An exercise on source and sink mutations of acyclic quivers*

Darij Grinberg

January 10, 2019

In this note, we will use the following notations (which come from Lampe's notes [Lampe, §2.1.1]):

- A *quiver* means a tuple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are two finite sets and where s and t are two maps from Q_1 to Q_0. We call the elements of Q_0 the *vertices* of the quiver Q, and we call the elements of Q_1 the *arrows* of the quiver Q. For every $e \in Q_1$, we call $s(e)$ the *starting point* of e (and we say that *e starts at $s(e)$*), and we call $t(e)$ the *terminal point* of e (and we say that *e ends at $t(e)$*). Furthermore, if $e \in Q_1$, then we say that e is an *arrow from $s(e)$ to $t(e)$*.

  So the notion of a quiver is one of many different versions of the notion of a finite directed graph. (Notice that it is a version which allows multiple arrows, and which distinguishes between them – i.e., the quiver stores not just the information of how many arrows there are from a vertex to another, but it actually has them all as distinguishable objects in Q_1. Lampe himself seems to later tacitly switch to a different notion of quivers, where edges from a given to vertex to another are indistinguishable and only exist as a number. This does not matter for the next exercise, which works just as well with either notion of a quiver; but I just wanted to have it mentioned.)

- The *underlying undirected graph* of a quiver $Q = (Q_0, Q_1, s, t)$ is defined as the undirected multigraph with vertex set Q_0 and edge multiset

  $$\{\{s(e), t(e)\} \mid e \in Q_1\}_{\text{multiset}}.$$

  ("Multigraph" means that multiple edges are allowed, but we do not make them distinguishable.)

*This used to be Chapter 7 of my notes "Notes on the combinatorial fundamentals of algebra" (version of 7 November 2018), but has since been removed from the latter notes.

- A quiver $Q = (Q_0, Q_1, s, t)$ is said to be *acyclic* if there is no sequence $(a_0, a_1, \ldots, a_n)$ of elements of Q_0 such that $n > 0$ and $a_0 = a_n$ and such that Q has an arrow from a_i to a_{i+1} for every $i \in \{0, 1, \ldots, n-1\}$. (This is equivalent to [Lampe, Definition 2.1.7].) Notice that this does not mean that the *underlying undirected graph* of Q has no cycles.

- Let $Q = (Q_0, Q_1, s, t)$. Then, a *sink* of Q means a vertex $v \in Q_0$ such that no $e \in Q_1$ starts at v (in other words, no arrow of Q starts at v). A *source* of Q means a vertex $v \in Q_0$ such that no $e \in Q_1$ ends at v (in other words, no arrow of Q ends at v).

- Let $Q = (Q_0, Q_1, s, t)$. If $i \in Q_0$ is a sink of Q, then the *mutation* $\mu_i(Q)$ of Q at i is the quiver obtained from Q simply by turning[1] all arrows ending at i. (To be really pedantic: We define $\mu_i(Q)$ as the quiver (Q_0, Q_1, s', t'), where

$$s'(e) = \begin{cases} t(e), & \text{if } t(e) = i; \\ s(e), & \text{if } t(e) \neq i \end{cases} \qquad \text{for each } e \in Q_1$$

$$\text{and} \qquad t'(e) = \begin{cases} s(e), & \text{if } t(e) = i; \\ t(e), & \text{if } t(e) \neq i \end{cases} \qquad \text{for each } e \in Q_1.$$

) If $i \in Q_0$ is a source of Q, then the *mutation* $\mu_i(Q)$ of Q at i is the quiver obtained from Q by turning all arrows starting at i. (Notice that if i is both a source and a sink of Q, then these two definitions give the same result; namely, $\mu_i(Q) = Q$ in this case.)

 If Q is an acyclic quiver, then $\mu_i(Q)$ is acyclic as well (whenever $i \in Q_0$ is a sink or a source of Q).

 We use the word "mutation" not only for the quiver $\mu_i(Q)$, but also for the operation that transforms Q into $\mu_i(Q)$. (We have defined this operation only if i is a sink or a source of Q. It can be viewed as a particular case of the more general definition of mutation given in [Lampe, Definition 2.2.1], at least if one gives up the ability to distinguish different arrows from one vertex to another.)

[1] To *turn* an arrow e means to reverse its direction, i.e., to switch the values of $s(e)$ and $t(e)$. We model this as a change to the functions s and t, not as a change to the arrow itself.

Exercise 0.1. Let $Q = (Q_0, Q_1, s, t)$ be an acyclic quiver.

(a) Let A and B be two subsets of Q_0 such that $A \cap B = \varnothing$ and $A \cup B = Q_0$. Assume that there exists no arrow of Q that starts at a vertex in B and ends at a vertex in A. Then, by turning all arrows of Q which start at a vertex in A and end at a vertex in B, we obtain a new acyclic quiver $\operatorname{mut}_{A,B} Q$.

(When we say "turning all arrows of Q which start at a vertex in A and end at a vertex in B", we mean "turning all arrows e of Q which satisfy $s(e) \in A$ and $t(e) \in B$". We do **not** mean that we fix a vertex a in A and a vertex b in B, and only turn the arrows from a to b.)

For example, if $Q = \begin{array}{ccc} 3 & \longrightarrow & 4 \\ \uparrow & \nearrow & \uparrow \\ 1 & \longrightarrow & 2 \end{array}$ and $A = \{1, 3\}$ and $B = \{2, 4\}$, then

$$\operatorname{mut}_{A,B} Q = \begin{array}{ccc} 3 & \longleftarrow & 4 \\ \uparrow & \swarrow & \uparrow \\ 1 & \longleftarrow & 2 \end{array}.$$

Prove that $\operatorname{mut}_{A,B} Q$ can be obtained from Q by a sequence of mutations at sinks. (More precisely, there exists a sequence $\left(Q^{(0)}, Q^{(1)}, \ldots, Q^{(\ell)}\right)$ of acyclic quivers such that $Q^{(0)} = Q$, $Q^{(\ell)} = \operatorname{mut}_{A,B} Q$, and for every $i \in \{1, 2, \ldots, \ell\}$, the quiver $Q^{(i)}$ is obtained from $Q^{(i-1)}$ by mutation at a sink of $Q^{(i-1)}$.)

[In our above example, we can mutate at 4 first and then at 2.]

(b) If $i \in Q_0$ is a **source** of Q, then show that the mutation $\mu_i(Q)$ can be obtained from Q by a sequence of mutations at sinks.

(c) Assume now that the underlying **undirected** graph of Q is a tree. (In particular, Q cannot have more than one edge between two vertices, as these would form a cycle in the underlying undirected graph!) Show that any acyclic quiver which can be obtained from Q by turning some of its arrows can also be obtained from Q by a sequence of mutations at sinks.

Remark 0.1. More general results than those of Exercise 0.1 are stated (for directed graphs rather than quivers, but it is easy to translate from one language into another) in [Pretzel]. An equivalent version of Exercise 0.1 **(c)** also appears as Exercise 6 in [GrRaOg] (because a quiver Q whose underlying undirected graph is a tree can be regarded as an orientation of the latter tree, and because the concept of "pushing sources" in [GrRaOg] corresponds precisely to our concept of mutations at sinks, except that all arrows need to be reversed).

Solution to Exercise 0.1. **(a)** We prove the claim by induction over $|B|$.

Induction base: Assume that $|B| = 0$. Thus, $B = \varnothing$, and thus there are no arrows of Q which start at a vertex in A and at a vertex in B. Hence, $\operatorname{mut}_{A,B} Q = Q$, and this can clearly be obtained from Q by a sequence of mutations at sinks (namely, by the empty sequence). Thus, Exercise 0.1 **(a)** holds if $|B| = 0$. This completes the induction base.[2]

Induction step:[3] Let $N \in \mathbb{N}$. Assume that Exercise 0.1 **(a)** holds whenever $|B| = N$. We now need to prove that Exercise 0.1 **(a)** holds whenever $|B| = N+1$.

So let A and B be two subsets of Q_0 such that $A \cap B = \varnothing$ and $A \cup B = Q_0$. Assume that there exists no arrow of Q that starts at a vertex in B and ends at a vertex in A. Assume further that $|B| = N+1$. We need to prove that $\operatorname{mut}_{A,B} Q$ can be obtained from Q by a sequence of mutations at sinks.

Notice that $B = Q_0 \setminus A$ (since $A \cap B = \varnothing$ and $A \cup B = Q_0$).

It is easy to see that there exists some $b \in B$ such that

$$\text{there is no } e \in Q_1 \text{ satisfying } t(e) = b \text{ and } s(e) \in B \tag{1}$$

[4]. Fix such a b. Clearly, $b \notin A$ (since $b \in B = Q_0 \setminus A$).

Now, $A \cup \{b\}$ and $B \setminus \{b\}$ are two subsets of Q_0 such that $(A \cup \{b\}) \cap (B \setminus \{b\}) = \varnothing$ and $(A \cup \{b\}) \cup (B \setminus \{b\}) = Q_0$ [5]. Furthermore, there exists no arrow of Q that starts at a vertex in $B \setminus \{b\}$ and ends at a vertex in $A \cup \{b\}$ [6]. Hence, $\operatorname{mut}_{A \cup \{b\}, B \setminus \{b\}} Q$ is a well-defined acyclic quiver. Moreover, since $b \in B$, we have $|B \setminus \{b\}| = \underbrace{|B|}_{=N+1} - 1 = N + 1 - 1 = N$. Thus, Exercise 0.1 **(a)** can be applied to $A \cup \{b\}$ and $B \setminus \{b\}$ instead of A and B (by the induction hypothesis). As a consequence, we conclude that $\operatorname{mut}_{A \cup \{b\}, B \setminus \{b\}} Q$ can be obtained from Q by a sequence of mutations at sinks.

---

[2] Yes, this was a completely honest induction base. You don't need to start at $|B| = 1$ unless you want to use something like $|B| > 1$ in the induction step (but even then, you should also handle the case $|B| = 0$ separately).

[3] The letter $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ here.

[4] *Proof.* Assume the contrary. Thus, for every $b \in B$, there is an $e \in Q_1$ satisfying $t(e) = b$ and $s(e) \in B$. Let us fix such an e (for each $b \in B$), and denote it by e_b.

Thus, for every $b \in B$, we have $e_b \in Q_1$ and $t(e_b) = b$ and $s(e_b) \in B$. We can thus define a sequence $(b_0, b_1, b_2, \ldots)$ of vertices in B recursively as follows: Set $b_0 = b$, and set $b_{i+1} = s(e_{b_i})$ for every $i \in \mathbb{N}$. Thus, $(b_0, b_1, b_2, \ldots)$ is an infinite sequence of elements of B. Since B is a finite set, this sequence must thus pass through an element twice (to say the least). In other words, there are two positive integers u and v such that $u < v$ and $b_u = b_v$. Consider these u and v.

Now, for every $i \in \mathbb{N}$, we have $t(e_{b_i}) = b_i$ (by the definition of e_{b_i}) and $s(e_{b_i}) = b_{i+1}$. Thus, for every $i \in \mathbb{N}$, the arrow e_{b_i} is an arrow from b_{i+1} to b_i. Thus, there is an arrow from b_{i+1} to b_i for every $i \in \mathbb{N}$. In particular, we have an arrow from b_v to b_{v-1}, an arrow from b_{v-1} to b_{v-2}, etc., and an arrow from b_{u+1} to b_u. Since $b_u = b_v$, these arrows form a cycle in Q, which contradicts the hypothesis that the quiver Q is acyclic. This contradiction proves that our assumption was wrong, qed.

[5] *Proof.* These are easy exercises in set theory. Use $A \cap B = \varnothing$ and $A \cup B = Q_0$ and $b \in B$.

[6] *Proof.* Assume the contrary. Thus, there exists an arrow of Q that starts at a vertex in $B \setminus \{b\}$ and ends at a vertex in $A \cup \{b\}$. Let e be such an arrow. Then, $s(e) \in B \setminus \{b\}$ and $t(e) \in A \cup \{b\}$.

We have $s(e) \in B \setminus \{b\} \subseteq B$. Thus, $t(e) \neq b$ (because having $t(e) = b$ would contradict (1)). Combined with $t(e) \in A \cup \{b\}$, this yields $t(e) \in (A \cup \{b\}) \setminus \{b\} \subseteq A$. Thus, e is an arrow of Q that starts at a vertex in B (since $s(e) \in B$) and ends at a vertex in A (since $t(e) \in A$). This contradicts our hypothesis that there exists no arrow of Q that starts at a vertex in B and ends at a vertex in A. This is the desired contradiction, and so we are done.

We shall now prove that $\operatorname{mut}_{A,B} Q$ can be obtained from $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ by a mutation at a sink. In fact, b is a sink of $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ [7]. Hence, the mutation $\mu_b\left(\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q\right)$ is well-defined. We now have

$$\operatorname{mut}_{A,B} Q = \mu_b\left(\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q\right) \tag{2}$$

[8]. Therefore, $\operatorname{mut}_{A,B} Q$ can be obtained from $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ by a single mutation

---

[7] *Proof.* Assume the contrary. Thus, there exists an arrow e of $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ which starts at b. Consider this e.

Recall that $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ was obtained from Q by turning all arrows of Q which start at a vertex in $A\cup\{b\}$ and end at a vertex in $B\setminus\{b\}$. Thus, every arrow of $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ which starts at a vertex in $B\setminus\{b\}$ and ends at a vertex in $A\cup\{b\}$ has originally been going in the opposite direction in Q (because there exists no arrow of Q that starts at a vertex in $B\setminus\{b\}$ and ends at a vertex in $A\cup\{b\}$), while all the other arrows of $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ have been copied over unchanged from Q. The arrow e of $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ starts at b (which is not an element of $B\setminus\{b\}$), so it does **not** start at a vertex in $B\setminus\{b\}$ and end at a vertex in $A\cup\{b\}$; therefore, the preceding sentence shows that this arrow e has been copied over unchanged from Q. In other words, the arrow e starts at b when considered as an arrow of Q as well. In other words, $s(e) = b$. (Recall that the functions s and t are part of the quiver Q; thus, they map every arrow of Q to its starting point and its terminal point, respectively. The same arrows might have different starting points and terminal points when regarded as arrows of $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$.)

Recall that there exists no arrow of Q that starts at a vertex in B and ends at a vertex in A. Thus, an arrow of Q which starts at a vertex in B must not end at a vertex in A. In particular, the arrow e of Q must not end at a vertex in A (because it starts at $b\in B$). Hence, the arrow e of Q ends at a vertex in $Q_0\setminus A = B$. In other words, $t(e)\in B$.

We cannot have $t(e) = s(e)$ (because otherwise, the arrow e would form a cycle, but the quiver Q is acyclic). Hence, $t(e) \neq s(e) = b$ (since e starts at b). Combined with $t(e)\in B$, this yields $t(e)\in B\setminus\{b\}$.

Thus, the arrow e of Q starts at a vertex in $A\cup\{b\}$ (since $s(e) = b\in A\cup\{b\}$) and ends at a vertex in $B\setminus\{b\}$ (since $t(e)\in B\setminus\{b\}$). As we know, $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ was obtained from Q by turning all such arrows. Hence, the arrow e must have been turned when it became an arrow of $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$. But this contradicts the fact that the arrow e has been copied over unchanged from Q. This contradiction proves that our assumption was wrong, qed.

[8] *Proof of (2):* We have $Q_0 = A\cup \underbrace{B}_{=\{b\}\cup(B\setminus\{b\})} = A\cup\{b\}\cup(B\setminus\{b\})$.

Recall that the quiver $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ was obtained from Q by turning all arrows of Q which start at a vertex in $A\cup\{b\}$ and end at a vertex in $B\setminus\{b\}$. Furthermore, the quiver $\mu_b\left(\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q\right)$ was obtained from $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ by turning all arrows ending at b. Thus, $\mu_b\left(\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q\right)$ can be obtained from Q by a two-step process, where

- in the first step, we turn all arrows of Q which start at a vertex in $A\cup\{b\}$ and end at a vertex in $B\setminus\{b\}$;
- in the second step, we turn all arrows ending at b.

Now, let us analyze what this two-step process does to an arrow of Q, depending on where the arrow starts and ends:

1. If e is an arrow of Q which ends at a vertex in A, then this arrow never gets turned during our process. Indeed, let e be such an arrow. Then, e ends at a vertex in A, and thus does

at a sink (namely, at the sink b). Since $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$ (in turn) can be obtained from Q by a sequence of mutations at sinks, this shows that $\operatorname{mut}_{A,B} Q$ can be obtained from Q by a sequence of mutations at sinks (namely, we first need to mutate

---

not end at a vertex in B (since $A \cap B = \varnothing$); therefore, it does not end at a vertex in $B \setminus \{b\}$ either. Hence, the first step does not turn it. Therefore, after the first step, it still does not end at a vertex in B (since it did not end at a vertex in B originally). In particular, it does not end at b (since $b \in B$). Hence, it does not get turned at the second step either. So, e never turns, and thus retains its original direction throughout the process.

2. If e is an arrow of Q which ends at b, then this arrow gets turned once (namely, at the second step). Thus, its direction is reversed at the end of the process.

3. If e is an arrow of Q which starts at a vertex in A and ends at a vertex in $B \setminus \{b\}$, then this arrow gets turned once (namely, at the first step). Here is why: Let e be an arrow of Q which starts at a vertex in A and ends at a vertex in $B \setminus \{b\}$. Then, e starts at a vertex in $A \cup \{b\}$ and ends at a vertex in $B \setminus \{b\}$ (since $A \subseteq A \cup \{b\}$). Thus, it gets turned at the first step. After this, it becomes an arrow which ends at a vertex in A (because originally it started at a vertex in A), and so it does not end at b (because $b \notin A$). Therefore, it does not turn at the second step; hence, it has turned exactly once altogether. Its direction is therefore reversed at the end of the process.

4. If e is an arrow of Q which starts at b and ends at a vertex in $B \setminus \{b\}$, then this arrow gets turned twice (once at each step). Indeed, let e be such an arrow. Then, e starts at a vertex in $A \cup \{b\}$ (namely, at b) and ends at a vertex in $B \setminus \{b\}$. Hence, it gets turned at the first step. After that, it ends at b (because it used to start at b before it was turned), and therefore it gets turned again at the second step. Hence, the direction of e at the end of the two-step process is again the same as it was in Q.

5. If e is an arrow of Q which starts at a vertex in $B \setminus \{b\}$ and ends at a vertex in $B \setminus \{b\}$, then this arrow never gets turned. Indeed, it starts at a vertex in $B \setminus \{b\}$; thus, it does **not** start at a vertex in $A \cup \{b\}$ (since $\underbrace{B}_{=Q_0 \setminus A} \setminus \{b\} = (Q_0 \setminus A) \setminus \{b\} = Q_0 \setminus (A \cup \{b\})$). Hence, it does not get turned at the first step. Moreover, in Q, this arrow e does not end at b (because it ends at a vertex in $B \setminus \{b\}$); thus it does not end at b after the first step either (since it does not get turned at the first step). Hence, it does not get turned at the second step either. Therefore, e never gets turned, and thus retains its original direction from Q after the two-step process.

The five cases we have just considered cover all possibilities (because every arrow e either ends at a vertex in A or ends at b or ends at a vertex in $B \setminus \{b\}$; and in the latter case, it either starts at a vertex in A, or starts at b, or starts at a vertex in $B \setminus \{b\}$ (since $Q_0 = A \cup \{b\} \cup (B \setminus \{b\})$)). From our case analysis, we can draw the following conclusions:

- If e is an arrow of Q which starts at a vertex in A and ends at a vertex in B, then the arrow e has reversed its orientation at the end of the two-step process. (This follows from our Cases 2 and 3 above.)

- If e is an arrow of Q which starts at a vertex in B or ends at a vertex in A, then this arrow e has the same orientation at the end of the two-step process as it did in Q. (Indeed, let us prove this. Let e be an arrow of Q which starts at a vertex in B or ends at a vertex in A. We need to show that e has the same orientation at the end of the two-step process as it did in Q. If e ends at a vertex in A, then this follows from our analysis of Case 1. So let us assume that e does not end at a vertex in A. Hence, e must start at a vertex in B (since e starts at a vertex in B or ends at a vertex in A). In other words, $s(e) \in B$. Hence, at the sinks that give us $\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q$, and then we have to mutate at b). This proves that Exercise 0.1 **(a)** holds whenever $|B| = N+1$. The induction step is complete, and thus Exercise 0.1 **(a)** is solved.

(b) Let $i \in Q_0$ be a source in Q. Let $A = \{i\}$ and $B = Q_0 \setminus A$. Then, A and B are two subsets of Q_0 such that $A \cap B = \varnothing$ and $A \cup B = Q_0$. There exists no arrow of Q that starts at a vertex in B and ends at a vertex in A [9]. Hence, the quiver $\operatorname{mut}_{A,B} Q$ is well-defined. Moreover, this quiver $\operatorname{mut}_{A,B} Q$ is obtained by turning all arrows of Q which start at a vertex in A and end at a vertex in B. But these arrows are precisely the arrows of Q starting at i [10]. Hence, $\operatorname{mut}_{A,B} Q$ is obtained by turning all arrows of Q starting at i. But this is exactly how we defined $\mu_i(Q)$. Therefore, $\operatorname{mut}_{A,B} Q = \mu_i(Q)$. Now, Exercise 0.1 **(a)** shows that $\operatorname{mut}_{A,B} Q$ can be obtained from Q by a sequence of mutations at sinks. Hence, $\mu_i(Q)$ can be obtained from Q by a sequence of mutations at sinks (since $\operatorname{mut}_{A,B} Q = \mu_i(Q)$). Exercise 0.1 **(b)** is proven.

(c) Let Q' be any acyclic quiver which can be obtained from Q by turning some of its arrows. We need to prove that Q' can also be obtained from Q by a sequence of mutations at sinks. But [Lampe, proof of Proposition 2.2.8] shows that Q' can be obtained from Q by a sequence of mutations at sinks and sources. Since every

$t(e) \neq b$ (because if we had $t(e) = b$, then e would contradict (1)). But also $t(e) \notin A$ (since e does not end at a vertex in A), so that $t(e) \in Q_0 \setminus A = B$ and thus $t(e) \in B \setminus \{b\}$ (since $t(e) \neq b$). Hence, the arrow e ends at a vertex in $B \setminus \{b\}$. It also starts at a vertex in B; thus, it either starts at b or it starts at a vertex in $B \setminus \{b\}$. Our claim now follows from our analysis of Case 4 (in the case when e starts at b) and from our analysis of Case 5 (in the case when e starts at a vertex in $B \setminus \{b\}$). In either case, our claim is proven.)

To summarize, the outcome of our two-step process is that every arrow e of Q which starts at a vertex in A and ends at a vertex in B reverses its orientation, while all other arrows preserve their orientation. In other words, the outcome of our two-step process is the same as the outcome of turning all arrows of Q which start at a vertex in A and end at a vertex in B. But the latter outcome is $\operatorname{mut}_{A,B} Q$ (because this is how $\operatorname{mut}_{A,B} Q$ was defined), while the former outcome is $\mu_b\left(\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q\right)$ (since we know that $\mu_b\left(\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q\right)$ can be obtained from Q by our two-step process). Thus, we have obtained $\mu_b\left(\operatorname{mut}_{A\cup\{b\},B\setminus\{b\}} Q\right) = \operatorname{mut}_{A,B} Q$. This proves (2).

[9] *Proof.* Assume the contrary. Then, there exists an arrow of Q which starts at a vertex in B and ends at a vertex in A. Let e be such an arrow. Then, e ends at a vertex in A. In other words, $t(e) \in A = \{i\}$, so that $t(e) = i$. In other words, e ends at i. But this is impossible, since i is a source. This contradiction proves that our assumption was wrong, qed.

[10] *Proof.* Each arrow of Q which starts at a vertex in A and ends at a vertex in B must be an arrow starting at i (because it starts at a vertex in $A = \{i\}$, but the only vertex in $\{i\}$ is i). It thus remains to prove the converse – i.e., to prove that each arrow of Q starting at i is an arrow of Q which starts at a vertex in A and ends at a vertex in B. So let e be an arrow of Q starting at i. Then, e clearly starts at a vertex in A (since $i \in \{i\} = A$). It remains to prove that e ends at a vertex in B. But Q is acyclic, and thus we cannot have $s(e) = t(e)$ (since otherwise, the arrow e would form a trivial cycle). Hence, $s(e) \neq t(e)$. But $s(e) = i$ (since e starts at i), so that $t(e) \neq s(e) = i$ and thus $t(e) \in Q_0 \setminus \underbrace{\{i\}}_{=A} = Q_0 \setminus A = B$. Hence, e ends at a vertex in B. This completes our proof.

mutation at a source can be simulated by a sequence of mutations at sinks (by Exercise 0.1 **(b)**), this yields that Q' can be obtained from Q by a sequence of mutations at sinks. This solves Exercise 0.1 **(c)**. □

References

[GrRaOg] Darij Grinberg, *UMN Spring 2017 Math 5707 Midterm #3*, with solutions by Nicholas Rancourt and Jacob Ogden.
`http://www.cip.ifi.lmu.de/~grinberg/t/17s/`

[Lampe] Philipp Lampe, *Cluster algebras*,
`http://www.math.uni-bielefeld.de/~lampe/teaching/cluster/cluster.pdf` .

[Pretzel] Oliver Pretzel, *On reorienting graphs by pushing down maximal vertices*, Order, 1986, Volume 3, Issue 2, pp. 135–153.