontal. Thus, we can write d in the form $d = \{(i, j), (i+1, j)\}$ for some i and j. Consider these i and j, and note that i is the rightness of d (by the definition of rightness). We have $\boxed{\mathbf{H}}(d) \notin D$ (since d is asymmetric). We have either $j = 1$ or $j = 2$ (in other words, the domino d lies either in the bottom half or in the top half of our rectangle $R_{n,2}$). We WLOG assume that $j = 1$, since the proof in the case $j = 2$ is analogous. Thus, $d = \{(i, 1), (i+1, 1)\}$, so that $\boxed{\mathbf{H}}(d) = \{(i, 2), (i+1, 2)\}$.

Now, the square $(i, 2)$ of $R_{n,2}$ must be covered by some domino $e \in D$ (since D is a domino tiling). Consider this e. If e was vertical, then e would contain $(i, 1)$ as well, which would contradict the fact that $(i, 1)$ is covered by the horizontal domino d. Hence, e must be horizontal. Thus, e is either $\{(i, 2), (i+1, 2)\}$ or $\{(i-1, 2), (i, 2)\}$.

But if e was $\{(i, 2), (i+1, 2)\}$, then we would have $e = \{(i, 2), (i+1, 2)\} = \boxed{\mathbf{H}}(d)$ and therefore $\boxed{\mathbf{H}}(d) = e \in D$, which would contradict $\boxed{\mathbf{H}}(d) \notin D$. Hence, e cannot be $\{(i, 2), (i+1, 2)\}$. Thus, e must be $\{(i-1, 2), (i, 2)\}$ (since e is either $\{(i, 2), (i+1, 2)\}$ or $\{(i-1, 2), (i, 2)\}$). We can thus visualize d and e as follows:



(where we are showing only columns $i-1, i, i+1$).

[5] Note the geometric meaning of this equality: It says that the reflection across the center is the composition of the reflection in the horizontal axis and the reflection in the vertical axis.

From $e = \{(i-1,2), (i,2)\}$, we obtain $\boxed{\mathbf{H}}(e) = \{(i-1,1), (i,1)\}$. Compare this with $d = \{(i,1), (i+1,1)\}$. Thus, the two dominos $\boxed{\mathbf{H}}(e)$ and d are distinct, but both contain the square $(i,1)$. Hence, the two dominos $\boxed{\mathbf{H}}(e)$ and d are distinct but not disjoint. Thus, $\boxed{\mathbf{H}}(e)$ and d cannot both belong to D (since D is a domino tiling). Since $d \in D$, we thus conclude that $\boxed{\mathbf{H}}(e) \notin D$.

Thus, $e \in D$ is a domino such that $\boxed{\mathbf{H}}(e) \notin D$. In other words, the domino e is asymmetric. Hence, the rightness of e must be $\geq$ to the rightness of d (since d was defined to be an asymmetric domino with minimum rightness). But this contradicts the fact that the rightness of e is $<$ to the rightness of d (indeed, the rightness of e is $i-1$, while the rightness of d is i). This contradiction shows that our assumption was false. Hence, Claim 1 is proven.]

Next, we claim the following:

> *Claim 2:* The axisymmetric domino tilings of $R_{n,2}$ are exactly the centrosymmetric domino tilings.

[*Proof of Claim 2:* Let D be a domino tiling. We need to show that D is axisymmetric if and only if D is centrosymmetric.

Claim 1 shows that D is horisymmetric. In other words, $\boxed{\boxed{\mathbf{H}}}(D) = D$.

Recall that D is axisymmetric if and only if $\boxed{\boxed{\mathbf{V}}}(D) = D$. Thus, we have the following chain of equivalences:

$$\begin{aligned}
(D \text{ is axisymmetric}) &\iff \left(\boxed{\boxed{\mathbf{V}}}(D) = D\right) \\
&\iff \left(\boxed{\boxed{\mathbf{V}}}\left(\boxed{\boxed{\mathbf{H}}}(D)\right) = D\right) \qquad \left(\text{since } D = \boxed{\boxed{\mathbf{H}}}(D)\right) \\
&\iff \left(\boxed{\boxed{\mathbf{C}}}(D) = D\right) \\
&\qquad \left(\text{since (11) yields } \boxed{\boxed{\mathbf{V}}}\left(\boxed{\boxed{\mathbf{H}}}(D)\right) = \boxed{\boxed{\mathbf{C}}}(D)\right) \\
&\iff (D \text{ is centrosymmetric})
\end{aligned}$$

(since D is centrosymmetric if and only if $\boxed{\boxed{\mathbf{C}}}(D) = D$). In other words, D is axisymmetric if and only if D is centrosymmetric. This proves Claim 2.]

It is now easy to complete the solution: We have defined s_n as the number of centrosymmetric domino tilings of $R_{n,2}$. Because of Claim 2, this shows that s_n is the number of axisymmetric domino tilings of $R_{n,2}$. Thus, our s_n is precisely the same number as the s_n from UMN Spring 2018 Math 4707 Exercise 5. Hence, the two claims of our exercise follow from the two claims of the latter exercise.

---

5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Let $S_{n,2}$ be the set

$$([n+1] \times [2]) \setminus \{(1,2), (n+1,1)\}.$$

For example, here is how $S_{6,2}$ looks like:

Find the number of domino tilings of $S_{n,2}$.

5.2 Solution sketch

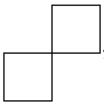
Let s_n denote the number of domino tilings of $S_{n,2}$. Then, we claim that

$$s_n = [n \text{ is even}] . \tag{12}$$

(Here, we are using the Iverson bracket notation; thus, $[n \text{ is even}]$ equals 1 if n is even, and 0 otherwise.)

It remains to prove (12).

To do so, we first notice that $s_0 = 1$ (indeed, the set $S_{0,2}$ is the empty set, and thus has exactly 1 domino tiling).

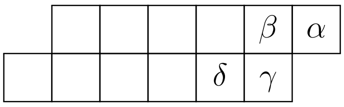
Furthermore, $s_1 = 0$ (indeed, the set $S_{1,2}$ has the form , and thus has 0 domino tilings).

Next, we shall show that

$$s_n = s_{n-2} \qquad \text{for every integer } n \geq 2. \tag{13}$$

[*Proof of* (13)*:* Let $n \geq 2$ be an integer.

We shall refer to the elements of $S_{n,2}$ as "squares", although they are just pairs of numbers. We denote the four squares $(n+1, 2)$, $(n, 2)$, $(n, 1)$ and $(n-1, 1)$ as α, β, γ and δ, respectively. All these four squares belong to $S_{n,2}$ (since $n \geq 2$). Let us show their locations on a picture (in the case $n = 6$ as an example):



Let D be a domino tiling of $S_{n,2}$. Then, the square α must belong to some domino in D. This domino must be a horizontal domino (since a vertical domino containing α would also contain the square $(n+1, 1)$, which is however not in $S_{n,2}$), and thus must be the domino $\{\alpha, \beta\}$ (since the other alternative would be a domino that contains the square $(n+2, 2)$, which is however not in $S_{n,2}$). Hence, the domino $\{\alpha, \beta\}$ must belong to D. Furthermore, the square γ must belong to some domino in D. This domino must be a horizontal domino (since a vertical domino containing γ would also contain the square β, which is however already contained in the horizontal domino $\{\alpha, \beta\}$), and thus must be the domino $\{\gamma, \delta\}$ (since the other alternative would be a domino that contains the square $(n+1, 1)$, which is however not in $S_{n,2}$). Hence, the domino $\{\gamma, \delta\}$ must belong to D. Thus, we have shown that the domino tiling D contains the two dominos $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$. Hence, $D \setminus \{\{\alpha, \beta\}, \{\gamma, \delta\}\}$ is a domino tiling of the set $S_{n,2} \setminus \{\alpha, \beta, \gamma, \delta\} = S_{n-2,2}$.

Now, forget that we fixed D. We thus have shown that if D is a domino tiling of $S_{n,2}$, then D contains the two dominos $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$, and the set $D \setminus \{\{\alpha, \beta\}, \{\gamma, \delta\}\}$ is a domino tiling of the set $S_{n-2,2}$. Hence, the map

$$\begin{aligned} \{\text{domino tilings of } S_{n,2}\} &\to \{\text{domino tilings of } S_{n-2,2}\} , \\ D &\mapsto D \setminus \{\{\alpha, \beta\}, \{\gamma, \delta\}\} \end{aligned}$$

is well-defined. This map is also invertible[6], and thus is a bijection. Hence,

$$|\{\text{domino tilings of } S_{n,2}\}| = |\{\text{domino tilings of } S_{n-2,2}\}| .$$

In view of

$$|\{\text{domino tilings of } S_{n,2}\}| = (\text{the number of domino tilings of } S_{n,2}) = s_n$$

(by the definition of s_n) and

$$|\{\text{domino tilings of } S_{n-2,2}\}| = s_{n-2}$$

(for similar reasons), this rewrites as $s_n = s_{n-2}$. This proves (13).]

[*Proof of* (12)*:* Using the equality (12) repeatedly, we see that

- each even $n \in \mathbb{N}$ satisfies $s_n = s_{n-2} = s_{n-4} = \cdots = s_2 = s_0 = 1$;
- each odd $n \in \mathbb{N}$ satisfies $s_n = s_{n-2} = s_{n-4} = \cdots = s_3 = s_1 = 0$.

Combining these two results, we obtain precisely (12).]

---

6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. If S is a finite nonempty set of integers, then $\max S$ denotes the maximum of S (that is, the largest element of S).

(a) Find the number of nonempty subsets S of $[n]$ satisfying $\max S = |S|$.

(b) Find the number of nonempty subsets S of $[n]$ satisfying $\max S = |S| + 1$.

6.2 Solution sketch

Before we solve this exercise, let us observe the following: Every integer $m \geq -1$ satisfies

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}. \qquad (14)$$

(Indeed, this formula is well-known in the case when $m \in \mathbb{N}$, and boils down to $0 = 0$ in the case when $m = -1$.)

(a) This number is n.

[*Proof.* We shall first prove the following claim:

[6] Indeed, its inverse is

$$\{\text{domino tilings of } S_{n-2,2}\} \to \{\text{domino tilings of } S_{n,2}\},$$
$$E \mapsto E \cup \{\{\alpha, \beta\}, \{\gamma, \delta\}\}.$$

> *Claim 1:* Let $i \in [n]$. Then, the number of nonempty i-element subsets S of $[n]$ satisfying $\max S = |S|$ is 1.

[*Proof of Claim 1:* If S is a nonempty i-element subset of $[n]$ satisfying $\max S = |S|$, then $\max S = |S| = i$ (since S is an i-element set), and therefore all elements of S are $\leq i$; but this entails that $S \subseteq [i]$, and therefore $S = [i]$ (since the only i-element subset of $[i]$ is $[i]$ itself). Thus, there exists **at most one** nonempty i-element subset S of $[n]$ satisfying $\max S = |S|$. But it is also clear that there exists **at least one** such subset (namely, $[i]$). Hence, there exists **exactly one** such subset. In other words, the number of such subsets is 1. This proves Claim 1.]

Clearly, if S is a nonempty subset of $[n]$, then $|S| \in [n]$. Hence,

$$\begin{aligned}
&(\text{the number of nonempty subsets } S \text{ of } [n] \text{ satisfying } \max S = |S|) \\
&= \sum_{i \in [n]} \underbrace{(\text{the number of nonempty subsets } S \text{ of } [n] \text{ satisfying } \max S = |S| \text{ and } |S| = i)}_{\substack{=(\text{the number of nonempty } i\text{-element subsets } S \text{ of } [n] \text{ satisfying } \max S = |S|)=1 \\ (\text{by Claim 1})}} \\
&= \sum_{i \in [n]} 1 = |[n]| \cdot 1 = |[n]| = n.
\end{aligned}$$

This concludes our proof.]

(b) This number is $n(n-1)/2$.

[*Proof.* We shall first prove the following claim:

> *Claim 2:* Let $i \in [n-1]$. Then, the number of nonempty i-element subsets S of $[n]$ satisfying $\max S = |S| + 1$ is i.

[*Proof of Claim 2:* Let S be a nonempty i-element subset of $[n]$ satisfying $\max S = |S|+1$. Then, $|S| = i$ (since S is an i-element set) and $\max S = |S| + 1 = i + 1$ (since $|S| = i$). Hence, all elements of S are $\leq i+1$; in other words, $S \subseteq [i+1]$. Thus, $|[i+1] \setminus S| = \underbrace{|[i+1]|}_{=i+1} - \underbrace{|S|}_{=i} = (i+1) - i = 1$. Hence, $[i+1] \setminus S$ is a 1-element set; in other words, $[i+1] \setminus S = \{j\}$ for some element j. Consider this j. We have $j \in [i+1] \setminus S$; in other words, $j \in [i+1]$ and $j \notin S$. We have $j \neq i+1$ (since $j \notin S$ but $i+1 = \max S \in S$). Combining this with $j \in [i+1]$, we obtain $j \in [i+1] \setminus \{i+1\} = [i]$. Also, from $S \subseteq [i+1]$, we obtain $S = [i+1] \setminus \underbrace{([i+1] \setminus S)}_{=\{j\}} = [i+1] \setminus \{j\}$.

Now, forget that we fixed S. We thus have shown that if S is a nonempty i-element subset of $[n]$ satisfying $\max S = |S| + 1$, then there exists some $j \in [i]$ such that $S = [i+1] \setminus \{j\}$. In other words, if S is a nonempty i-element subset of $[n]$ satisfying $\max S = |S| + 1$, then S must be one of the i sets $S = [i+1] \setminus \{j\}$ with $j \in [i]$. Hence, there exist **at most** i nonempty i-element subsets S of $[n]$ satisfying $\max S = |S| + 1$. But it is also clear that there exist **at least** i such subsets (namely, the i distinct subsets $[i+1] \setminus \{j\}$ with $j \in [i]$ [7]). Hence, there exist **exactly** i such subsets. In other words, the number of such subsets is i. This proves Claim 2.]

[7] You need to check that these i subsets are actually distinct and that they do fit the bill (i.e., that they are nonempty i-element subsets S of $[n]$ satisfying $\max S = |S| + 1$). This is straightforward, so I leave it to the reader.

Now, if S is a nonempty subset of $[n]$ satisfying $\max S = |S| + 1$, then $|S| \in [n-1]$ [8]. Hence,

$$\begin{aligned}
&(\text{the number of nonempty subsets } S \text{ of } [n] \text{ satisfying } \max S = |S| + 1) \\
&= \sum_{i \in [n-1]} \underbrace{(\text{the number of nonempty subsets } S \text{ of } [n] \text{ satisfying } \max S = |S| + 1 \text{ and } |S| = i)}_{\substack{=(\text{the number of nonempty } i\text{-element subsets } S \text{ of } [n] \text{ satisfying } \max S = |S|+1) = i \\ (\text{by Claim 2})}} \\
&= \sum_{i \in [n-1]} i = \sum_{i=1}^{n-1} i = \frac{(n-1)((n-1)+1)}{2} \qquad (\text{by (14)}) \\
&= n(n-1)/2.
\end{aligned}$$

This concludes our proof.]

6.3 Remark

Exercise 7 from Spring 2018 Math 4707 Homework set #1 looks similar to part **(a)** of this exercise, but uses the minimum instead of the maximum. The answers, however, are not similar at all.

---

7 Exercise 7

7.1 Problem

For any nonnegative integers a and b and any real x, prove that

$$x^{\underline{a}} x^{\underline{b}} = \sum_{r=\max\{a,b\}}^{a+b} \frac{a!b!}{(r-a)!(r-b)!(a+b-r)!} x^{\underline{r}}. \qquad (15)$$

7.2 Solution sketch

First, we prove an auxiliary fact:

Claim 1: Let $k \in \mathbb{N}$. Let x be any real. Then,

$$x^{\underline{k+1}} = x^{\underline{k}} (x-k) \qquad (16)$$

and

$$x x^{\underline{k}} = x^{\underline{k+1}} + k x^{\underline{k}}. \qquad (17)$$

[*Proof of Claim 1:* The definition of $x^{\underline{k}}$ yields $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1)$. But the definition of $x^{\underline{k+1}}$ yields

$$\begin{aligned}
x^{\underline{k+1}} &= x(x-1)(x-2)\cdots(x-(k+1)+1) \\
&= x(x-1)(x-2)\cdots(x-k) \\
&= \underbrace{(x(x-1)(x-2)\cdots(x-k+1))}_{=x^{\underline{k}}}(x-k) = x^{\underline{k}}(x-k).
\end{aligned}$$

[8] *Proof.* Let S be a nonempty subset of $[n]$ satisfying $\max S = |S| + 1$. Then, $|S| > 0$ (since S is nonempty). Also, $\max S \in S \subseteq [n]$, so that $\max S \leq n$ and thus $|S| + 1 = \max S \leq n$, so that $|S| \leq n - 1$. Combining this with $|S| > 0$, we obtain $|S| \in [n-1]$. Qed.

This proves (16).

Furthermore, (16) becomes

$$x^{\underline{k+1}} = x^{\underline{k}} (x-k) = (x-k) x^{\underline{k}} = x x^{\underline{k}} - k x^{\underline{k}}.$$

Solving this equation for $x x^{\underline{k}}$, we obtain $x x^{\underline{k}} = x^{\underline{k+1}} + k x^{\underline{k}}$. This proves (17). Thus, Claim 1 is proven.]

Next, let us show that the fraction on the right hand side of (15) is well-defined:

> *Claim 2:* Let $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ be such that $\max\{a,b\} \leq r \leq a+b$. Then, $(r-a)!\,(r-b)!\,(a+b-r)!$ is a well-defined nonzero integer.

[*Proof of Claim 2:* We have $a \leq \max\{a,b\} \leq r$, so that $r \geq a$ and thus $r-a \in \mathbb{N}$. Hence, $(r-a)!$ is a well-defined positive integer. Similarly, $(r-b)!$ is a well-defined positive integer. Also, $r \leq a+b$, so that $a+b-r \geq 0$ and thus $a+b-r \in \mathbb{N}$. Hence, $(a+b-r)!$ is a well-defined positive integer.

Now, we know that $(r-a)!$, $(r-b)!$ and $(a+b-r)!$ are well-defined positive integers. Hence, their product $(r-a)!\,(r-b)!\,(a+b-r)!$ is a well-defined positive integer as well – and therefore a well-defined nonzero integer. This proves Claim 2.]

We now introduce a notation that will shorten our computations:

For any $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfying $\max\{a,b\} \leq r \leq a+b$, we define a rational number $g_{a,b,r}$ by

$$g_{a,b,r} = \frac{a!b!}{(r-a)!\,(r-b)!\,(a+b-r)!}. \qquad (18)$$

This is well-defined, because Claim 2 shows that $(r-a)!\,(r-b)!\,(a+b-r)!$ is a well-defined nonzero integer.

For ease of access, we state a few simple properties of the numbers we have just defined:

> *Claim 3:*
>
> **(a)** If $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfy $\max\{a,b\} \leq r \leq a+b$, then $g_{a,b,r}$ is a positive integer.
>
> **(b)** Any $b \in \mathbb{N}$ satisfies $g_{0,b,b} = 1$.
>
> **(c)** If $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfy $a+1 \leq b$ and $b+1 \leq r \leq a+b+1$, then
>
> $$g_{a,b,r-1} = \frac{r-b}{a+1} g_{a+1,b,r}. \qquad (19)$$
>
> (In particular, both $g_{a,b,r-1}$ and $g_{a+1,b,r}$ are well-defined.)
>
> **(d)** If $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfy $a+1 \leq b$ and $b \leq r \leq a+b$, then
>
> $$(r-a)\, g_{a,b,r} = \frac{a+b+1-r}{a+1} g_{a+1,b,r}. \qquad (20)$$
>
> (In particular, both $g_{a,b,r}$ and $g_{a+1,b,r}$ are well-defined.)

(We won't actually need part **(a)** of this.)

[*Proof of Claim 3:* **(a)** Let $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfy $\max\{a,b\} \leq r \leq a+b$. We know that $g_{a,b,r}$ is well-defined. Also, (18) shows that $g_{a,b,r}$ is a quotient of products of factorials; hence, $g_{a,b,r}$ is positive (since factorials are positive).

We have $r \leq a+b$, thus $a+b \geq r$, and therefore $a \geq r-b$ and $a+b-r \geq 0$. The latter inequality yields $a+b-r \in \mathbb{N}$; thus, $(a+b-r)!$ is a well-defined positive integer. Also, $b \leq \max\{a,b\} \leq r$, so that $r-b \geq 0$ and thus $r-b \in \mathbb{N}$.

Recall that $\dbinom{n}{k} \in \mathbb{N}$ for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Applying this to $n=a$ and $k=r-b$, we obtain $\dbinom{a}{r-b} \in \mathbb{N}$ (since $a \in \mathbb{N}$ and $r-b \in \mathbb{N}$). In other words, $\dbinom{a}{r-b}$ is a nonnegative integer. The same argument (with the roles of a and b interchanged) shows that $\dbinom{b}{r-a}$ is a nonnegative integer.

Recall the classical formula which says that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \text{for any } n \in \mathbb{N} \text{ and } k \in \mathbb{N} \text{ satisfying } n \geq k. \tag{21}$$

Applying this formula to $n=a$ and $k=r-b$, we obtain

$$\binom{a}{r-b} = \frac{a!}{(r-b)!\,(a-(r-b))!} \qquad (\text{since } a \in \mathbb{N} \text{ and } r-b \in \mathbb{N} \text{ and } a \geq r-b)$$
$$= \frac{a!}{(r-b)!\,(a+b-r)!}.$$

The same argument (with the roles of a and b interchanged) shows that

$$\binom{b}{r-a} = \frac{b!}{(r-a)!\,(b+a-r)!} = \frac{b!}{(r-a)!\,(a+b-r)!}.$$

Multiplying the last two equalities, we obtain

$$\binom{a}{r-b}\binom{b}{r-a} = \frac{a!}{(r-b)!\,(a+b-r)!} \cdot \frac{b!}{(r-a)!\,(a+b-r)!}.$$

Multiplying both sides of this equality by $(a+b-r)!$, we obtain

$$(a+b-r)!\binom{a}{r-b}\binom{b}{r-a} = (a+b-r)! \cdot \frac{a!}{(r-b)!\,(a+b-r)!} \cdot \frac{b!}{(r-a)!\,(a+b-r)!}$$
$$= \frac{a!b!}{(r-a)!\,(r-b)!\,(a+b-r)!} = g_{a,b,r} \qquad (\text{by } (18)).$$

But the left hand side of this equality is an integer (since all of $(a+b-r)!$, $\dbinom{a}{r-b}$ and $\dbinom{b}{r-a}$ are integers). Hence, its right hand side is an integer as well. In other words, $g_{a,b,r}$ is an integer. Thus, $g_{a,b,r}$ is a positive integer (since we know that $g_{a,b,r}$ is positive). This proves Claim 3 **(a)**.

(b) Let $b \in \mathbb{N}$. Thus, $b \geq 0$, so that $\max\{0,b\} = b \leq b \leq 0+b$. Hence, $g_{0,b,b}$ is well-defined. The definition of $g_{0,b,b}$ yields

$$g_{0,b,b} = \frac{0!b!}{(b-0)!\,(b-b)!\,(0+b-b)!} = \frac{0!b!}{b!0!0!} = \frac{1}{0!} = 1$$

(since $0! = 1$). This proves Claim 3 **(b)**.

(c) Let $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfy $a+1 \leq b$ and $b+1 \leq r \leq a+b+1$.

Let us first show that all the expressions appearing in (19) are well-defined.

From $a \leq a+1 \leq b$, we obtain $\max\{a,b\} = b$. Also, from $b+1 \leq r$, we obtain $b \leq r-1$. Hence, $\max\{a,b\} = b \leq r-1 \leq a+b$ (since $r \leq a+b+1$). Also, from $a+1 \leq b$, we obtain $\max\{a+1,b\} = b \leq r-1 \leq r \leq a+b+1 = (a+1)+b$.

From $b \leq r-1$, we obtain $r-1 \geq b \geq 0$ and thus $r-1 \in \mathbb{N}$. Also, recall that $\max\{a,b\} \leq r-1 \leq a+b$. Hence, $g_{a,b,r-1}$ is well-defined.

Also, recall that $\max\{a+1,b\} \leq r \leq (a+1)+b$. Hence, $g_{a+1,b,r}$ is well-defined.

Also, the quotient $\dfrac{r-b}{a+1}$ is well-defined, since $a+1>0$ (because $a \in \mathbb{N}$).

Thus, all the expressions appearing in (19) are well-defined. It remains to prove (19) itself.

We have $b+1 \leq r$, thus $r \geq b+1$ and therefore $r-b \geq 1$. Hence, $(r-b)! = (r-b)\cdot(r-b-1)!$.

The definition of $g_{a+1,b,r}$ yields

$$\begin{aligned} g_{a+1,b,r} &= \frac{(a+1)!b!}{(r-(a+1))!\,(r-b)!\,((a+1)+b-r)!} = \frac{(a+1)!b!}{(r-a-1)!\,(r-b)!\,(a+b-r+1)!} \\ &= \frac{(a+1)\cdot a!b!}{(r-a-1)!\,(r-b)\cdot(r-b-1)!\,(a+b-r+1)!} \\ &\qquad \left(\text{since } (a+1)! = (a+1)\cdot a! \text{ and } (r-b)! = (r-b)\cdot(r-b-1)!\right). \end{aligned}$$

Multiplying both sides of this equality by $\dfrac{r-b}{a+1}$, we find

$$\begin{aligned} \frac{r-b}{a+1} g_{a+1,b,r} &= \frac{r-b}{a+1}\cdot\frac{(a+1)\cdot a!b!}{(r-a-1)!\,(r-b)\cdot(r-b-1)!\,(a+b-r+1)!} \\ &= \frac{a!b!}{(r-a-1)!\,(r-b-1)!\,(a+b-r+1)!}. \end{aligned}$$

Comparing this with

$$\begin{aligned} g_{a,b,r-1} &= \frac{a!b!}{((r-1)-a)!\,((r-1)-b)!\,(a+b-(r-1))!} \qquad \left(\text{by the definition of } g_{a,b,r-1}\right) \\ &= \frac{a!b!}{(r-a-1)!\,(r-b-1)!\,(a+b-r+1)!}, \end{aligned}$$

we obtain $g_{a,b,r-1} = \dfrac{r-b}{a+1} g_{a+1,b,r}$. This proves (19). Thus, Claim 3 **(c)** is proven.

(d) Let $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfy $a+1 \leq b$ and $b \leq r \leq a+b$.

Let us first show that all the expressions appearing in (20) are well-defined.

From $a \leq a+1 \leq b$, we obtain $\max\{a,b\} = b$. Hence, $\max\{a,b\} = b \leq r \leq a+b$. Thus, $g_{a,b,r}$ is well-defined.

Also, from $a+1 \leq b$, we obtain $\max\{a+1,b\} = b \leq r \leq a+b \leq a+b+1 = (a+1)+b$. Hence, $g_{a+1,b,r}$ is well-defined.

Also, the quotient $\dfrac{a+b+1-r}{a+1}$ is well-defined, since $a+1>0$ (because $a \in \mathbb{N}$).

Thus, all the expressions appearing in (20) are well-defined. It remains to prove (20) itself.

We have $a+b+1-\underbrace{r}_{\leq a+b} \geq a+b+1-(a+b) = 1$. Thus,

$$(a+b+1-r)! = (a+b+1-r)\cdot((a+b+1-r)-1)! = (a+b+1-r)\cdot(a+b-r)!$$

(since $(a+b+1-r)-1 = a+b-r$).

From $a+1 \leq b \leq r$, we obtain $r \geq a+1$, thus $r-a \geq 1$. Hence, $(r-a)! = (r-a) \cdot (r-a-1)!$.

The definition of $g_{a+1,b,r}$ yields

$$g_{a+1,b,r} = \frac{(a+1)!b!}{(r-(a+1))!\,(r-b)!\,((a+1)+b-r)!} = \frac{(a+1)!b!}{(r-a-1)!\,(r-b)!\,(a+b-r+1)!}$$
$$= \frac{(a+1)\cdot a!b!}{(r-a-1)!\,(r-b)!\,(a+b+1-r)\cdot(a+b-r)!}$$
$$\left(\text{since } (a+1)! = (a+1)\cdot a! \text{ and } (a+b+1-r)! = (a+b+1-r)\cdot(a+b-r)!\right).$$

Multiplying both sides of this equality by $\dfrac{a+b+1-r}{a+1}$, we find

$$\frac{a+b+1-r}{a+1} g_{a+1,b,r} = \frac{a+b+1-r}{a+1} \cdot \frac{(a+1)\cdot a!b!}{(r-a-1)!\,(r-b)!\,(a+b+1-r)\cdot(a+b-r)!}$$
$$= \frac{a!b!}{(r-a-1)!\,(r-b)!\,(a+b-r)!}.$$

Comparing this with

$$(r-a)\,g_{a,b,r} = (r-a)\,\frac{a!b!}{(r-a)!\,(r-b)!\,(a+b-r)!} \qquad \text{(by (18))}$$
$$= (r-a)\,\frac{a!b!}{(r-a)\cdot(r-a-1)!\,(r-b)!\,(a+b-r)!}$$
$$\left(\text{since } (r-a)! = (r-a)\cdot(r-a-1)!\right)$$
$$= \frac{a!b!}{(r-a-1)!\,(r-b)!\,(a+b-r)!},$$

we obtain $(r-a)\,g_{a,b,r} = \dfrac{a+b+1-r}{a+1} g_{a+1,b,r}$. This proves (20). Thus, Claim 3 **(d)** is proven.]

Next, we observe that a and b play symmetric roles in the exercise. Thus, it suffices to solve the exercise in the case when $a \leq b$ (since the other case then follows by the same argument, with the roles of a and b interchanged). To do so, we shall prove the following claim:

Claim 4: Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be such that $a \leq b$. Let x be any real. Then,

$$x^{\underline{a}} x^{\underline{b}} = \sum_{r=b}^{a+b} g_{a,b,r} x^{\underline{r}}. \qquad (22)$$

(In particular, the numbers $g_{a,b,r}$ are well-defined for all $r \in \{b, b+1, \ldots, a+b\}$.)

[*Proof of Claim 4:* We first notice that for every $r \in \{b, b+1, \ldots, a+b\}$, the number $g_{a,b,r}$ is well-defined[9]. Thus, the equality (22) makes sense.

Now, forget that we fixed a. Our goal is to prove the equality (22) for each $a \in \mathbb{N}$ satisfying $a \leq b$. In other words, our goal is to prove the equality (22) for each $a \in \{0, 1, \ldots, b\}$. We shall prove (22) by induction on a:

[9] *Proof.* Let $r \in \{b, b+1, \ldots, a+b\}$. Thus, $r \geq b$ and $r \leq a+b$. But $a \leq b$, so that $\max\{a,b\} = b \leq r$ (since $r \geq b$). Hence, $\max\{a,b\} \leq r \leq a+b$. Thus, the number $g_{a,b,r}$ is well-defined, qed.

Induction base: The definition of $x^{\underline{0}}$ yields

$$x^{\underline{0}} = x(x-1)(x-2)\cdots(x-0+1) = (\text{empty product}) = 1.$$

Thus,

$$\underbrace{x^{\underline{0}}}_{=1} x^{\underline{b}} = x^{\underline{b}}.$$

Comparing this with

$$\sum_{r=b}^{0+b} g_{0,b,r} x^{\underline{r}} = \sum_{r=b}^{b} g_{0,b,r} x^{\underline{r}} = \underbrace{g_{0,b,b}}_{\substack{=1 \\ \text{(by Claim 3 (b))}}} x^{\underline{b}} = x^{\underline{b}},$$

we obtain

$$x^{\underline{0}} x^{\underline{b}} = \sum_{r=b}^{0+b} g_{0,b,r} x^{\underline{r}}.$$

In other words, (22) holds for $a = 0$. This completes the induction base.

Induction step: Fix some $c \in \mathbb{N}$ such that $c+1 \in \{0, 1, \ldots, b\}$. Assume that (22) holds for $a = c$. We must show that (22) holds for $a = c+1$.

Each $r \in \mathbb{N}$ satisfies

$$\begin{aligned} (x-c) x^{\underline{r}} &= \underbrace{x x^{\underline{r}}}_{\substack{= x^{\underline{r+1}} + r x^{\underline{r}} \\ \text{(by (17))}}} - c x^{\underline{r}} = x^{\underline{r+1}} + r x^{\underline{r}} - c x^{\underline{r}} \\ &= x^{\underline{r+1}} + (r-c) x^{\underline{r}}. \end{aligned} \tag{23}$$

We have $c \leq c+1 \leq b$ (since $c+1 \in \{0, 1, \ldots, b\}$). Hence,

$$x^{\underline{c}} x^{\underline{b}} = \sum_{r=b}^{c+b} g_{c,b,r} x^{\underline{r}} \tag{24}$$

(since we assumed that (22) holds for $a = c$).

Now, (16) (applied to $k = c$) yields

$$x^{\underline{c+1}} = x^{\underline{c}} (x-c).$$

Hence,

$$\begin{aligned} x^{\underline{c+1}} x^{\underline{b}} &= x^{\underline{c}} (x-c) x^{\underline{b}} = (x-c) x^{\underline{c}} x^{\underline{b}} = (x-c) \sum_{r=b}^{c+b} g_{c,b,r} x^{\underline{r}} \qquad \text{(by (24))} \\ &= \sum_{r=b}^{c+b} g_{c,b,r} \underbrace{(x-c) x^{\underline{r}}}_{\substack{= x^{\underline{r+1}} + (r-c) x^{\underline{r}} \\ \text{(by (23))}}} = \sum_{r=b}^{c+b} g_{c,b,r} \left(x^{\underline{r+1}} + (r-c) x^{\underline{r}} \right) \\ &= \sum_{r=b}^{c+b} g_{c,b,r} x^{\underline{r+1}} + \sum_{r=b}^{c+b} g_{c,b,r} (r-c) x^{\underline{r}}. \end{aligned} \tag{25}$$

We shall now analyze the two addends on the right hand side of this equation separately.

Substituting $r-1$ for r in the sum $\sum\limits_{r=b}^{c+b} g_{c,b,r} x^{\underline{r+1}}$, we obtain

$$\sum_{r=b}^{c+b} g_{c,b,r} x^{\underline{r+1}} = \sum_{r=b+1}^{c+b+1} \underbrace{g_{c,b,r-1}}_{\substack{=\frac{r-b}{c+1} g_{c+1,b,r} \\ \text{(by (19), applied to } a=c)}} \underbrace{x^{\underline{(r-1)+1}}}_{=x^{\underline{r}}} = \sum_{r=b+1}^{c+b+1} \frac{r-b}{c+1} g_{c+1,b,r} x^{\underline{r}}.$$

Comparing this with

$$\sum_{r=b}^{c+b+1} \frac{r-b}{c+1} g_{c+1,b,r} x^{\underline{r}} = \underbrace{\frac{b-b}{c+1}}_{=0} g_{c+1,b,b} x^{\underline{b}} + \sum_{r=b+1}^{c+b+1} \frac{r-b}{c+1} g_{c+1,b,r} x^{\underline{r}}$$

(here, we have split off the addend for $r=b$ from the sum)

$$= \sum_{r=b+1}^{c+b+1} \frac{r-b}{c+1} g_{c+1,b,r} x^{\underline{r}},$$

we obtain

$$\sum_{r=b}^{c+b} g_{c,b,r} x^{\underline{r+1}} = \sum_{r=b}^{c+b+1} \frac{r-b}{c+1} g_{c+1,b,r} x^{\underline{r}}. \tag{26}$$

On the other hand,

$$\sum_{r=b}^{c+b} g_{c,b,r} (r-c) x^{\underline{r}} = \sum_{r=b}^{c+b} \underbrace{(r-c) g_{c,b,r}}_{\substack{=\frac{c+b+1-r}{c+1} g_{c+1,b,r} \\ \text{(by (20), applied to } a=c)}} x^{\underline{r}} = \sum_{r=b}^{c+b} \frac{c+b+1-r}{c+1} g_{c+1,b,r} x^{\underline{r}}.$$

Comparing this with

$$\sum_{r=b}^{c+b+1} \frac{c+b+1-r}{c+1} g_{c+1,b,r} x^{\underline{r}}$$
$$= \sum_{r=b}^{c+b} \frac{c+b+1-r}{c+1} g_{c+1,b,r} x^{\underline{r}} + \underbrace{\frac{c+b+1-(c+b+1)}{c+1}}_{=0} g_{c+1,b,c+b+1} x^{\underline{c+b+1}}$$

(here, we have split off the addend for $r=c+b+1$ from the sum)

$$= \sum_{r=b}^{c+b} \frac{c+b+1-r}{c+1} g_{c+1,b,r} x^{\underline{r}},$$

we obtain

$$\sum_{r=b}^{c+b} g_{c,b,r} (r-c) x^{\underline{r}} = \sum_{r=b}^{c+b+1} \frac{c+b+1-r}{c+1} g_{c+1,b,r} x^{\underline{r}}. \tag{27}$$

Adding this equality to (26), we find

$$\begin{aligned}
&\sum_{r=b}^{c+b} g_{c,b,r} x^{\underline{r+1}} + \sum_{r=b}^{c+b} g_{c,b,r} \left(r-c\right) x^{\underline{r}} \\
&= \sum_{r=b}^{c+b+1} \frac{r-b}{c+1} g_{c+1,b,r} x^{\underline{r}} + \sum_{r=b}^{c+b+1} \frac{c+b+1-r}{c+1} g_{c+1,b,r} x^{\underline{r}} \\
&= \sum_{r=b}^{c+b+1} \underbrace{\left(\frac{r-b}{c+1} + \frac{c+b+1-r}{c+1}\right)}_{=1} g_{c+1,b,r} x^{\underline{r}} = \sum_{r=b}^{c+b+1} g_{c+1,b,r} x^{\underline{r}}.
\end{aligned}$$

Hence, (25) becomes

$$x^{\underline{c+1}} x^{\underline{b}} = \sum_{r=b}^{c+b} g_{c,b,r} x^{\underline{r+1}} + \sum_{r=b}^{c+b} g_{c,b,r} \left(r-c\right) x^{\underline{r}} = \sum_{r=b}^{c+b+1} g_{c+1,b,r} x^{\underline{r}} = \sum_{r=b}^{(c+1)+b} g_{c+1,b,r} x^{\underline{r}}$$

(since $c+b+1 = (c+1)+b$). In other words, (22) holds for $a = c+1$. This completes the induction step. Hence, (22) is proven by induction. Thus, Claim 4 is proven.]

Now, let us solve the actual exercise. Let a and b be nonnegative integers, and let x be a real. We must prove the equality (15). The variables a and b play symmetric roles in this equality (that is, if we swap a with b, then the meaning of (15) does not change). Thus, we can WLOG assume that $a \leq b$ (since otherwise, we can simply swap a with b). Assume this. Hence, Claim 4 yields

$$x^{\underline{a}} x^{\underline{b}} = \sum_{r=b}^{a+b} g_{a,b,r} x^{\underline{r}}.$$

Comparing this with

$$\sum_{r=\max\{a,b\}}^{a+b} \underbrace{\frac{a!b!}{(r-a)!\,(r-b)!\,(a+b-r)!}}_{\substack{=g_{a,b,r} \\ \text{(by (18))}}} x^{\underline{r}} = \sum_{r=\max\{a,b\}}^{a+b} g_{a,b,r} x^{\underline{r}} = \sum_{r=b}^{a+b} g_{a,b,r} x^{\underline{r}}$$

$$\left(\text{since } \max\{a,b\} = b \text{ (since } a \leq b\text{)}\right),$$

we obtain

$$x^{\underline{a}} x^{\underline{b}} = \sum_{r=\max\{a,b\}}^{a+b} \frac{a!b!}{(r-a)!\,(r-b)!\,(a+b-r)!} x^{\underline{r}}.$$

This solves the exercise.