Notes on graph theory

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Thursday 6th April, 2023 at 10:42pm (unfinished draft!)
These notes are **frozen** in a (very) unfinished state.
Currently, only two chapters (beyond the preface) exist, and they too are incomplete (although hopefully readable).

Contents

1. Preface

These are lecture notes on graph theory – the part of mathematics involved with graphs. They are currently work in progress (but the parts that are finished are self-contained!); once finished, they should contain a semester's worth of material. I have tried to keep the presentation as self-contained and elementary as possible; the reader is nevertheless assumed to possess some mathematical maturity (in particular, to know how to read combinatorial proofs, filling in simple details)[1] and know how to work with modular arithmetic (i.e., congruences between integers modulo a positive integer n) and summation signs (such as $\sum\limits_{i=1}^{n}$ and $\sum\limits_{a \in A}$, where n is an integer and A is a finite set)[2]. In some chapters, familiarity with matrices (and their products), permutations (and their signs)[3] and polynomials will be required. I hope that the proofs are readable; feel free to contact me (at `darijgrinberg@gmail.com`) for any clarifications.

The choice of material surveyed in these notes is idiosyncratic (sometimes even purposefully trying to wander some seldom trodden paths). Some standard material (Eulerian walks, Hamiltonian paths, trees, bipartite matching theory, network flows) is present (at least in the eventual final form of these notes), whereas other

[1] I believe that the MIT lecture notes [LeLeMe16] are a good source for achieving this mathematical maturity. (Actually, there is some intersection between [LeLeMe16, Chapters 10 and 12] and our notes, but [LeLeMe16] mostly keeps to the basics of graph theory.) Two other resources to familiarize oneself with proofs are [Hammac15] and [Day16]. Generally, most good books about "reading and writing mathematics" or "introductions to abstract mathematics" should convey these skills, although the extent to which they actually do so may differ.

[2] See [Grinbe16, §1.4] for a list of important properties of summation signs.

[3] A summary of the most fundamental results about signs of permutations can be found in [LaNaSc16, §8.1]. These results appear with proofs in [Day16, Chapter 6.B]. For an even more detailed treatment (also including proofs), see [Grinbe16, §5.1–5.3]. Some of the most important properties of signs of permutations are also proven in [Strick13, Appendix B], which also gives one of the best introductions to determinants I have seen in the literature. Another treatment can be found in [Conrad], but this requires some familiarity with group theory.

popular topics (planar graphs, random graphs, adjacency matrices and spectral graph theory) are missing. Some of these omissions have specific reasons (e.g., many of the omitted topics would make it much harder to keep the notes self-contained), whereas others are merely due to my tastes and lack of time.

These notes are accompanying a class on graph theory (Math 5707) I am giving at the University of Minneapolis in Spring 2017. They contain both the material of the class (although with no promise of timeliness!) and the homework exercises (and possibly some additional exercises). Sections marked with an asterisk (*) are not part of the Math 5707 course.

Various other texts on graph theory are [Bollob79], [Bollob98], [Harary69], [Harju14], [Balakr97], [Jungni13], [Martin16], [ThuSwa92], [BonMur76], [Ore74], [BehCha71], [BeChZh15], [BonMur08], [Ruohon13], [Dieste16], [Ore90], [HaHiMo08], [Berge91], [ChaLes15], [Griffi15], [Wilson96]. (This is a haphazard list; I have barely touched most of these texts.) Also, texts on combinatorics and on discrete mathematics (such as [BenWil12], [KelTro15], [PoTaWo83], [Bona11], [Guicha16], or the introductory [LoPeVe03]) often contain sections on graph theory, since graph theory is considered to be part of both. Material on graph theory can also be found in large quantities on mathematical contests for students (such as the International Mathematical Olympiad) and, consequently, in collections of problems from these contests, such as the AoPS collection of IMO Shortlist problems [AoPS-ISL]. Finally, some elementary results in graph theory double as puzzles (or are related to puzzles), which often has the consequence that they appear on puzzle websites such as Cut-the-Knot [Bogomoln].

The notes you are reading are under construction, and will remain so for at least the whole Spring of 2017. Please let me know of any errors and unclarities you encounter (my email address is `darijgrinberg@gmail.com`)[4]. Thank you!

1.1. Acknowledgments

Thanks to Victor Reiner and Travis Scrimshaw for helpful conversations.

[Your name could be in here!]

2. Introduction

In this chapter, we shall define a first notion of graphs ("first" because there are several others to follow) and various basic notions related to it, and prove some elementary properties thereof. This chapter is meant to give a taste of the whole theory, although (not unexpectedly) it is not a representative sample.

[4]The sourcecode of the notes is also publicly available at `https://github.com/darijgr/nogra` .

2.1. Notations and conventions

Before we get to anything interesting, let me get some technicalities out of the way. Namely, I shall be using the following notations:

- In the following, we use the symbol $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$. (Be warned that some other authors use this symbol for $\{1, 2, 3, \ldots\}$ instead.)
- We let $\mathbb{Q}$ denote the set of all rational numbers; we let $\mathbb{R}$ be the set of all real numbers.
- If X and Y are two sets, then we shall use the notation "$X \to Y,\ x \mapsto E$" (where x is some symbol which has no specific meaning in the current context, and where E is some expression which usually involves x) for "the map from X to Y that sends every $x \in X$ to E". For example, "$\mathbb{N} \to \mathbb{N},\ x \mapsto x^2 + x + 6$" means the map from $\mathbb{N}$ to $\mathbb{N}$ that sends every $x \in \mathbb{N}$ to $x^2 + x + 6$. For another example, "$\mathbb{N} \to \mathbb{Q},\ x \mapsto \dfrac{x}{1+x}$" denotes the map from $\mathbb{N}$ to $\mathbb{Q}$ that sends every $x \in \mathbb{N}$ to $\dfrac{x}{1+x}$. [5]
- If S is a set, then the *powerset* of S means the set of all subsets of S. This powerset will be denoted by $\mathcal{P}(S)$. For example, the powerset of $\{1, 2\}$ is $\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

  Furthermore, if S is a set and k is an integer, then $\mathcal{P}_k(S)$ shall mean the set of all k-element subsets of S. (This is empty if $k < 0$.)

2.2. Simple graphs

As already hinted above, there is not one single concept of a "graph". Instead, there are several mutually related (but not equivalent) concepts of "graph", which are often kept apart by adorning them with adjectives (e.g., "simple graph", "directed

[5] A word of warning: Of course, the notation "$X \to Y,\ x \mapsto E$" does not always make sense; indeed, the map that it stands for might sometimes not exist. For instance, the notation "$\mathbb{N} \to \mathbb{Q},\ x \mapsto \dfrac{x}{1-x}$" does not actually define a map, because the map that it is supposed to define (i.e., the map from $\mathbb{N}$ to $\mathbb{Q}$ that sends every $x \in \mathbb{N}$ to $\dfrac{x}{1-x}$) does not exist (since $\dfrac{x}{1-x}$ is not defined for $x = 1$). For another example, the notation "$\mathbb{N} \to \mathbb{Z},\ x \mapsto \dfrac{x}{1+x}$" does not define a map, because the map that it is supposed to define (i.e., the map from $\mathbb{N}$ to $\mathbb{Z}$ that sends every $x \in \mathbb{N}$ to $\dfrac{x}{1+x}$) does not exist (for $x = 2$, we have $\dfrac{x}{1+x} = \dfrac{2}{1+2} \notin \mathbb{Z}$, which shows that a map from $\mathbb{N}$ to $\mathbb{Z}$ cannot send this x to this $\dfrac{x}{1+x}$). Thus, when defining a map from X to Y (using whatever notation), do not forget to check that it is well-defined (i.e., that your definition specifies precisely one image for each $x \in X$, and that these images all lie in Y). In many cases, this is obvious or very easy to check (I will usually not even mention this check), but in some cases, this is a difficult task.

graph", "loopless graph", "loopless weighted undirected graph", "infinite graph", etc.) or prefixes ("digraph", "multigraph", etc.). Let me first define the simplest one of these:

Definition 2.2.1. A *simple graph* is a pair (V, E), where V is a finite set, and where E is a subset of $\mathcal{P}_2(V)$.

Let us unpack this definition first. The word "simple" in "simple graph" (roughly speaking) has the meaning of "with no bells and whistles"; i.e., it says that the notion of "simple graph" is one of the crudest, most primitive notions of a graph known. It does not mean that everything that can be said about simple graphs is simple (this is far from the case, as we will see below). The condition "E is a subset of $\mathcal{P}_2(V)$" in Definition 2.2.1 can be rewritten as "E is a set of 2-element subsets of V" (since $\mathcal{P}_2(V)$ is the set of all 2-element subsets of V). Thus, a simple graph is a pair consisting of a finite set V, and a set of 2-element subsets of V. For example, $(\{1,2,3\}, \{\{1,3\}, \{3,2\}\})$ and $(\{2,5\}, \{\{2,5\}\})$ and $(\varnothing, \varnothing)$ are three simple graphs.

Warning 2.2.2. **(a)** Our Definition 2.2.1 differs from the definition of a "simple graph" in many sources, in that we are requiring V to be finite.

(b) Simple graphs are often just called "graphs". But then again, some other concepts of graphs (such as multigraphs, which we will encounter further below) are also often just called "graphs". Thus, the precise meaning of the word "graph" depends on the context in which it appears. For example, Bollobás (in [Bollob79]) uses the word "graph" for "simple graph", whereas Bondy and Murty (in [BonMur08]) use it for "multigraph with loops allowed" (a concept we will define further below). When reading literature, always check the definitions (and, if these are missing, try to take an educated guess, ruling out options that make some of the claims false).

So far, we have not explained how we should intuitively think of simple graphs, and why they are interesting. We will spend a significant part of these notes answering the latter question; but let us first comment on the former.

Simple graphs can be used to model symmetric relations between different objects. For example, if you have n integers (for some $n \in \mathbb{N}$), then you can define a graph (V, E) for which V is the set of these n integers, and E is the set of all 2-element subsets $\{u, v\}$ of V for which $|u - v| \leq 3$. (Notice that $\{u, u\}$ does not count as a 2-element subset.) For a non-mathematical example, consider an (idealized) group P of (finitely many) people, each two of which are either mutual friends or not[6]. Then, you can define a graph (P, E), where E is the set of all 2-element subsets $\{u, v\}$ of P for which u and v are mutual friends. This graph then models the friendships between the people in the group P; in a sense, it is a social network

[6] We assume that if a person u is a friend of a person v, then v is a friend of u. We also assume that no person u is a friend of u itself (or, at least, we don't count this as friendship).

(similar to the ones widespread on the Internet, but much more rudimentary, since it only knows who is a friend of whom).[7]

The following notations provide a quick way to reference the elements of V and E when given a graph (V, E):

Definition 2.2.3. Let $G = (V, E)$ be a simple graph.

(a) The set V is called the *vertex set* of G; it is denoted by $\mathrm{V}(G)$. (Notice that the letter "V" in "$\mathrm{V}(G)$" is upright, as opposed to the letter "V" in "(V, E)", which is italic. These are two different symbols, and have different meanings: The letter V stands for the specific set V which is the first component of the pair G, whereas the letter V is part of the notation $\mathrm{V}(G)$ for the vertex set of any graph. Thus, if $H = (W, F)$ is another graph, then $\mathrm{V}(H)$ is W, not V.)

The elements of V are called the *vertices* (or the *nodes*) of G.

(b) The set E is called the *edge set* of G; it is denoted by $\mathrm{E}(G)$. (Again, the letter "E" in "$\mathrm{E}(G)$" is upright, and stands for a different thing than the "E".)

The elements of E are called the *edges* of G. When u and v are two elements of V, we shall often use the notation uv for $\{u, v\}$; thus, each edge of G has the form uv for two distinct elements u and v of V. Of course, we always have $uv = vu$.

Notice that each simple graph G satisfies $G = (\mathrm{V}(G), \mathrm{E}(G))$.

(c) Two vertices u and v of G are said to be *adjacent* (or *connected by an edge*) if $uv \in E$ (that is, if uv is an edge of G). In this case, the edge uv is said to *connect* u and v; the vertices u and v are called the *endpoints* of this edge. When the graph G is not obvious from the context, we shall often say "adjacent in G" instead of just "adjacent".

Two vertices u and v of G are said to be *non-adjacent* if they are not adjacent (i.e., if $uv \notin E$).

We say that a vertex u of G is *adjacent to* a vertex v of G if the vertices u and v are adjacent (i.e., if $uv \in E$). Similarly, we say that a vertex u of G is *non-adjacent to* a vertex v of G if the vertices u and v are non-adjacent (i.e., if $uv \notin E$).

(d) Let v be a vertex of G (that is, $v \in V$). Then, the *neighbors* of v are the vertices u of G that satisfy $vu \in E$. In other words, the *neighbors* of v are the vertices of G that are adjacent to v. (Of course, these neighbors depend on both v and G. When G is not clear from the context, we shall call them the "neighbors of v in G" instead of just "neighbors of v".)

[7] This kind of "social graphs" has been used for many years as a language for stating theorems about graphs without saying the word "graph" (and without using mathematical notation): Just speak of people and their mutual friendships. This language was in use long before the Internet and actual social networks came about.

Of course, the relation of adjacency is symmetric[8]. The same holds for the relation of non-adjacency.

Example 2.2.4. Let U be the 5-element set $\{1,2,3,4,5\}$. Let E be the subset $\{\{u,v\} \in \mathcal{P}_2(U) \mid u+v \geq 5\}$ of $\mathcal{P}_2(U)$. This set E is well-defined, because the sum $u+v$ of two integers u and v depends only on the set $\{u,v\}$ and not on how this set is written (since $u+v=v+u$). (This is important, because if we had used the condition $u-v \geq 3$ instead of $u+v \geq 5$, then the set E would not be well-defined, because it would not be clear whether $\{1,5\}$ should be inside it or not – indeed, if we write $\{1,5\}$ as $\{u,v\}$ with $u=5$ and $v=1$, then $u-v \geq 3$ is satisfied, but if we write $\{1,5\}$ as $\{u,v\}$ with $u=1$ and $v=5$, then $u-v \geq 3$ is not satisfied.)

Let G be the simple graph (U,E). Then, $\mathrm{V}(G) = U = \{1,2,3,4,5\}$ and

$$\begin{aligned}\mathrm{E}(G) = E &= \{\{u,v\} \in \mathcal{P}_2(U) \mid u+v \geq 5\} \\ &= \{\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}. \qquad (1)\end{aligned}$$

Thus, G has $|\mathrm{V}(G)| = |U| = 5$ vertices and $|\mathrm{E}(G)| = |E| = 8$ edges.

Using the shorthand notation uv for $\{u,v\}$ (introduced in Definition 2.2.3 **(b)**), the equality (1) rewrites as

$$\mathrm{E}(G) = \{14,15,23,24,25,34,35,45\}.$$

The vertices 2 and 4 of G are adjacent (since $24 \in E$). In other words, 4 is a neighbor of 2. Equivalently, 2 is a neighbor of 4. On the other hand, the vertices 1 and 3 of G are not adjacent (since $13 \notin E$); thus, 1 is not a neighbor of 3. The neighbors of 1 are 4 and 5.

2.3. Drawing graphs

There is a common method to represent graphs visually: Namely, a graph can be drawn as a set of points in the plane and a set of curves connecting some of these points with each other.

More precisely:

Definition 2.3.1. A simple graph G can be visually represented by *drawing* it on the plane (or on some other surface). To do so, we represent each vertex of G by a point (at which we put the name of the vertex), and then, for each edge uv of G, we draw a curve that connects the point representing u with the point representing v. The positions of the points and the shapes of the curves can be chosen freely, as long as they allow the reader to unambigously reconstruct the graph G from the picture. (Thus, for example:

[8]This means the following: Given two vertices u and v of a simple graph G, the vertex u is adjacent to v if and only if v is adjacent to u.

- the points should be distinct and spaced reasonably far apart;
- the curves should not pass through points other than the ones they connect;
- the curves should not be too jagged (and they should certainly be contiguous);
- the curves should intersect in such a way that one can easily see which strand is the continuation of which one.

Various examples of graphs being drawn are given below.)

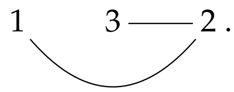
Example 2.3.2. Let us draw some simple graphs.

(a) The simple graph $(\{1,2,3\}, \{12,23\})$ (where we are again using the shorthand notation uv for $\{u,v\}$) can be drawn as follows:

$$1 \text{——} 2 \text{——} 3 .$$

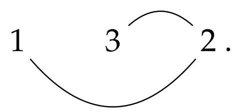
This is (in a sense) the simplest way to draw this graph: The edges are represented by straight lines (rather than tortuous curves).

There are other options as well. For example, we can draw the same graph as follows:



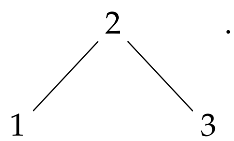
Here, we have placed the points representing the vertices $1,2,3$ differently. As a consequence, we were not able to draw the edge 12 as a straight line, because it would then have overlapped with the vertex 3, which would make the structure of the graph ambiguous (the edge 12 could be mistaken for two edges 13 and 32).

We can just as well draw both edges as curves:

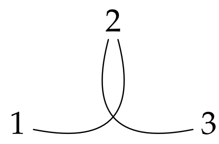


This drawing still represents the same graph $(\{1,2,3\}, \{12,23\})$.

We do not have to place the three vertices all along the same line. We can just as well draw the graph $(\{1,2,3\}, \{12,23\})$ as follows:



So far we have drawn our edges in such a way that they did not intersect. But this is not a requirement. We could just as well have represented our graph $(\{1,2,3\},\{12,23\})$ as follows:



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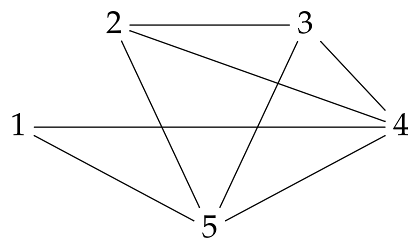
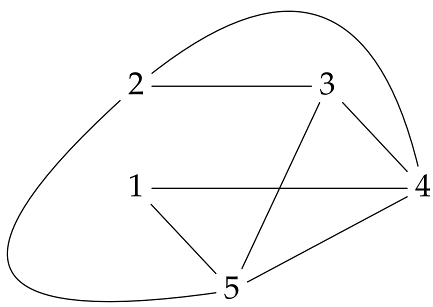
This last drawing is, of course, needlessly complicated for such a graph, but it is perfectly legitimate.

(b) Let U, E and G be as in Example 2.2.4. Here is one way to draw the graph G:



Here is another way to draw the same graph G, with fewer intersections between edges:



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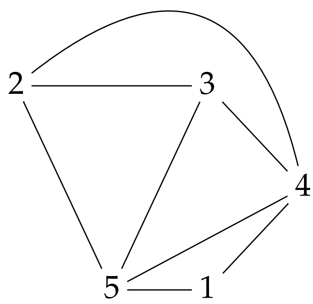
By appropriately repositioning the points corresponding to the five vertices of G, we can actually get rid of all intersections; namely, we can draw the graph G as follows:



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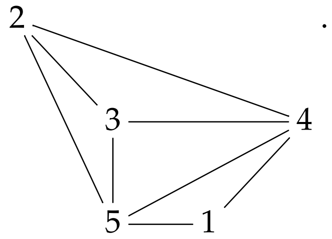
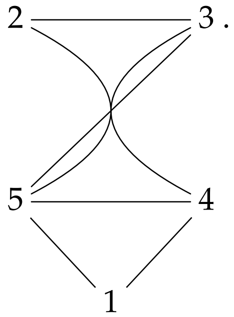
If we reposition the vertices further, we can even achieve a drawing without intersecting curves that uses only straight lines as curves:



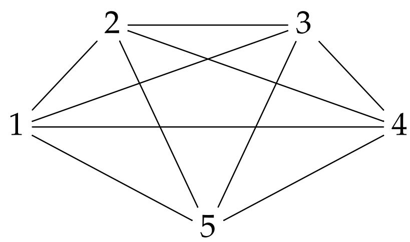
All of these drawings are equally legitimate; some are more convenient for certain purposes while others are less so.

Let us now show a **bad** drawing of the graph G (that is, a drawing that fails the "allow the reader to unambigously reconstruct the graph G from the picture" criterion):

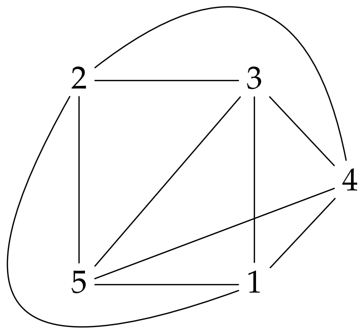


This drawing is bad because of the muddle that happens in the center of the rectangle formed by the vertices 2, 3, 4 and 5. Two curves are touching each other at that center, making the reader wonder whether they are two curves representing the edges 24 and 35 "passing through each other", or two curves representing the edges 25 and 34 "bouncing off each other", or both, or maybe even three curves. (The answer is "both": they represent all four edges 24, 35, 25 and 34. But this is not clear from the drawing.)

(c) Let us draw one further graph: the simple graph $(\{1,2,3,4,5\}, \mathcal{P}_2(\{1,2,3,4,5\}))$. This is the simple graph whose vertices are $1,2,3,4,5$, and whose edges are all possible two-element sets consisting of its vertices (i.e., each pair of two distinct vertices is adjacent). We shall later (in Definition 2.7.1) refer to this graph as the "complete graph K_5". Here is a simple way to draw this graph:



This drawing is useful for many purposes; for example, it makes the abstract symmetry of this graph (i.e., the fact that, roughly speaking, its vertices $1, 2, 3, 4, 5$ are "equal in rights") obvious. But sometimes, you might want to draw it differently, to minimize the number of intersecting curves. Here is a drawing with fewer intersections:



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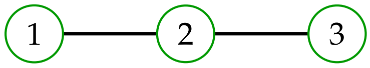
In this drawing, we have only one intersection between two curves left. Can we get rid of all intersections?

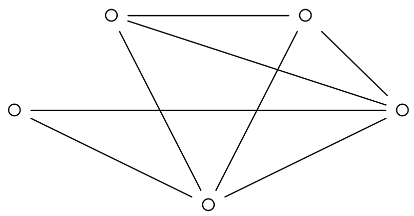
This is a non-combinatorial question, since it really is about curves in the plane rather than about finite sets and graphs. The answer is "no". (That is, no matter how you draw this graph in the plane, you will always have at least one pair of curves intersect.) This is a classical result (one of the first theorems in the theory of *planar graphs*), and proofs of it can be found in various textbooks on graph theory (e.g., [Harary69, Corollary 11.1(d)] or [BonMur08, Theorem 10.2]). These proofs, however, are usually not self-contained; they rely on some basic facts from the topology of the real plane (mostly, the Jordan curve theorem). We shall not study geometrical questions like this in these notes, but instead refer the reader to texts such as [FriFri98] for careful and complete treatments of the topological technicalities. (If you are willing to take certain intuitively obvious topological facts about curves for granted, you can also read the chapters about planar graphs appearing in most books on graph theory.)

We note that some authors prefer to put the labels on the nodes in little circles when drawing a graph. For example, the graph from Example 2.3.2 **(a)** then looks as follows:



Remark 2.3.3. When drawing a simple graph G, we have so far labelled the points by the names of the vertices that they represent. However, often, the names of the vertices will be unimportant. In such cases, we can just as well label the points by little circles. For example, the simple graph G from Example 2.2.4

can thus be drawn as follows:



Of course, from such a drawing, we **cannot** unambiguously reconstruct the graph, since we do not know which edge connects which vertices. But often, all we want to know about the graph is visible on such a drawing, and the names of the vertices would only be distracting.

2.4. A first fact: The Ramsey number $R(3,3) = 6$

After these definitions, it might be time for a first result. The following classical fact (which is actually the beginning of a deep theory – the so-called *Ramsey theory*) should neatly illustrate the concepts introduced above:

Proposition 2.4.1. Let G be a simple graph with $|\mathrm{V}(G)| \geq 6$ (that is, G has at least 6 vertices). Then, at least one of the following two statements holds:

- *Statement 1:* There exist three distinct vertices a, b and c of G such that ab, bc and ca are edges of G.
- *Statement 2:* There exist three distinct vertices a, b and c of G such that none of ab, bc and ca is an edge of G.

In other words, Proposition 2.4.1 says that if a graph G has at least 6 vertices, then we can either find three distinct vertices that are mutually adjacent[9] or find three distinct vertices that are mutually non-adjacent (i.e., no two of them are adjacent), or both.

Proposition 2.4.1 can be stated more tersely as follows: "In any graph containing at least six vertices, you can always find three vertices that are mutually adjacent, or three vertices that are mutually non-adjacent". It is also often restated as follows: "In any group of at least six people, you can always find three that are (pairwise) friends to each other, or three no two of whom are friends" (provided that friendship is a symmetric relation). This follows the old paradigm of restating facts about graphs in terms of people and friendship.

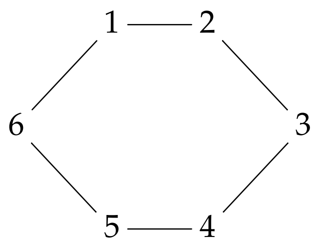
[9]by which we mean (of course) that any two **distinct** ones among these three vertices are adjacent

Example 2.4.2. Let us show four examples of graphs G to which Proposition 2.4.1 applies, as well as an example to which it does not (because it fails to satisfy the condition $|\operatorname{V}(G)| \geq 6$):

(a) Let G be the graph (V, E), where

$$V = \{1,2,3,4,5,6\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,1\}\}.$$

(This graph can be drawn in such a way as to look like a hexagon:



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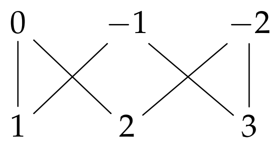
) This graph satisfies Statement 2 of Proposition 2.4.1 for $a = 1$, $b = 3$ and $c = 5$, because its three vertices 1, 3 and 5 are mutually non-adjacent (i.e., none of 13, 35 and 51 is an edge of G). It also satisfies Statement 2 of Proposition 2.4.1 for $a = 2$, $b = 4$ and $c = 6$. So in this situation, we witness something slightly stronger than what Proposition 2.4.1 says: There are at least *two* choices of a, b and c making one of the Statements 1 and 2 valid (and these two choices are not the same up to order). See Exercise 2.4.6 **(a)** below.

(b) Let G be the simple graph (V, E), where

$$V = \{-2, -1, 0, 1, 2, 3\} \qquad \text{and}$$
$$E = \{\{u, v\} \in \mathcal{P}_2(V) \mid u \not\equiv v \mod 3,$$
$$\text{and exactly one of } u \text{ and } v \text{ is positive}\}.$$

(This graph can be drawn as follows:



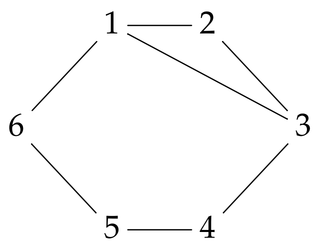
.

) This graph satisfies Statement 2 of Proposition 2.4.1 for $a = 1$, $b = 2$ and $c = 3$. (Again, other choices of a, b and c are also possible.)

(c) Let G be the graph (V, E), where

$$V = \{1,2,3,4,5,6\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,1\}, \{1,3\}\}.$$

(This graph can be drawn in such a way as to look like a hexagon with one extra diagonal:



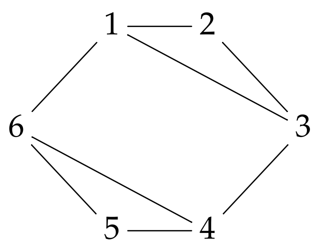
.

) This graph satisfies Statement 1 of Proposition 2.4.1 for $a = 1$, $b = 2$ and $c = 3$. It also satisfies Statement 2 of Proposition 2.4.1 for $a = 2$, $b = 4$ and $c = 6$.

(d) Let G be the graph (V, E), where

$$V = \{1, 2, 3, 4, 5, 6\} \qquad \text{and}$$
$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}, \{1, 3\}, \{4, 6\}\}.$$

(This graph can be drawn in such a way as to look like a hexagon with two extra diagonals:



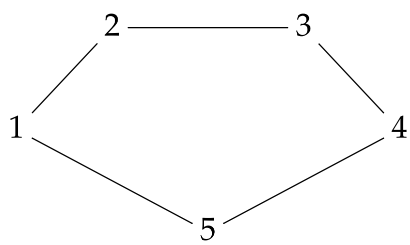
.

) This graph no longer satisfies Statement 2 of Proposition 2.4.1, but it satisfies Statement 1 for $a = 1$, $b = 2$ and $c = 3$.

(e) Let G be the graph (V, E), where

$$V = \{1, 2, 3, 4, 5\} \qquad \text{and}$$
$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}.$$

(This graph can be drawn to look like a pentagon:



) Proposition 2.4.1 says nothing about this graph, since this graph does not satisfy the assumption of Proposition 2.4.1 (in fact, its number of vertices $|\mathrm{V}(G)|$ fails to be ≥ 6). By itself, this does not yield that the claim of Proposition 2.4.1 is false for this graph. However, it is easy to check that the claim actually **is** false for this graph: Neither Statement 1 nor Statement 2 hold.

Proof of Proposition 2.4.1. We need to prove that either Statement 1 holds or Statement 2 holds (or both).

Choose any vertex $u \in \mathrm{V}(G)$. (This is clearly possible, since $|\mathrm{V}(G)| \geq 6 \geq 1$.) Then, $|\mathrm{V}(G) \setminus \{u\}| = |\mathrm{V}(G)| - 1 \geq 5$ (since $|\mathrm{V}(G)| \geq 6$). We are in one of the following two cases:

Case 1: At least 3 vertices in $\mathrm{V}(G) \setminus \{u\}$ are adjacent to u.

Case 2: At most 2 vertices in $\mathrm{V}(G) \setminus \{u\}$ are adjacent to u.

Let us consider Case 1 first. In this case, at least 3 vertices in $\mathrm{V}(G) \setminus \{u\}$ are adjacent to u. Hence, we can find three distinct vertices p, q and r in $\mathrm{V}(G) \setminus \{u\}$ that are adjacent to u. Consider these p, q and r. If none of pq, qr and rp is an edge of G, then Statement 2 holds (in fact, we can just take $a = p$, $b = q$ and $c = r$). Thus, if none of pq, qr and rp is an edge of G, then our proof is complete[10]. Thus, we WLOG[11] assume that at least one pq, qr and rp is an edge of G. In other words, we can pick two distinct elements g and h of $\{p, q, r\}$ such that gh is an edge of G. Consider these g and h.

The vertex g is one of p, q and r (since $g \in \{p, q, r\}$). The vertices p, q and r are adjacent to u. Hence, the vertex g is adjacent to u (since the vertex g is one of p, q and r). In other words, ug is an edge of G. Similarly, uh is an edge of G. In other words, hu is an edge of G (since $hu = uh$).

We have $g \in \{p, q, r\} \subseteq \mathrm{V}(G) \setminus \{u\}$ (since p, q and r belong to $\mathrm{V}(G) \setminus \{u\}$). Hence, $g \neq u$. In other words, $u \neq g$. Similarly, $u \neq h$. Hence, $h \neq u$. Finally, $g \neq h$ (since g and h are distinct). Now, we know that the three vertices u, g and h are distinct (since $u \neq g$, $g \neq h$ and $h \neq u$), and have the property that ug, gh and hu are edges of G. Therefore, Statement 1 holds (in fact, we can just take $a = u$, $b = g$ and $c = h$). Hence, the proof is complete in Case 1.

Let us now consider Case 2. In this case, at most 2 vertices in $\mathrm{V}(G) \setminus \{u\}$ are adjacent to u. Thus, at least 3 vertices in $\mathrm{V}(G) \setminus \{u\}$ are non-adjacent to u [12].

---

[10] because our goal in this proof is to show that either Statement 1 holds or Statement 2 holds (or both)

[11] The word "WLOG" means "without loss of generality". See, e.g., the corresponding Wikipedia page `https://en.wikipedia.org/wiki/Without_loss_of_generality` for its meaning.

[12] *Proof.* Let k be the number of vertices in $\mathrm{V}(G) \setminus \{u\}$ that are adjacent to u. Let ℓ be the number of vertices in $\mathrm{V}(G) \setminus \{u\}$ that are non-adjacent to u. Then, $k + \ell = |\mathrm{V}(G) \setminus \{u\}|$ (since each vertex in $\mathrm{V}(G) \setminus \{u\}$ is either adjacent to u or non-adjacent to u, but not both at the same time). But $k \leq 2$ (since at most 2 vertices in $\mathrm{V}(G) \setminus \{u\}$ are adjacent to u). Hence, $k + \ell \leq 2 + \ell = \ell + 2$, so that $\ell + 2 \geq k + \ell = |\mathrm{V}(G) \setminus \{u\}| \geq 5$ and thus $\ell \geq 3$. In other words, at least 3 vertices in $\mathrm{V}(G) \setminus \{u\}$ are non-adjacent to u. Qed.

Hence, we can find three distinct vertices p, q and r in $\operatorname{V}(G) \setminus \{u\}$ that are non-adjacent to u. Consider these p, q and r. If all of pq, qr and rp are edges of G, then Statement 1 holds (in fact, we can just take $a = p$, $b = q$ and $c = r$). Thus, if all of pq, qr and rp are edges of G, then our proof is complete. Thus, we WLOG assume that not all of pq, qr and rp are edges of G. In other words, at least one of pq, qr and rp is **not** an edge of G. In other words, we can pick two distinct elements g and h of $\{p, q, r\}$ such that gh is not an edge of G. Consider these g and h.

The vertex g is one of p, q and r (since $g \in \{p, q, r\}$). The vertices p, q and r are non-adjacent to u. Hence, the vertex g is non-adjacent to u (since the vertex g is one of p, q and r). In other words, ug is not an edge of G. Similarly, uh is not an edge of G. In other words, hu is not an edge of G (since $hu = uh$).

We have $g \in \{p, q, r\} \subseteq \operatorname{V}(G) \setminus \{u\}$ (since p, q and r belong to $\operatorname{V}(G) \setminus \{u\}$). Thus, $g \neq u$. In other words, $u \neq g$. Similarly, $u \neq h$. Finally, $g \neq h$ (since g and h are distinct). Now, we know that the three vertices u, g and h are distinct (since $u \neq g$, $g \neq h$ and $h \neq u$), and have the property that none of ug, gh and hu is an edge of G (since ug is not an edge of G, since gh is not an edge of G, and since hu is not an edge of G). Therefore, Statement 2 holds (in fact, we can just take $a = u$, $b = g$ and $c = h$). Hence, the proof is complete in Case 2.

We have now proven Proposition 2.4.1 in each of the two Cases 1 and 2. Thus, the proof of Proposition 2.4.1 is complete. □

Remark 2.4.3. I have written the above proof in much detail, since it is the first proof in these notes. I could have easily made it much shorter if I had relied on the reader to fill in some details (and in fact, I **will** rely on the reader in similar situations further below). The proof can further be shortened by noticing that part of the argument for Case 2 was a "mirror version" of the argument for Case 1, with the only difference that "adjacent" is replaced by "non-adjacent" (and vice versa), and "is an edge" is replaced by "is not an edge" (and vice versa).

Remark 2.4.4. Let me observe that Proposition 2.4.1 could be proven by brute force as well (using a computer). Indeed, here is how such a proof would proceed: Let $x_1, x_2, x_3, x_4, x_5, x_6$ be six distinct vertices of G. (Such six vertices exist, since $|\operatorname{V}(G)| \geq 6$.) Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the set of these six vertices. Notice that the set X has 6 elements, and thus the set $\mathcal{P}_2(X)$ has $\dbinom{6}{2} = 15$ elements. Let F be the set of all edges uv of G for which both u and v belong to X. (In other words, F is the set of all edges of G having the form $x_i x_j$.) Clearly, it suffices to prove Proposition 2.4.1 for the graph (X, F) instead of G (because if we have found, for example, three distinct vertices a, b and c of (X, F) such that ab, bc and ca are edges of (X, F), then these a, b and c are obviously also three vertices of G such that ab, bc and ca are edges of G). However, F is a subset of $\mathcal{P}_2(X)$. Since there are only finitely many subsets of $\mathcal{P}_2(X)$ (in fact, there are 2^{15} such subsets, since $\mathcal{P}_2(X)$ has 15 elements), we thus see that there are only finitely many choices for F (when X is being regarded as fixed). We can check, for each of these choices, whether the graph (X, F) satisfies Proposition 2.4.1. (Just try each possible choice of three distinct vertices a, b and c of this graph (X, F), and check that at least one of these choices satisfies either Statement 1 or Statement 2.) After a huge but finite amount of checking (which you can automate), you will see that Proposition 2.4.1 holds for (X, F). Thus, as we have already mentioned, Proposition 2.4.1 also holds for the original graph G.

Proposition 2.4.1 is the first result in a field of graph theory known as *Ramsey theory*. I shall not dwell on this field in these notes, but let me make a few more remarks. The first step beyond Proposition 2.4.1 is the following generalization:

Proposition 2.4.5. Let r and s be two positive integers. Let G be a simple graph with $|\operatorname{V}(G)| \geq \dbinom{r+s-2}{r-1}$. Then, at least one of the following two statements holds:

- *Statement 1:* There exist r distinct vertices of G that are mutually adjacent (i.e., each two distinct ones among these r vertices are adjacent).
- *Statement 2:* There exist s distinct vertices of G that are mutually non-adjacent (i.e., no two distinct ones among these s vertices are adjacent).

Applying Proposition 2.4.5 to $r = 3$ and $s = 3$, we can recover Proposition 2.4.1.

One might wonder whether the number $\dbinom{r+s-2}{r-1}$ in Proposition 2.4.5 can be improved – i.e., whether we can replace it by a smaller number without making Proposition 2.4.5 false. In the case of $r = 3$ and $s = 3$, this is impossible, because the number 6 in Proposition 2.4.1 cannot be made smaller[13]. However, for some other values of r and s, the value $\dbinom{r+s-2}{r-1}$ can be improved. (For example, for $r = 4$ and $s = 4$, the best possible value is 18 rather than $\dbinom{4+4-2}{4-1} = 20$.) The smallest possible value that could stand in place of $\dbinom{r+s-2}{r-1}$ in Proposition 2.4.5 is called the *Ramsey number* $R(r, s)$; thus, we have just showed that $R(3, 3) = 6$. Finding $R(r, s)$ for higher values of r and s is a hard computational challenge; here are some values that have been found with the help of computers:

$$\begin{array}{llll} R(3,4) = 9; & R(3,5) = 14; & R(3,6) = 18; & R(3,7) = 23; \\ R(3,8) = 28; & R(3,9) = 36; & R(4,4) = 18; & R(4,5) = 25. \end{array}$$

(We are only considering the cases $r \leq s$, since it is easy to see that $R(r, s) = R(s, r)$ for all r and s. Also, the trivial values $R(1, s) = 1$ and $R(2, s) = s + 1$ for $s \geq 2$

[13] Indeed, there is a graph with 5 vertices (namely, the graph G constructed in Example 2.4.2 **(e)**) that satisfies neither Statement 1 nor Statement 2.

are omitted.) The Ramsey number $R(5,5)$ is still unknown (although it is known that $43 \leq R(5,5) \leq 48$). See [Radzis21] for more about the current state of affairs in computing Ramsey numbers.

Proposition 2.4.5 can be further generalized to a result called *Ramsey's theorem*. The idea behind the generalization is to slightly change the point of view, and replace the simple graph G by a complete graph (i.e., a simple graph in which every two distinct vertices are adjacent) whose edges are colored in two colors (say, blue and red). This is a completely equivalent concept, because the concepts of "adjacent" and "non-adjacent" in G can be identified with the concepts of "adjacent through a blue edge" (i.e., the edge connecting them is colored blue) and "adjacent through a red edge", respectively. Statements 1 and 2 then turn into "there exist r distinct vertices that are mutually adjacent through blue edges" and "there exist s distinct vertices that are mutually adjacent through red edges", respectively. From this point of view, it is only logical to generalize Proposition 2.4.5 further to the case when the edges of a complete graph are colored in k (rather than two) colors. The corresponding generalization is known as Ramsey's theorem. We refer to the well-written Wikipedia page `https://en.wikipedia.org/wiki/Ramsey's_theorem` for a treatment of this generalization with proof, as well as a table of known Ramsey numbers $R(r,s)$ and a self-contained (if somewhat terse) proof of Proposition 2.4.5. Ramsey's theorem can be generalized further, and many variations of it can be defined, which are usually subsumed under the label "Ramsey theory".[14] There are many papers and at least one textbook [GrRoSp90] available on Ramsey theory. For elementary introductions, see the Cut-the-knot page `http://www.cut-the-knot.org/Curriculum/Combinatorics/ThreeOrThree.shtml` in [Bogomoln], the above-mentioned Wikipedia article, as well as [Harju14, §4.2], [Bollob79, Chapter VI] and [West01, Section 8.3].

Exercise 2.4.6. Let G be a simple graph. A *triangle* in G means a set $\{a,b,c\}$ of three distinct vertices a, b and c of G such that ab, bc and ca are edges of G. An *anti-triangle* in G means a set $\{a,b,c\}$ of three distinct vertices a, b and c of G such that none of ab, bc and ca is an edge of G. A *triangle-or-anti-triangle* in G is a set that is either a triangle or an anti-triangle. (Of the three words I have just introduced, only "triangle" is standard.)

(a) Assume that $|\mathrm{V}(G)| \geq 6$. Prove that G has at least two triangle-or-anti-triangles. (For comparison: Proposition 2.4.1 shows that G has at least one triangle-or-anti-triangle.)

(b) Assume that $|\mathrm{V}(G)| = m + 6$ for some $m \in \mathbb{N}$. Prove that G has at least $m + 1$ triangle-or-anti-triangles.

---

[14] See [RaWiRa] for a popular view on the philosophy of Ramsey theory (in the wide sense of this word). It should probably be said that mathematicians usually define the word "Ramsey theory" somewhat more restrictively, and not every result of the form "you can find a pattern in any sufficiently large structure" belongs to Ramsey theory; but the rough idea is correct.

2.5. Degrees

Next, we introduce the notion of the *degree* of a vertex of a graph. This is simply the number of edges containing this vertex:

Definition 2.5.1. Let $G = (V, E)$ be a simple graph. Let $v \in V$ be a vertex of G. Then, the *degree* of v (with respect to G) is defined as the number of all edges of G that contain v. This degree is a nonnegative integer, and is denoted by $\deg v$. It is also denoted by $\deg_G v$, when the graph G is not clear from the context.

Thus,

$$\begin{aligned}
\deg v = \deg_G v &= (\text{the number of all edges of } G \text{ that contain } v) && (2)\\
&= (\text{the number of all } e \in E \text{ that contain } v) && (3)\\
&\qquad (\text{since the edges of } G \text{ are precisely the } e \in E)\\
&= |\{e \in E \ \mid \ v \in e\}| . && (4)
\end{aligned}$$

Example 2.5.2. Define the sets U and E as in Example 2.2.4. Let G be the graph (U, E). Then, $\deg 1$ is the number of all edges of G that contain 1. These edges are 14 and 15; hence, their number is 2. Thus, $\deg 1 = 2$. Similarly, $\deg 4 = 4$ (since $\deg 4$ is the number of all edges of G that contain 4, but these edges are 14, 24, 34 and 45). Similarly, $\deg 2 = 3$, $\deg 3 = 3$ and $\deg 5 = 4$.

Here are some different characterizations of degrees in a simple graph:

Proposition 2.5.3. Let $G = (V, E)$ be a simple graph. Let $v \in V$ be a vertex of G. Then,

$$\begin{aligned}
\deg v &= \deg_G v && (5)\\
&= (\text{the number of all neighbors of } v) && (6)\\
&= |\{u \in V \ \mid \ u \text{ is a neighbor of } v\}| && (7)\\
&= |\{u \in V \ \mid \ u \text{ is adjacent to } v\}| && (8)\\
&= |\{u \in V \ \mid \ uv \text{ is an edge of } G\}| && (9)\\
&= |\{u \in V \ \mid \ uv \in E\}| . && (10)
\end{aligned}$$

Proposition 2.5.3 is essentially obvious: It follows from observing that each edge of E that contains v contains exactly one neighbor of v, and that different such edges contain different neighbors of v. For the mere sake of completeness, let me include a formal proof.

Proof of Proposition 2.5.3. Both $\deg v$ and $\deg_G v$ denote the degree of v. Thus, $\deg v = \deg_G v$. Hence, (5) is proven.

Let U be the set of all neighbors of v. Then,

$$|U| = (\text{the number of all neighbors of } v).$$

Let E_v be the subset $\{e \in E \mid v \in e\}$ of E. This is the set of all edges $e \in E$ that contain v.

For each $u \in U$, we have $vu \in E_v$ [15]. Hence, we can define a map $\alpha : U \to E_v$ by setting

$$\alpha(u) = vu \qquad \text{for each } u \in U.$$

Consider this map α. This map α is injective[16] and surjective[17]. Hence, the map α is bijective. Thus, we have found a bijective map α from U to E_v. Therefore, $|E_v| = |U|$.

But $E_v = \{e \in E \mid v \in e\}$ and thus $|E_v| = |\{e \in E \mid v \in e\}| = \deg v$ (by (4)). Hence, $\deg v = |E_v| = |U| =$ (the number of all neighbors of v). This proves (6).

---

[15] *Proof.* Let $u \in U$. Thus, u is a neighbor of v (by the definition of U). In other words, $vu \in E$ (by the definition of a "neighbor"). Furthermore, $v \in \{v, u\} = vu$. Hence, vu is an element $e \in E$ satisfying $v \in e$ (since $vu \in E$ and $v \in vu$). In other words, $vu \in \{e \in E \mid v \in e\} = E_v$. Qed.

[16] *Proof.* Let u_1 and u_2 be two elements of U satisfying $\alpha(u_1) = \alpha(u_2)$. We shall show that $u_1 = u_2$.

The definition of α yields $\alpha(u_1) = vu_1 = \{v, u_1\}$. Hence, $\{v, u_1\} = \alpha(u_1) \in E_v \subseteq E \subseteq \mathcal{P}_2(V)$. Therefore, $\{v, u_1\}$ is a two-element set. Hence, $v \neq u_1$. Therefore, $\{v, u_1\} \setminus \{v\} = \{u_1\}$. Hence, $\{u_1\} = \underbrace{\{v, u_1\}}_{=\alpha(u_1)} \setminus \{v\} = \alpha(u_1) \setminus \{v\}$. The same argument (but applied to u_2 instead of u_1) shows that $\{u_2\} = \alpha(u_2) \setminus \{v\}$. Now,

$$\{u_1\} = \underbrace{\alpha(u_1)}_{=\alpha(u_2)} \setminus \{v\} = \alpha(u_2) \setminus \{v\} = \{u_2\},$$

so that $u_1 \in \{u_1\} = \{u_2\}$ and therefore $u_1 = u_2$.

Now, forget that we fixed u_1 and u_2. We thus have proven that if u_1 and u_2 are two elements of U satisfying $\alpha(u_1) = \alpha(u_2)$, then $u_1 = u_2$. In other words, the map α is injective.

[17] *Proof.* Let $f \in E_v$. Thus, $f \in E_v = \{e \in E \mid v \in e\}$. In other words, f is an element of E satisfying $v \in f$.

We have $f \in E \subseteq \mathcal{P}_2(V)$. Thus, f is a 2-element subset of V. Thus, $|f| = 2$. Since $v \in f$, we have $|f \setminus \{v\}| = |f| - 1 = 1$ (since $|f| = 2$). Hence, $f \setminus \{v\}$ is a 1-element set. Therefore, $f \setminus \{v\} = \{u\}$ for some u. Consider this u. Now, $u \in \{u\} = f \setminus \{v\} \subseteq f \subseteq V$; thus, u is a vertex of G. Moreover, $v \in f$ and thus $f = \underbrace{(f \setminus \{v\})}_{=\{u\}} \cup \{v\} = \{u\} \cup \{v\} = \{u, v\} = \{v, u\} = vu$. Hence, $vu = f \in E$. In other words, u is a neighbor of v (by the definition of "neighbor"). In other words, $u \in U$ (since U is the set of all neighbors of v). The definition of α now yields $\alpha(u) = vu = f$. Hence, $f = \alpha(u) \in \alpha(U)$.

Now, forget that we fixed f. We thus have shown that $f \in \alpha(U)$ for each $f \in E_v$. In other words, $E_v \subseteq \alpha(U)$. In other words, the map α is surjective.

Now,

$$\begin{aligned}
\deg v &= (\text{the number of all neighbors of } v) \\
&= \left| \underbrace{\text{the set of all neighbors of } v}_{=\{u \in V \ \mid \ u \text{ is a neighbor of } v\}} \right| \\
&= \left| \{u \in V \ \mid \ u \text{ is a neighbor of } v\} \right| \\
&= \left| \{u \in V \ \mid \ u \text{ is adjacent to } v\} \right|
\end{aligned}$$

(because for any vertex $u \in V$, the condition "u is a neighbor of v" is equivalent to the condition "u is adjacent to v"). Thus, (7) and (8) are proven.

Furthermore,

$$\begin{aligned}
\deg v &= \left| \{u \in V \ \mid \ u \text{ is a neighbor of } v\} \right| \\
&= \left| \{u \in V \ \mid \ vu \in E\} \right|
\end{aligned}$$

(because for any vertex $u \in V$, the condition "u is a neighbor of v" is equivalent to the condition "$vu \in E$"). Hence,

$$\begin{aligned}
\deg v &= \left| \left\{ u \in V \ \mid \ \underbrace{vu}_{=uv} \in E \right\} \right| = \left| \{u \in V \ \mid \ uv \in E\} \right| \\
&= \left| \{u \in V \ \mid \ uv \text{ is an edge of } G\} \right|
\end{aligned}$$

(because for any vertex $u \in V$, the condition "$uv \in E$" is equivalent to the condition "uv is an edge of G"). Thus, we have proven (10) and (9). The proof of Proposition 2.5.3 is thus complete. □

Remark 2.5.4. Different sources use different notations for the degree of a vertex v of a simple graph G. We call it $\deg v$ (and so do Ore's [Ore90] and the introductory notes [LeLeMe16]); Ore's [Ore74] calls it $\rho(v)$; Bollobás's [Bollob79] and Bondy's and Murty's [BonMur08] and [BonMur76] call it $d(v)$.

At this point, we can state a few simple facts about degrees:

Proposition 2.5.5. Let G be a simple graph. Let $n = |\operatorname{V}(G)|$. Let v be a vertex of G. Then, $\deg v \in \{0, 1, \ldots, n-1\}$.

Proof of Proposition 2.5.5. Let U be the set of all neighbors of v. Then,

$$|U| = (\text{the number of all neighbors of } v).$$

Comparing this with (6), we obtain $|U| = \deg v$.

But $U \subseteq \operatorname{V}(G) \setminus \{v\}$ [18]. Hence, $|U| \leq |\operatorname{V}(G) \setminus \{v\}| = |\operatorname{V}(G)| - 1$ (since $v \in \operatorname{V}(G)$). Since $|\operatorname{V}(G)| = n$, this inequality can be rewritten as $|U| \leq n-1$. Since $|U|$ is a nonnegative integer, we thus have $|U| \in \{0, 1, \ldots, n-1\}$. Since $|U| = \deg v$, we can rewrite this as $\deg v \in \{0, 1, \ldots, n-1\}$. This proves Proposition 2.5.5. □

Proposition 2.5.6. Let G be a simple graph. The sum of the degrees of all vertices of G equals twice the number of edges of G. In other words,

$$\sum_{v \in \operatorname{V}(G)} \deg v = 2 |\operatorname{E}(G)| . \tag{11}$$

Proof of Proposition 2.5.6. Write the simple graph G in the form $G = (V, E)$. Thus, $\operatorname{V}(G) = V$ and $\operatorname{E}(G) = E$.

Now, let N be the number of all pairs $(v, e) \in V \times E$ such that $v \in e$. Then, we can express N in the following two ways:

- We can obtain N by computing, for each $v \in V$, the number of all $e \in E$ satisfying $v \in e$, and then adding up these numbers over all $v \in V$. Thus, we obtain

$$N = \sum_{v \in V} \underbrace{(\text{the number of all } e \in E \text{ satisfying } v \in e)}_{\substack{=(\text{the number of all } e \in E \text{ that contain } v) = \deg v \\ (\text{by (3)})}} = \sum_{v \in V} \deg v. \tag{12}$$

- We can also obtain N by computing, for each $e \in E$, the number of all $v \in V$ satisfying $v \in e$, and then adding up these numbers over all $e \in E$. Thus, we obtain

$$N = \sum_{e \in E} (\text{the number of all } v \in V \text{ satisfying } v \in e) . \tag{13}$$

  But for each $e \in E$, we have

$$(\text{the number of all } v \in V \text{ satisfying } v \in e) = 2$$

[18] *Proof.* Let $u \in U$. Thus, u is a neighbor of v (by the definition of U). In other words, u is a vertex of G such that $vu \in \operatorname{E}(G)$ (by the definition of "neighbor"). Thus, $vu \in \operatorname{E}(G) \subseteq \mathcal{P}_2(\operatorname{V}(G))$ (since each edge of G is a 2-element subset of $\operatorname{V}(G)$). In particular, vu is a 2-element set. Hence, $u \neq v$. But $u \in \operatorname{V}(G)$ (since u is a vertex of G). Combining this with $u \neq v$, we obtain $u \in \operatorname{V}(G) \setminus \{v\}$.

Now, forget that we fixed u. We thus have shown that $u \in \operatorname{V}(G) \setminus \{v\}$ for each $u \in U$. In other words, $U \subseteq \operatorname{V}(G) \setminus \{v\}$.

[19]. Hence, (13) becomes

$$N = \sum_{e \in E} \underbrace{(\text{the number of all } v \in V \text{ satisfying } v \in e)}_{=2} \qquad (14)$$

$$= \sum_{e \in E} 2 = |E| \cdot 2 \qquad (15)$$

(because the sum $\sum_{e \in E} 2$ consists of $|E|$ addends, each of which equals 2).

Comparing (12) with (15), we obtain $\sum_{v \in V} \deg v = |E| \cdot 2 = 2|E|$. Since $\mathrm{V}(G) = V$ and $\mathrm{E}(G) = E$, we can rewrite this as $\sum_{v \in \mathrm{V}(G)} \deg v = 2|\mathrm{E}(G)|$. Thus, Proposition 2.5.6 is proven. □

Remark 2.5.7. The above proof of Proposition 2.5.6 is a classical example of the technique of *double-counting*: i.e., finding two different expressions for one and the same value (which, in itself, is not of much interest), and then obtaining an equality by comparing these two different expressions. In the above proof, the value was N, and the two expressions were (12) and (15).

Proposition 2.5.8. Let G be a simple graph. Then, the number of vertices v of G whose degree $\deg v$ is odd is even.

Proof of Proposition 2.5.8. From (11), we obtain

$$\sum_{v \in \mathrm{V}(G)} \deg v = 2|\mathrm{E}(G)| \equiv 0 \mod 2.$$

Hence,

$$\begin{aligned}
0 &\equiv \sum_{v \in \mathrm{V}(G)} \deg v = \sum_{\substack{v \in \mathrm{V}(G);\\ \deg v \text{ is even}}} \underbrace{\deg v}_{\substack{\equiv 0 \mod 2\\ (\text{since } \deg v \text{ is even})}} + \sum_{\substack{v \in \mathrm{V}(G);\\ \deg v \text{ is odd}}} \underbrace{\deg v}_{\substack{\equiv 1 \mod 2\\ (\text{since } \deg v \text{ is odd})}} \\
&\equiv \underbrace{\sum_{\substack{v \in \mathrm{V}(G);\\ \deg v \text{ is even}}} 0}_{=0} + \sum_{\substack{v \in \mathrm{V}(G);\\ \deg v \text{ is odd}}} 1 \\
&= \sum_{\substack{v \in \mathrm{V}(G);\\ \deg v \text{ is odd}}} 1 = |\{v \in \mathrm{V}(G) \mid \deg v \text{ is odd}\}| \cdot 1 \\
&= |\{v \in \mathrm{V}(G) \mid \deg v \text{ is odd}\}| \\
&= (\text{the number of vertices } v \text{ of } G \text{ whose degree } \deg v \text{ is odd}) \mod 2.
\end{aligned}$$

[19]*Proof.* Let $e \in E$. Then, $e \in E \subseteq \mathcal{P}_2(V)$. In other words, e is a 2-element subset of V. Hence, $|e| = 2$ (since e is a 2-element set).

But the $v \in V$ satisfying $v \in e$ are precisely the $v \in e$ (because e is a subset of V). Hence,

$$(\text{the number of all } v \in V \text{ satisfying } v \in e) = (\text{the number of all } v \in e) = |e| = 2,$$

qed.

In other words, the number of vertices v of G whose degree $\deg v$ is odd is even. This proves Proposition 2.5.8. $\square$

As usual, Proposition 2.5.8 can be restated in terms of friendships among people; this restatement takes the following form: "In a group of (finitely many) people, the number of people having an odd number of friends is even". (Once again, this assumes that friendship is a symmetric relation, and that nobody counts as a friend of his own.)

Proposition 2.5.9. Let G be a simple graph with at least two vertices. Then, there exist two distinct vertices v and w of G having the same degree (that is, having $\deg v = \deg w$).

Proof of Proposition 2.5.9. Assume the contrary. Then, there exist **no** two distinct vertices v and w of G having the same degree. Hence, the degrees of all vertices of G are distinct. In other words, any two distinct vertices v and w of G satisfy

$$\deg v \neq \deg w. \tag{16}$$

Let $n = |\operatorname{V}(G)|$. Proposition 2.5.5 shows that $\deg v \in \{0, 1, \ldots, n-1\}$ for each $v \in \operatorname{V}(G)$. Hence, we can define a map $d : \operatorname{V}(G) \to \{0, 1, \ldots, n-1\}$ by

$$d(v) = \deg v \qquad \text{for each } v \in \operatorname{V}(G).$$

Consider this map d. The map d is injective[20].

Recall the following elementary fact: If X and Y are two finite sets satisfying $|X| = |Y|$, then each injective map from X to Y is bijective[21]. Applying this to $X = \operatorname{V}(G)$, $Y = \{0, 1, \ldots, n-1\}$ and $f = d$, we conclude that the map d is bijective (since $\operatorname{V}(G)$ and $\{0, 1, \ldots, n-1\}$ are finite sets satisfying $|\operatorname{V}(G)| = n = |\{0, 1, \ldots, n-1\}|$, and since $d : \operatorname{V}(G) \to \{0, 1, \ldots, n-1\}$ is an injective map). In particular, d is surjective.

Now, $n = |\operatorname{V}(G)| \geq 2$ (since G has at least two vertices). Hence, $0 \in \{0, 1, \ldots, n-1\}$. Since d is surjective, this shows that there exists some vertex $a \in \operatorname{V}(G)$ such that $d(a) = 0$. Consider this a. The definition of d yields $d(a) = \deg a$, so that $\deg a = d(a) = 0$.

[20] *Proof.* Let v and w be two distinct elements of $\operatorname{V}(G)$. Then, $\deg v \neq \deg w$ (by (16)). But the definition of d yields $d(v) = \deg v$ and $d(w) = \deg w$. Now, $d(v) = \deg v \neq \deg w = d(w)$.

Now, forget that we fixed v and w. We thus have shown that if v and w are two distinct elements of $\operatorname{V}(G)$, then $d(v) \neq d(w)$. In other words, the map d is injective.

[21] Here is a quick *proof* of this fact: Let X and Y be two finite sets satisfying $|X| = |Y|$. Let f be an injective map from X to Y. We must show that f is bijective.

The set Y is finite. Hence, the only subset of Y that has as many elements as Y is Y itself.

The set $f(X)$ consists of the images of the elements of X under the map f. These images are pairwise distinct (since f is injective), and there are precisely $|X|$ many of them. Hence, $|f(X)| = |X|$. Now, $f(X)$ is a subset of Y that has as many elements as Y (since $|f(X)| = |X| = |Y|$). Therefore, $f(X)$ must be Y itself (since the only subset of Y that has as many elements as Y is Y itself). In other words, $Y = f(X)$. Hence, the map f is surjective. Thus, f is bijective (since f is injective and surjective). This proves the fact that we wanted to prove.

Also, $n-1 \in \{0, 1, \ldots, n-1\}$ (since $n \geq 2$). Since d is surjective, this shows that there exists some vertex $b \in \operatorname{V}(G)$ such that $d(b) = n-1$. Consider this b. The definition of d yields $d(b) = \deg b$, so that $\deg b = d(b) = n-1$.

From $n \geq 2$, we obtain $n-1 \geq 1 > 0$, so that $n-1 \neq 0$, and thus $\deg a = 0 \neq n-1 = \deg b$. Hence, $a \neq b$.

Now, let B be the set of all neighbors of b. Then,

$$|B| = (\text{the number of all neighbors of } b).$$

Comparing this with

$$\deg b = (\text{the number of all neighbors of } b) \qquad (\text{by (6), applied to } v = b),$$

we obtain $|B| = \deg b = n-1$.

But $B \subseteq \operatorname{V}(G) \setminus \{a, b\}$ [22]. Hence,

$$\begin{aligned} |B| &\leq |\operatorname{V}(G) \setminus \{a,b\}| = \underbrace{|\operatorname{V}(G)|}_{=n} - \underbrace{|\{a,b\}|}_{\substack{=2 \\ (\text{since } a \neq b)}} \qquad (\text{since } \{a,b\} \subseteq \operatorname{V}(G)) \\ &= n-2 < n-1. \end{aligned}$$

This contradicts $|B| = n-1$. This contradiction shows that our assumption was wrong. Proposition 2.5.9 is proven. □

Degrees of vertices can be used to prove various facts about graphs. For an example, let us show *Mantel's theorem*:

Theorem 2.5.10. Let G be a simple graph. Let $n = |\operatorname{V}(G)|$ be the number of vertices of G. Assume that $|\operatorname{E}(G)| > n^2/4$. (In other words, assume that G has more than $n^2/4$ edges.) Then, there exist three distinct vertices a, b and c of G such that ab, bc and ca are edges of G.

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[22] *Proof.* Write the graph G in the form $G = (V, E)$. Let $v \in B$. Then, v is a neighbor of b (since B is the set of all neighbors of b). In other words, $bv \in E$ (by the definition of "neighbor"). Hence, $bv \in E \subseteq \mathcal{P}_2(V)$. Therefore, bv is a 2-element set. Hence, $v \neq b$.

Also, $vb = bv \in E$. Thus, b is a neighbor of v. Hence, the vertex v has at least one neighbor (namely, b). Now, (6) yields

$$\deg v = (\text{the number of all neighbors of } v) \geq 1$$

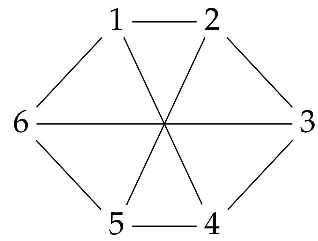
(since the vertex v has at least one neighbor), so that $\deg v \geq 1 > 0 = \deg a$ and thus $\deg v \neq \deg a$ and therefore $v \neq a$. Combining this with $v \neq b$, we find $v \notin \{a, b\}$. Hence, $v \in \operatorname{V}(G) \setminus \{a, b\}$.

Now, forget that we fixed v. We thus have proven that each $v \in B$ satisfies $v \in \operatorname{V}(G) \setminus \{a, b\}$. In other words, $B \subseteq \operatorname{V}(G) \setminus \{a, b\}$.

Example 2.5.11. Let G be the graph (V, E), where

$$V = \{1,2,3,4,5,6\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,1\}, \{1,4\}, \{2,5\}, \{3,6\}\}.$$

(This graph can be drawn in such a way as to look like a hexagon with its three longest diagonals:



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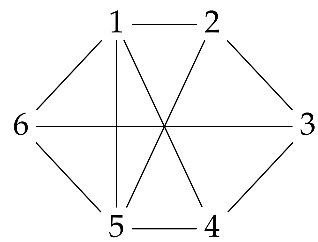
This G does not satisfy the conditions of Theorem 2.5.10, because its number of edges is $|\mathrm{E}(G)| = |E| = 9 = 6^2/4$ (whereas Theorem 2.5.10 requires it to be $> 6^2/4$). Therefore, it should come as no surprise that there exist no three distinct vertices a, b and c of G such that ab, bc and ca are edges of G. (There is a particularly quick way to convince yourself of this: Observe that if u and v are two adjacent vertices of G, then the integer $u + v$ is odd. Therefore, if there were three distinct vertices a, b and c of G such that ab, bc and ca are edges of G, then the integers $a + b$, $b + c$ and $c + a$ would all be odd, and therefore their sum $(a + b) + (b + c) + (c + a)$ would be odd as well. But this is impossible, since this sum is even (in fact, it equals $2(a + b + c)$). Hence, no such three vertices a, b and c exist.)

However, if we add any new edge to G, then the resulting graph G' satisfies $|\mathrm{E}(G')| = 9 + 1 > 6^2/4$, and therefore Theorem 2.5.10 can be applied to G' instead of G. Hence, there exist three distinct vertices a, b and c of G' such that ab, bc and ca are edges of G'. Which three vertices they will be depends on which edge we have added. For example, if we have added the edge 15, then (for example) the three vertices $a = 1$, $b = 5$ and $c = 6$ work. Here is a drawing of this G':



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Of course, the converse of Theorem 2.5.10 does not hold. There are many simple graphs G which do not satisfy the assumption of Theorem 2.5.10, but nevertheless have three distinct vertices a, b and c such that ab, bc and ca are edges of G. The reader can easily construct such a graph.

For proofs of Theorem 2.5.10, see (for example) the rather well-explained [Choo16] (note that the fourth proof is incomplete, as the existence of the maximum S^* needs to be proven).

TODO 2.5.12. Proof.

Remark 2.5.13. Let us contrast Proposition 2.4.1 with Theorem 2.5.10. The former guarantees that a graph G with sufficiently many vertices (namely, at least 6 vertices) must have a triangle or an anti-triangle (where we are using the terminology from Exercise 2.4.6). Theorem 2.5.10 says that any graph G with sufficiently many edges (namely, more than $n^2/4$ edges, where $n = |\operatorname{V}(G)|$) must have a triangle. These two facts are similar and yet different in nature (e.g., the number 6 in Proposition 2.4.1 is a constant, whereas the $n^2/4$ in Theorem 2.5.10 is not). We can also wonder how many edges a graph must have in order to be guaranteed an anti-triangle; this is answered by Exercise 2.5.14 below.

Exercise 2.5.14. Let G be a simple graph. Let $n = |\operatorname{V}(G)|$ be the number of vertices of G. Assume that $|\operatorname{E}(G)| < n(n-2)/4$. (In other words, assume that G has less than $n(n-2)/4$ edges.) Prove that there exist three distinct vertices a, b and c of G such that none of ab, bc and ca are edges of G.

Just as Proposition 2.4.1 is merely the tip of a deep iceberg called Ramsey theory, Theorem 2.5.10 is the beginning of a longer story. The most famous piece of this story is the following fact, known as *Turán's theorem*:

Theorem 2.5.15. Let r be a positive integer. Let G be a simple graph. Let $n = |\operatorname{V}(G)|$ be the number of vertices of G. Assume that $|\operatorname{E}(G)| > \dfrac{r-1}{r} \cdot \dfrac{n^2}{2}$. Then, there exist $r+1$ distinct vertices of G that are mutually adjacent (i.e., each two distinct ones among these $r+1$ vertices are adjacent).

Theorem 2.5.10 is the particular case of Theorem 2.5.15 for $r = 2$. See [Jukna11, Chapter 4, Theorem 4.8] or [Aigner95] or [AigZie, Chapter 41] for proofs of Theorem 2.5.15. Various proofs are also sketched in `https://en.wikipedia.org/wiki/Tur%C3%A1n's_theorem` .

Results like Theorem 2.5.10 (and sometimes also like Proposition 2.4.5) are commonly regarded as part of a subject called *extremal graph theory*. (The word "extremal" refers to the appearance of bounds, such as the $\dfrac{r-1}{r} \cdot \dfrac{n^2}{2}$.) There are textbooks on this subject, such as [Jukna11].

2.6. Graph isomorphisms

Two graphs can be distinct and yet "the same up to the names of their vertices". For instance, the two graphs $(\{1,2,3\}, \{\{1,2\}\})$ and $(\{1,2,3\}, \{\{1,3\}\})$ are distinct (since 12 is an edge of the former graph but not of the latter), but if we rename the vertices 2 and 3 of the former graph as 3 and 2, respectively, then it becomes the latter graph. This kind of relation between two graphs is weaker than (literal) equality, but still strong enough to ensure that (roughly speaking) the two graphs have the same properties (as long as the properties don't refer to specific vertices). Thus, it is worth giving this relation a rigorous definition and a name:

Definition 2.6.1. Let G and H be two simple graphs.

(a) A *graph isomorphism* from G to H means a bijection $\phi : \operatorname{V}(G) \to \operatorname{V}(H)$ such that for every two vertices u and v of G, the following logical equivalence holds:
$$(uv \in \operatorname{E}(G)) \iff (\phi(u)\,\phi(v) \in \operatorname{E}(H)). \qquad (17)$$
(At this point, let me remind you that uv is shorthand for $\{u, v\}$, and similarly $\phi(u)\,\phi(v)$ is shorthand for $\{\phi(u), \phi(v)\}$.)

When it is clear what we mean, we shall abbreviate "graph isomorphism" as "*isomorphism*". We also often write "graph isomorphism $G \to H$" for "graph isomorphism from G to H".

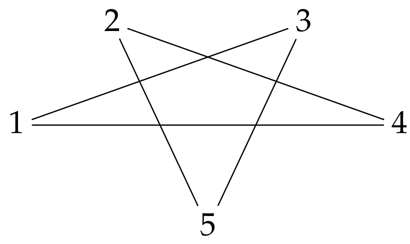
(b) If there exists a graph isomorphism from G to H, then we say the graphs G and H are *isomorphic*, and we write $G \cong H$. Sometimes, the relation $G \cong H$ itself will be called an "isomorphism" (although it is not a map).

Given this definition, we can now rigorously state the relation between the two graphs $(\{1,2,3\}, \{\{1,2\}\})$ and $(\{1,2,3\}, \{\{1,3\}\})$ we observed above. Namely, if we denote these two graphs by G and H, respectively, then the map $\{1,2,3\} \to \{1,2,3\}$ that sends $1,2,3$ to $1,3,2$ (respectively) is a graph isomorphism from G to H. Thus, these two graphs are isomorphic. Note that we did not have to pick two graphs with the same vertex set; isomorphisms are often observed between graphs with completely different vertex sets and different definitions. Also, it sometimes happens that there are several isomorphisms between two graphs.

Example 2.6.2. Consider the graph $G = (V, E)$, where
$$V = \{1,2,3,4,5\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}.$$
(This graph has already been introduced in Example 2.4.2 **(e)**. It can be drawn to look like a pentagon.)

Consider furthermore the graph $H = (V, F)$, where V is as before, and where
$$F = \{\{1,3\}, \{2,4\}, \{3,5\}, \{4,1\}, \{5,2\}\}.$$

(If we draw the vertices $1, 2, 3, 4, 5 \in V$ as the five vertices of a pentagon, then the graph H is the pentagram formed by the diagonals of this pentagon:



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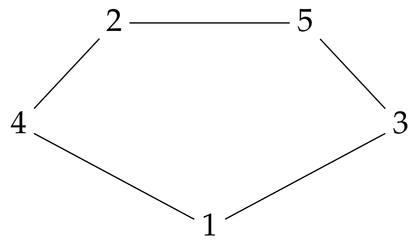
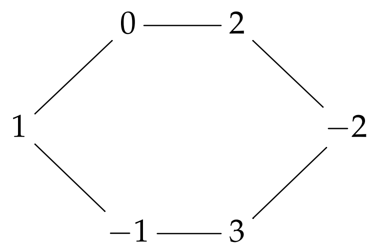
The graphs G and H are distinct, yet isomorphic. One isomorphism $G \to H$ is the bijection $V \to V$ sending $1, 2, 3, 4, 5$ to $1, 3, 5, 2, 4$, respectively. Another is the bijection $V \to V$ sending $1, 2, 3, 4, 5$ to $4, 2, 5, 3, 1$, respectively. There are 10 isomorphisms in total.

One way to quickly (visually) observe that two graphs are isomorphic is to draw them in such a way that the drawings look exactly the same but for the names of the vertices. Given such drawings, one can simply "overlay them" to obtain a graph isomorphism. For example, we could have spotted the isomorphism $G \to H$ in Example 2.6.2 by drawing the graph H as follows:



and overlaying this with the drawing of G made in Example 2.4.2 **(e)**. (Of course, such drawings are not always easy to find: For example, our first drawing of H in Example 2.6.2 looked completely different from our drawing of G.)

Example 2.6.3. The graph G constructed in Example 2.4.2 **(a)** is isomorphic to the graph G constructed in Example 2.4.2 **(b)**, even though this might not be obvious from their definitions. Probably the easiest way to see this is to draw the latter graph G as follows:



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This drawing makes it clear that it is isomorphic to the former graph (after all, the picture is exactly the same; it's just that the vertices are different numbers now). Formally speaking, the map sending $1, 2, 3, 4, 5, 6$ to $0, 2, -2, 3, -1, 1$ (respectively) is a graph isomorphism from the former graph to the latter graph.

Any isomorphism between two graphs is invertible (since, by its definition, it is a bijection). Its inverse, too, is an isomorphism:

Proposition 2.6.4. Let G and H be two simple graphs. Let ϕ be a graph isomorphism from G to H. Then, its inverse ϕ^{-1} is a graph isomorphism from H to G.

Proof of Proposition 2.6.4. The map ϕ is a graph isomorphism from G to H. By the definition of "graph isomorphism", this means that ϕ is a bijection $\operatorname{V}(G) \to \operatorname{V}(H)$ such that for every two vertices u and v of G, the following logical equivalence holds:

$$(uv \in \operatorname{E}(G)) \iff (\phi(u)\,\phi(v) \in \operatorname{E}(H)). \tag{18}$$

Now, for every two vertices u and v of H, the following equivalence holds:

$$(uv \in \operatorname{E}(H)) \iff \left(\phi^{-1}(u)\,\phi^{-1}(v) \in \operatorname{E}(G)\right) \tag{19}$$

(because we have the following chain of equivalences:

$$\begin{aligned}
&\left(\phi^{-1}(u)\,\phi^{-1}(v) \in \operatorname{E}(G)\right) \\
&\iff \left(\underbrace{\phi\left(\phi^{-1}(u)\right)}_{=u}\underbrace{\phi\left(\phi^{-1}(v)\right)}_{=v} \in \operatorname{E}(H)\right) \\
&\qquad \left(\text{by (18), applied to } \phi^{-1}(u) \text{ and } \phi^{-1}(v) \text{ instead of } u \text{ and } v\right) \\
&\iff (uv \in \operatorname{E}(H))
\end{aligned}$$

). Thus, ϕ^{-1} is a bijection $\operatorname{V}(H) \to \operatorname{V}(G)$ such that for every two vertices u and v of H, the equivalence (19) holds. By the definition of "graph isomorphism", this means precisely that ϕ^{-1} is a graph isomorphism from H to G. This proves Proposition 2.6.4. $\square$

The following fact is similarly easy to check:

Proposition 2.6.5. Let G, H and I be three simple graphs. Let ϕ be a graph isomorphism from G to H. Let ψ be a graph isomorphism from H to I. Then, the composition $\psi \circ \phi$ is a graph isomorphism from G to I.

As we have already alluded to, isomorphic graphs are "equal in all but names", and thus share all properties that do not depend on the names of vertices. For example, the following holds:

Proposition 2.6.6. Let G and H be two simple graphs. Let ϕ be a graph isomorphism from G to H.

(a) For every $v \in \operatorname{V}(G)$, we have $\deg_G v = \deg_H(\phi(v))$.

(b) We have $|\operatorname{E}(H)| = |\operatorname{E}(G)|$.

Proof of Proposition 2.6.6. The map ϕ is a graph isomorphism from G to H. By the definition of "graph isomorphism", this means that ϕ is a bijection $\operatorname{V}(G) \to \operatorname{V}(H)$ such that for every two vertices u and v of G, the following logical equivalence holds:

$$(uv \in \operatorname{E}(G)) \iff (\phi(u)\phi(v) \in \operatorname{E}(H)). \tag{20}$$

(a) Fix $v \in \operatorname{V}(G)$. Then, (10) (applied to $\operatorname{V}(G)$ and $\operatorname{E}(G)$ instead of V and E) yields $\deg_G v = |\{u \in \operatorname{V}(G) \mid uv \in \operatorname{E}(G)\}|$ (since $G = (\operatorname{V}(G), \operatorname{E}(G))$). Similarly, (10) (applied to H, $\operatorname{V}(H)$, $\operatorname{E}(H)$ and $\phi(v)$ instead of G, V, E and v) yields

$$\begin{aligned} \deg_H(\phi(v)) &= |\{u \in \operatorname{V}(H) \mid u\phi(v) \in \operatorname{E}(H)\}| \\ &= |\{w \in \operatorname{V}(H) \mid w\phi(v) \in \operatorname{E}(H)\}| \end{aligned} \tag{21}$$

(here, we have renamed the index u as w). But recall that $\phi : \operatorname{V}(G) \to \operatorname{V}(H)$ is a bijection. Thus, each $w \in \operatorname{V}(H)$ can be written uniquely in the form $\phi(u)$ for some $u \in \operatorname{V}(G)$. Thus, we can substitute $\phi(u)$ for w in $\{w \in \operatorname{V}(H) \mid w\phi(v) \in \operatorname{E}(H)\}$. We thus find

$$\{w \in \operatorname{V}(H) \mid w\phi(v) \in \operatorname{E}(H)\} = \{\phi(u) \mid u \in \operatorname{V}(G) \text{ and } \phi(u)\phi(v) \in \operatorname{E}(H)\}.$$

Hence, (21) becomes

$$\begin{aligned} \deg_H(\phi(v)) &= \left| \underbrace{\{w \in \operatorname{V}(H) \mid w\phi(v) \in \operatorname{E}(H)\}}_{=\{\phi(u) \mid u \in \operatorname{V}(G) \text{ and } \phi(u)\phi(v) \in \operatorname{E}(H)\}} \right| \\ &= |\{\phi(u) \mid u \in \operatorname{V}(G) \text{ and } \phi(u)\phi(v) \in \operatorname{E}(H)\}| \\ &= \left| \left\{ u \in \operatorname{V}(G) \mid \underbrace{\phi(u)\phi(v) \in \operatorname{E}(H)}_{\substack{\iff (uv \in \operatorname{E}(G)) \\ \text{(by (20))}}} \right\} \right| \qquad \text{(since } \phi \text{ is a bijection)} \\ &= |\{u \in \operatorname{V}(G) \mid uv \in \operatorname{E}(G)\}| = \deg_G v. \end{aligned}$$

This proves Proposition 2.6.6 **(a)**.

(b) Define a map $\phi' : \operatorname{E}(G) \to \operatorname{E}(H)$ as follows: Let $e \in \operatorname{E}(G)$. Write the edge e in the form uv for two distinct vertices u and v. Then, $uv = e \in \operatorname{E}(G)$.

Thus, (20) shows that $\phi(u)\phi(v) \in \operatorname{E}(H)$. Moreover, this new edge $\phi(u)\phi(v)$ is independent of the choice of u and v (as long as e is fixed)[23]. Hence, we can set $\phi'(e) = \phi(u)\phi(v)$. Doing so, we obtain a map $\phi' : \operatorname{E}(G) \to \operatorname{E}(H)$. This map ϕ' is a bijection[24]. Thus, we have found a bijection from $\operatorname{E}(G)$ to $\operatorname{E}(H)$. Consequently, $|\operatorname{E}(H)| = |\operatorname{E}(G)|$. This proves Proposition 2.6.6 **(b)**. $\square$

Results similar to Proposition 2.6.6 (saying that isomorphic graphs share the same properties) can be stated easily for any property of graphs you can imagine (as long as the property does not depend on the names of the vertices); the proofs are always straightforward like the one given above (the only idea being to apply bijectivity of ϕ and the equivalence (20) over and over in order to "transfer knowledge" from one graph to the other). I shall not state such results explicitly; instead, I shall merely refer to the overarching idea that isomorphic graphs share the same properties whenever it becomes useful.

One use of graph isomorphisms is to "relabel vertices" in a proof: Instead of proving some property of a given graph G, we can just as well prove the same property for a graph isomorphic to G (since isomorphic graphs share the same properties), i.e., a graph obtained from G by renaming its vertices. The vertices can be renamed as we wish; a popular choice is to rename them as $1, 2, \ldots, n$ (where $n = |\operatorname{V}(G)|$). Formally speaking, this renaming is possible because of the following fact:

Proposition 2.6.7. Let G be a simple graph. Let S be a finite set such that $|S| = |\operatorname{V}(G)|$. Then, there exists a simple graph H that is isomorphic to G and has vertex set $\operatorname{V}(H) = S$.

Proof of Proposition 2.6.7. We have $|S| = |\operatorname{V}(G)|$. Thus, there exists a bijection $\phi : \operatorname{V}(G) \to S$. Fix such a ϕ. Define a set F by

$$F = \{\phi(e) \mid e \in \operatorname{E}(G)\} = \{\phi(u)\phi(v) \mid uv \in \operatorname{E}(G)\}$$

(where $\phi(e)$, as usual, denotes the subset $\{\phi(x) \mid x \in e\}$ of S). Now, let H be the simple graph (S, F). Then, $\operatorname{V}(H) = S$. Hence, ϕ is a bijection $\operatorname{V}(G) \to \operatorname{V}(H)$. It is straightforward to check that ϕ is a graph isomorphism from G to H (indeed, the definition of F was tailored precisely to make the equivalence (17) hold). Thus,

[23]There are two ways of choosing u and v, since the elements of a 2-element set (like e) can be listed in two different orders. We are claiming that the edge $\phi(u)\phi(v)$ does not depend on the choice of order. The slickest way of proving this is to rewrite this edge in a form that makes no reference to the choice of u and v to begin with. Namely: The edge $\phi(u)\phi(v)$ can be rewritten as $\{\phi(x) \mid x \in e\}$ (because

$$\begin{aligned} \{\phi(x) \mid x \in e\} &= \{\phi(x) \mid x \in \{u, v\}\} \qquad (\text{since } e = uv = \{u, v\}) \\ &= \{\phi(u), \phi(v)\} = \phi(u)\phi(v) \end{aligned}$$

), and this is clearly independent of the choice of u and v.

[24]Indeed, the inverse map is constructed in the same way as ϕ', except that H, G and ϕ^{-1} take the roles of G, H and ϕ.

H is isomorphic to G. As we have already seen, H has vertex set $\operatorname{V}(H) = S$. The proof of Proposition 2.6.7 is thus complete. □

There is more to say about graph isomorphisms. For example, we can ask how to check whether two given graphs G and H are isomorphic. This is a famous problem in computation, known as the *graph isomorphism problem*, and has seen recent progress[25].

A related problem is to count, for a given $n \in \mathbb{N}$, the isomorphism classes of graphs with n vertices. In other words: If we pretend that isomorphic graphs are equal, then how many graphs are there with n vertices? The number is finite, since each graph with n vertices is isomorphic to a graph with vertex set $\{1, 2, \ldots, n\}$ (this follows from Proposition 2.6.7, applied to $S = \{1, 2, \ldots, n\}$), and the number of the latter graphs is clearly finite. Yet, computing this number exactly is hard, and there does not seem to be a closed-form formula. (There is a whole book [HarPal73] written about this. See also `http://oeis.org/A000088` for references and small values.)

2.7. Examples of graphs, and basic operations

2.7.1. Complete and empty graphs

Let us next define certain classes of graphs, as well as some specific graphs. These will serve as examples in the first place, but some of them will also reveal themselves to be useful later.

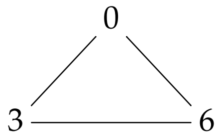
The simplest kinds of simple graphs are the *complete graphs* and the *simple graphs*:

Definition 2.7.1. The *complete graph* on a finite set V means the simple graph $(V, \mathcal{P}_2(V))$. It is the simple graph with vertex set V in which each two distinct vertices are adjacent. For each $n \in \mathbb{N}$, we denote the complete graph on $\{1, 2, \ldots, n\}$ by K_n. Often, any graph that is isomorphic to K_n is called "a K_n" (with the indefinite article, to signify that it isn't necessarily the exact graph K_n, but can be any graph isomorphic to it).

[25] See `https://en.wikipedia.org/wiki/Graph_isomorphism_problem` for a general overview. On the recent progress, see Lászlò Babai's preprint `arXiv:1512.03547` for the work and [Klarre17] for an amusing popularization.

In theory, the problem can be solved by brute force (just check all possible bijections $\operatorname{V}(G) \to \operatorname{V}(H)$ for being graph isomorphisms); but this is highly inefficient (the number of such bijections becomes forbiddingly large very quickly). In many cases, two non-isomorphic graphs G and H can be "told apart" by some property that holds for one and not the other (e.g., if their numbers of edges differ). But in general, a simple and fast algorithm is not known. The problem is in complexity class NP, and Babai's work claims to prove a quasipolynomial-time (but very complicated) algorithm. In practice, software written for solving the graph isomorphism problem makes tradeoffs between simplicity and efficiency.

Example 2.7.2. **(a)** Here is a drawing of the complete graph on the finite set $\{0, 3, 6\}$:



.

Of course, many other drawings are possible.

(b) The complete graph on the set $\{1, 2, 3, 4, 5\}$ is the graph $(\{1, 2, 3, 4, 5\}, \mathcal{P}_2(\{1, 2, 3, 4, 5\}))$, which we have drawn in Example 2.3.2 **(c)**. We are calling this graph K_5 now, according to Definition 2.7.1.

Note that a simple graph $G = (V, E)$ is a K_n (i.e., is isomorphic to K_n) if and only if it is the complete graph on V and satisfies $|V| = n$.

If V is a finite set, then each vertex v of the complete graph on V has degree $\deg v = |V| - 1$, because it is adjacent to the $|V| - 1$ remaining vertices of the complete graph.

Definition 2.7.3. The *empty graph* on a finite set V means the simple graph $(V, \varnothing)$. It has no edges; thus, it is the simple graph with vertex set V in which no two vertices are adjacent.

The empty graphs are, in a sense, the opposite to the complete graphs: In a complete graph, each two distinct vertices are adjacent, whereas in an empty graph, no two vertices are adjacent.

Example 2.7.4. Here is a drawing of the empty graph on the finite set $\{0, 3, 6\}$:

0 .

3 6

Remark 2.7.5. Let V be a finite set. Let G be the complete graph on this set V. Then, each permutation of V is a graph isomorphism $G \to G$. (This is straightforward to check.) The same holds if G is the empty graph on V instead of being the complete graph on V.

2.7.2. Path and cycle graphs

Next, we define some slightly more interesting graphs:

Definition 2.7.6. For each $n \in \mathbb{N}$, we define the *n-th path graph* to be the simple graph

$$\begin{aligned}&(\{1,2,\ldots,n\}, \{\{i,i+1\} \mid i \in \{1,2,\ldots,n-1\}\}) \\ &= (\{1,2,\ldots,n\}, \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}\}).\end{aligned}$$

This graph is denoted by P_n. It has n vertices and $n-1$ edges (unless $n = 0$, in which case it has 0 edges). Again, we say that a simple graph is "a P_n" if it is isomorphic to P_n.

Here is a drawing of P_4:

$$1 —— 2 —— 3 —— 4 . \qquad (22)$$

Similarly, for each $n \in \mathbb{N}$, we can draw P_n by placing n equally spaced points $1,2,\ldots,n$ on a straight line (in this order) and connecting each point $i \in \{1,2,\ldots,n-1\}$ to the next point $i+1$.

Path graphs provide a first mildly interesting example of a graph isomorphism:

Remark 2.7.7. Let $n \in \mathbb{N}$. The map

$$\{1,2,\ldots,n\} \to \{1,2,\ldots,n\}, \qquad i \mapsto n+1-i$$

[26] is an isomorphism $P_n \to P_n$. This isomorphism is called the *reflection* on the path P_n.

A look at the drawing (22) of P_4 should explain why the name "reflection" was chosen for the reflection on the path P_n.

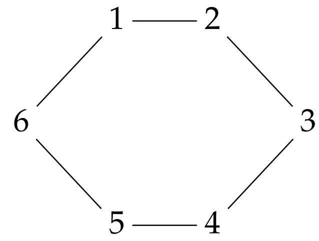
Definition 2.7.8. For each integer $n > 1$, we define the *n-th cycle graph* to be the simple graph

$$\begin{aligned}&(\{1,2,\ldots,n\}, \{\{i,i+1\} \mid i \in \{1,2,\ldots,n-1\}\} \cup \{\{n,1\}\}) \\ &= (\{1,2,\ldots,n\}, \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}\}).\end{aligned}$$

This graph is denoted by C_n. It has n vertices and $\begin{cases} n, & \text{if } n \geq 3; \\ 1, & \text{if } n = 2 \end{cases}$ edges. Again, we say that a simple graph is "a C_n" if it is isomorphic to C_n.

[26] This map is the permutation of $\{1,2,\ldots,n\}$ that sends the numbers $1,2,\ldots,n$ to $n,n-1,\ldots,1$, respectively.

Here is a drawing of C_6:



. (23)

Similarly, we can draw C_n for each $n > 2$ by drawing an n-gon and labelling its vertices by $1, 2, \ldots, n$. (Some authors like to use a regular n-gon for this purpose, but this is of course irrelevant for the purpose of illustrating the graph.)

Again, cycle graphs have nontrivial isomorphisms to themselves:

Remark 2.7.9. Let $n > 1$ be an integer. For each integer x, we define $x \%_1 n$ to be the unique element $p \in \{1, 2, \ldots, n\}$ that satisfies $p \equiv x \mod n$. For each $k \in \mathbb{Z}$, the map

$$\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}, \qquad i \mapsto (i + k) \%_1 n$$

[27] is an isomorphism $C_n \to C_n$. This isomorphism is called the *k-rotation* on the cycle C_n. Note that this k-rotation depends only on the remainder of k modulo n, not on the integer k itself; in total, we thus have n of these k-rotations. Furthermore, there is a reflection on the cycle C_n, defined exactly as the reflection on the path P_n. This reflection is also a graph isomorphism $C_n \to C_n$. Hence, any composition of this reflection with a k-rotation is also a graph isomorphism $C_n \to C_n$ (because of Proposition 2.6.5).

Again, the names "rotation" and "reflection" reflect the fact that if the cycle graph C_n is drawn as a regular n-gon, then the k-rotation actually corresponds to rotating the n-gon by an angle of $\dfrac{2\pi k}{n}$ around its center, whereas the reflection corresponds to reflecting the n-gon through a certain line.

2.7.3. A few more examples

Here are some further examples of simple graphs, mostly to illustrate certain ideas.

[27]If $k \in \{0, 1, \ldots, n\}$, then this map is the permutation of $\{1, 2, \ldots, n\}$ that sends the numbers $1, 2, \ldots, n$ to $k + 1, k + 2, \ldots, n, 1, 2, \ldots, k$, respectively.

Example 2.7.10. Let $n \in \mathbb{N}$. Let G be the graph (V, E), where $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, j\} \in \mathcal{P}_2(V) \mid i \equiv j \mod 3\}$. For example, if $n = 8$, then G can be drawn as follows:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[rr, dash, bend right] & 4 \arrow[r, dash] & 7 & 2 \arrow[r, dash] \arrow[rr, dash, bend right] & 5 \arrow[r, dash] & 8 & 3 \arrow[r, dash] & 6 \end{tikzcd}.$$

If $n = 9$, then G can be drawn as follows:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[rr, dash, bend right] & 4 \arrow[r, dash] & 7 & 2 \arrow[r, dash] \arrow[rr, dash, bend right] & 5 \arrow[r, dash] & 8 & 3 \arrow[r, dash] \arrow[rr, dash, bend right] & 6 \arrow[r, dash] & 9 \end{tikzcd}.$$

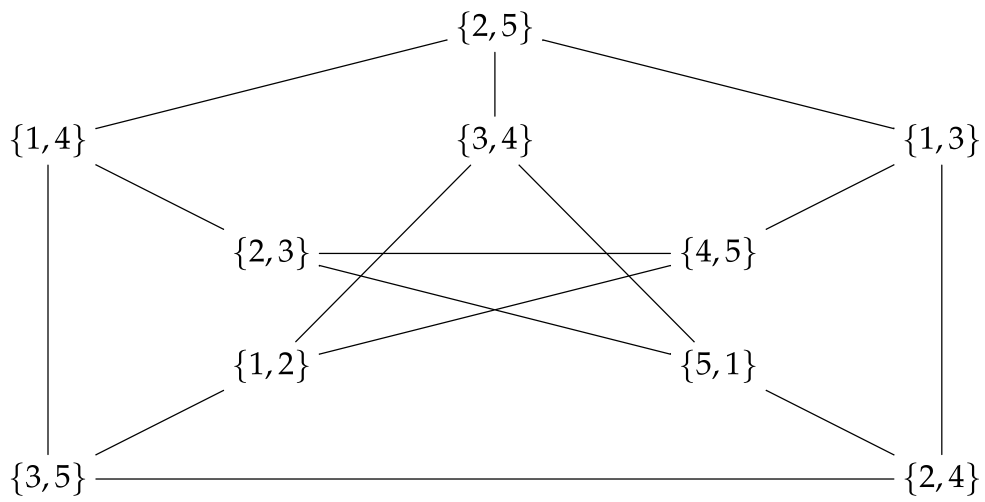
In both cases, we notice that the graph is "disconnected": It really consists of three smaller graphs, with no edges connecting one to another. We shall make this more rigorous later (in fact, we will formalize the concept of "disconnected" in Subsection 2.8.7, and we will formalize the concept of "consisting of three smaller graphs" in Subsection 2.7.5).

Example 2.7.11. If S is a finite set, and if $k \in \mathbb{N}$, then we let $KG_{S,k}$ be the simple graph

$$(\mathcal{P}_k(S), \{\{I, J\} \in \mathcal{P}_2(\mathcal{P}_k(S)) \mid I \cap J = \varnothing\}).$$

The vertices of this graph are the k-element subsets of S, and two such vertices are adjacent if they (regarded as subsets of S) are disjoint. The graph $KG_{S,k}$ is called the *k-th Kneser graph of the set S*.

The graph $KG_{\{1,2,3,4,5\},2}$ is known as the *Petersen graph*. Here is one way to draw it:



For $n \in \mathbb{N}$ and $k \in \mathbb{N}$, it is common to denote the Kneser graph $KG_{\{1,2,\ldots,n\},k}$ by $KG_{n,k}$. Thus, the Petersen graph is $KG_{5,2}$.

Example 2.7.12. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Define a graph $P_{n,k}$ by

$$P_{n,k} = \left(\{1,2,\ldots,n\}, \{\{i,j\} \in \mathcal{P}_2(\{1,2,\ldots,n\}) \mid |i-j| \leq k\}\right).$$

Thus, the vertices of this graph are $1,2,\ldots,n$, and two distinct vertices i and j are adjacent if and only if they satisfy $|i-j| \leq k$. Hence:

- The graph $P_{n,0}$ is the empty graph on the finite set $\{1,2,\ldots,n\}$. (It is empty because no two distinct vertices i and j satisfy $|i-j| \leq 0$, and thus no two distinct vertices i and j are adjacent.)

- The graph $P_{n,1}$ is the path graph P_n (because two distinct vertices i and j satisfy $|i-j| \leq 1$ if and only if i and j are consecutive integers in some order).

- Whenever $k \geq n-1$, the graph $P_{n,k}$ is the complete graph K_n. This is because whenever $k \geq n-1$, **any** pair of distinct vertices i and j of $P_{n,k}$ satisfies $|i-j| \leq k$.

- Here is how $P_{5,2}$ can be drawn:

$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[rr, dash, bend right] & 2 \arrow[r, dash] \arrow[rr, dash, bend right] & 3 \arrow[r, dash] \arrow[rr, dash, bend right] & 4 \arrow[r, dash] & 5
\end{tikzcd}.$$

2.7.4. Subgraphs

If we are given a graph G, we can construct a smaller graph H by picking some of the vertices of G and some of the edges that connect these vertices, and forgetting the rest. Such smaller graphs are known as *subgraphs* of G. Here is the formal definition:

Definition 2.7.13. Let $G = (V, E)$ be a simple graph.

(a) A *subgraph* of G means a simple graph of the form $H = (W, F)$ with $W \subseteq V$ and $F \subseteq E$. In other words, a *subgraph* of G means a simple graph whose vertices are vertices of G and whose edges are edges of G.

(b) Let S be a subset of V. The *induced subgraph of G on the set S* denotes the subgraph $(S, E \cap \mathcal{P}_2(S))$ of G. In other words, it denotes the subgraph of G whose vertices are the elements of S, and whose edges are precisely those edges of G that connect two vertices in S.

(c) An *induced subgraph* of G means a subgraph of G that is the induced subgraph of G on S for some subset S of V.

The definition of an induced subgraph readily leads to the following criterion:

Proposition 2.7.14. Let H be a subgraph of a simple graph G. Then, H is an induced subgraph of G if and only if every edge uv of G whose endpoints u and v both are vertices of H is an edge of H.

Proof sketch for Proposition 2.7.14. The "only if" part follows immediately from the definition of an "induced subgraph". The "if" part is almost as obvious (show that H is the induced subgraph of G on $\operatorname{V}(H)$). The details are left to the reader. $\square$

Example 2.7.15. Let $n > 1$ be an integer.

(a) The path graph P_n (defined in Definition 2.7.6) is a subgraph of the cycle graph C_n (defined in Definition 2.7.8), since all vertices of P_n are vertices of C_n and since all edges of P_n are edges of C_n. However, if $n > 2$, then P_n is not an **induced** subgraph of C_n, because not every edge uv of C_n whose endpoints u and v both are vertices of P_n is an edge of P_n. (In fact, the edge $n1$ of C_n is not an edge of P_n, although both its endpoints n and 1 are vertices of P_n.)

(b) The path graph P_{n-1} is an induced subgraph of P_n (namely, the induced subgraph of P_n on the set $\{1, 2, \ldots, n-1\}$).

(c) Assume that $n > 3$. Then, C_{n-1} is not a subgraph of C_n. Indeed, the edge $(n-1)\,1$ of C_{n-1} is not an edge of C_n.

TODO 2.7.16. Pictures for these examples.

2.7.5. Disjoint unions

Another basic operation that we might want to do with graphs is taking *disjoint unions*. The idea behind disjoint unions is simple: Take two simple graphs and place them "alongside each other", without creating any new edges. For example, the disjoint union of the two simple graphs

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[dr, dash] & 2 \arrow[d, dash] & \text{and} & 2 \arrow[r, dash] \arrow[d, dash] & 8 \arrow[d, dash] \\ & 3 & & 4 \arrow[r, dash] & 6 \end{tikzcd} \tag{24}$$

should be something like

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[dr, dash] & 2 \arrow[d, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 8 \arrow[d, dash] \\ & 3 & 4 \arrow[r, dash] & 6 \end{tikzcd} . \tag{25}$$

However, (25) does not represent a valid graph, because a graph cannot have two vertices with the same "name" (or, more precisely: there are two "vertices 2" in (25), which belong to different edges; but this makes no sense in a graph). So we have to rename the vertices of our original two graphs in order to ensure that there is no overlap between them. For example, we could rename each vertex v of the first graph as $(1, v)$, and each vertex w of the second graph as $(2, w)$; the disjoint union would then be

$$\begin{tikzcd}
(1,1) \arrow[r, dash] \arrow[dr, dash] & (1,2) \arrow[d, dash] & (2,2) \arrow[r, dash] \arrow[d, dash] & (2,8) \arrow[d, dash] \\
& (1,3) & (2,4) \arrow[r, dash] & (2,6)
\end{tikzcd}, \tag{26}$$

which is a perfectly well-defined simple graph with 7 vertices.

This is indeed how we shall define the disjoint union of two simple graphs. The formal definition is as follows:

Definition 2.7.17. Let $G = (V, E)$ and $H = (W, F)$ be two simple graphs. The *disjoint union* of G and H is defined to be the simple graph (X, R), where

$$X = \{(1, v) \mid v \in V\} \cup \{(2, w) \mid w \in W\} \qquad \text{and}$$
$$R = \{\{(1, v_1), (1, v_2)\} \mid \{v_1, v_2\} \in E\} \cup \{\{(2, w_1), (2, w_2)\} \mid \{w_1, w_2\} \in F\}.$$

(Informally speaking, X is the set consisting of all vertices v of G renamed as $(1, v)$ and all vertices w of H renamed as $(2, w)$, whereas R is the set of all edges of G "transplanted" onto X (that is, an edge $\{v_1, v_2\}$ becomes $\{(1, v_1), (1, v_2)\}$) and all edges of H "transplanted" onto X.)

We denote the disjoint union (X, R) of G and H as $G \sqcup H$.

For example, if G and H are the two simple graphs in (24) (in this order), then their disjoint union $G \sqcup H$ is the simple graph in (26).

Proposition 2.7.18. Let G and H be two simple graphs.

(a) Then, the disjoint union $G \sqcup H$ of G and H is a well-defined simple graph and satisfies $|\mathrm{V}(G \sqcup H)| = |\mathrm{V}(G)| + |\mathrm{V}(H)|$ and $|\mathrm{E}(G \sqcup H)| = |\mathrm{E}(G)| + |\mathrm{E}(H)|$.

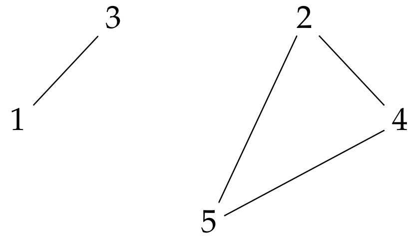
(b) The simple graphs $G \sqcup H$ and $H \sqcup G$ are isomorphic.

Notice that the simple graphs $G \sqcup H$ and $H \sqcup G$ in Proposition 2.7.18 **(b)** are generally not identical; the vertices of the form $(1, v)$ in the former correspond to the vertices $(2, v)$ in the latter.

Example 2.7.19. Let G be the graph

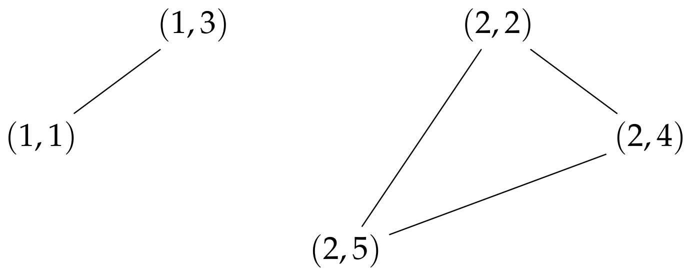
$$(\{1,2,3,4,5\}, \{13,24,45,25\}).$$

This can be drawn as follows:



.

This picture shows that the graph is "built from two disjoint pieces": One consisting of the vertices 1 and 3, and another consisting of the vertices 2, 4 and 5. No edge connects a vertex in one piece with a vertex in the other. Formally, this is saying that our graph G is isomorphic to the disjoint union $H \sqcup K$ of the graph $H = (\{1,3\}, \{13\})$ and the graph $K = (\{2,4,5\}, \{24,45,25\})$. To wit, the disjoint union $H \sqcup K$ can be drawn as follows:



.

The map that sends each vertex (i,v) of $H \sqcup K$ to the vertex v of G is a graph isomorphism.

We can extend Definition 2.7.17 to define a disjoint union of several graphs:

Definition 2.7.20. Let k be a nonnegative integer. For each $i \in \{1,2,\ldots,k\}$, let $G_i = (V_i, E_i)$ be a simple graph. The *disjoint union* of these k simple graphs $G_1, G_2, \ldots, G_k$ is defined to be the simple graph (X,R), where

$$X = \{(i,v) \mid i \in \{1,2,\ldots,k\} \text{ and } v \in V_i\} \qquad \text{and}$$
$$R = \{\{(i,v_1),(i,v_2)\} \mid i \in \{1,2,\ldots,k\} \text{ and } \{v_1,v_2\} \in E_i\}.$$

(Informally speaking, X is the set consisting of all vertices v_i of all graphs G_i renamed as (i,v_i), whereas R is the set of all edges of all graphs G_i "transplanted" onto X (that is, an edge $\{v_1,v_2\}$ of a graph G_i becomes $\{(i,v_1),(i,v_2)\}$).)

We denote the disjoint union (X,R) of $G_1, G_2, \ldots, G_k$ as $G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k$.

It is easy to see that Definition 2.7.17 is the particular case of Definition 2.7.20 for $k = 2$.

Example 2.7.21. Let G_1, G_2 and G_3 be the simple graphs

$$\begin{aligned}
G_1 &= \{\{1,2\}, \{12\}\}, \\
G_2 &= \{\{1,2\}, \{12\}\}, \\
G_3 &= \{\{1,2,3,4\}, \{12,23,34,13,14\}\}.
\end{aligned}$$

(Yes, G_1 is equal to G_2, but this does not matter.) Here are their drawings:

$$G_1 = \begin{tikzcd} 1 \arrow[r, dash] & 2 \end{tikzcd} \quad ; \quad G_2 = \begin{tikzcd} 1 \arrow[r, dash] & 2 \end{tikzcd} \quad ; \quad G_3 = \begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & 2 \arrow[dl, dash] \\ 3 \arrow[r, dash] & 4 \end{tikzcd} .$$

Then, the disjoint union $G_1 \sqcup G_2 \sqcup G_3$ is the following graph:

$$\begin{tikzcd} (1,1) \arrow[r, dash] & (1,2) & (2,1) \arrow[r, dash] & (2,2) & (3,1) \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & (3,2) \arrow[dl, dash] \\ & & & & (3,3) \arrow[r, dash] & (3,4) \end{tikzcd} .$$

Here are some further examples of simple graphs, mostly to illustrate certain ideas.

Example 2.7.22. Recall the graph G from Exercise 2.7.10, where $n = 8$. As we recall, G looks as follows:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[rr, dash, bend right] & 4 \arrow[r, dash] & 7 & 2 \arrow[r, dash] \arrow[rr, dash, bend right] & 5 \arrow[r, dash] & 8 & 3 \arrow[r, dash] & 6 \end{tikzcd} .$$

As we already noticed, this graph consists of three disjoint pieces. These three pieces are the graphs

$$G_1 = \begin{tikzcd} 1 \arrow[r, dash] \arrow[rr, dash, bend right] & 4 \arrow[r, dash] & 7 \end{tikzcd} ; \qquad G_2 = \begin{tikzcd} 2 \arrow[r, dash] \arrow[rr, dash, bend right] & 5 \arrow[r, dash] & 8 \end{tikzcd} ;$$

$$G_3 = \begin{tikzcd} 3 \arrow[r, dash] & 6 \end{tikzcd} .$$

Formally speaking, G is isomorphic to the disjoint union $G_1 \sqcup G_2 \sqcup G_3$. Namely, the map from $\operatorname{V}(G_1 \sqcup G_2 \sqcup G_3)$ to $\operatorname{V}(G)$ that sends each (i, v) to v is a graph isomorphism.

The disjoint union operation on graphs is associative up to isomorphism, in the following sense:

Proposition 2.7.23. Let G_1, G_2 and G_3 be three simple graphs. Then, the three graphs $(G_1 \sqcup G_2) \sqcup G_3$ and $G_1 \sqcup (G_2 \sqcup G_3)$ and $G_1 \sqcup G_2 \sqcup G_3$ are isomorphic. Specifically:

(a) The map from $\operatorname{V}((G_1 \sqcup G_2) \sqcup G_3)$ to $\operatorname{V}(G_1 \sqcup G_2 \sqcup G_3)$ that sends each $(1,(1,v))$ to $(1,v)$, sends each $(1,(2,v))$ to $(2,v)$, and sends each $(2,v)$ to $(3,v)$ is a graph isomorphism from $(G_1 \sqcup G_2) \sqcup G_3$ to $G_1 \sqcup G_2 \sqcup G_3$.

(b) The map from $\operatorname{V}(G_1 \sqcup (G_2 \sqcup G_3))$ to $\operatorname{V}(G_1 \sqcup G_2 \sqcup G_3)$ that sends each $(1,v)$ to $(1,v)$, sends each $(2,(1,v))$ to $(2,v)$, and sends each $(2,(2,v))$ to $(3,v)$ is a graph isomorphism from $G_1 \sqcup (G_2 \sqcup G_3)$ to $G_1 \sqcup G_2 \sqcup G_3$.

The proof of this proposition is a straightforward matter of checking the definition of a graph isomorphism; intuitively, the proposition is already obvious enough: All three graphs $(G_1 \sqcup G_2) \sqcup G_3$ and $G_1 \sqcup (G_2 \sqcup G_3)$ and $G_1 \sqcup G_2 \sqcup G_3$ are obtained by putting the original three graphs G_1, G_2 and G_3 alongside each other; they differ only in how the vertices are relabeled.

There are generalizations of Proposition 2.7.23 for disjoint unions of more than three graphs. The reader won't have much trouble stating and proving these.

2.8. Walks and paths

Walks and paths are among the most fundamental notions that can be defined for a graph; a large part of graph theory is concerned with these notions. Let us now introduce them.

2.8.1. Definitions and examples

Definition 2.8.1. Let G be a simple graph.

(a) A *walk* (in G) means a finite sequence $(v_0, v_1, \ldots, v_k)$ of vertices of G (with $k \geq 0$) such that all of $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ are edges of G. (We allow k to be 0, in which case the condition that "$v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ are edges of G" is vacuously true.)

(b) If $\mathbf{w} = (v_0, v_1, \ldots, v_k)$ is a walk in G, then:

- The *vertices of* $\mathbf{w}$ are defined to be the vertices $v_0, v_1, \ldots, v_k$.
- The *edges of* $\mathbf{w}$ are defined to be the edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ of G;
- The nonnegative integer k is called the *length* of $\mathbf{w}$. (This integer is the number of all edges of $\mathbf{w}$, counted with multiplicity. It is 1 smaller than the number of all vertices of $\mathbf{w}$, counted with multiplicity.)
- The vertex v_0 is said to be the *starting point* of $\mathbf{w}$. We say that the walk $\mathbf{w}$ *starts* at v_0 (or, equivalently, *begins* at v_0).

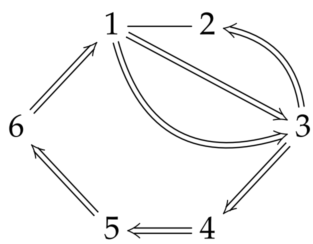
- The vertex v_k is said to be the *ending point* of $\mathbf{w}$. We say that the walk $\mathbf{w}$ *ends* at v_k.

(c) A *path* (in G) means a walk (in G) whose vertices are distinct. (In other words, a *path* in G means a walk $(v_0, v_1, \ldots, v_k)$ in G such that $v_0, v_1, \ldots, v_k$ are distinct.)

(e) Let p and q be two vertices of G. A *walk from p to q* (in G) means a walk (in G) whose starting point is p and whose ending point is q. A *path from p to q* (in G) means a path (in G) whose starting point is p and whose ending point is q.

Example 2.8.2. Let V and E be as in Example 2.4.2 **(c)**. Let G be the graph (V, E). Then:

- The sequence $(1, 3, 4, 5, 6, 1, 3, 2)$ of vertices of G is a walk (in G), since all of 13, 34, 45, 56, 61, 13, and 32 are edges of G. This walk is a walk from 1 to 2 (since its starting point is 1, and its ending point is 2). It is not a path (since 1, 3, 4, 5, 6, 1, 3 and 2 are not distinct). The length of this walk is 7. Let us visualize this walk by marking all the edges that it uses with double arrows:



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- The sequence $(1, 2, 4, 3)$ of vertices of G is not a walk (in fact, 12 and 43 are edges of G, but 24 is not). Hence, it is not a path either.
- The sequence $(1, 3, 2)$ of vertices of G is a walk, but also a path (from 1 to 2). The length of this path is 2.
- The sequence $(1, 3, 2, 1)$ of vertices of G is a walk (from 1 to 1), but not a path (since 1, 3, 2 and 1 are not distinct). The length of this walk is 3.
- The sequence $(1, 2, 1)$ of vertices of G is a walk (from 1 to 1). It is not a path. The length of this walk is 2.
- The sequence (5) of vertices of G is a walk (from 5 to 5) and a path as well. The length of this path is 0.
- The sequence $(5, 4)$ of vertices of G is a walk (from 5 to 4) and a path as well. The length of this path is 1.

Exercise 2.8.3. Let G be a simple graph. Let $\mathbf{w}$ be a path in G. Prove that the edges of $\mathbf{w}$ are distinct. (This may look obvious when you can point to a picture; but we ask you to give a rigorous proof!)

2.8.2. Composing and reverting walks

Let us discuss a few basic properties of walks and paths. First of all, if v is a vertex of a simple graph G, then the one-element list (v) is a path (in G) from v to v. This path has length 0. Hence, there exists a path (of length 0) from each vertex to itself. Furthermore, if $(v_0, v_1, \ldots, v_k)$ is a walk in a simple graph G, and if p and q are two elements of $\{0, 1, \ldots, k\}$ satisfying $p \leq q$, then $(v_p, v_{p+1}, \ldots, v_q)$ also is a walk in G. The same statement holds if the word "walk" is replaced by "path" both times. Simple facts like this will be used without further mention.

Let us now pass to a few facts which are less obvious (although intuitively clear). The first says that we can "splice" two walks together if the ending point of the first is the starting point of the second:

Proposition 2.8.4. Let G be a simple graph. Let u, v and w be three vertices of G. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a walk from u to v. Let $\mathbf{b} = (b_0, b_1, \ldots, b_\ell)$ be a walk from v to w. Then,

$$\begin{aligned}(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell) &= (a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell) \\ &= (a_0, a_1, \ldots, a_{k-1}, v, b_1, b_2, \ldots, b_\ell)\end{aligned}$$

is a walk from u to w.

This walk shall be denoted by $\mathbf{a} * \mathbf{b}$.

Example 2.8.5. Let V and E be as in Example 2.4.2 **(c)**. Let G be the graph (V, E). Let $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (3, 1, 6)$. These are two walks in G. Then, $\mathbf{a} * \mathbf{b} = (1, 2, 3, 1, 6)$. As Proposition 2.8.4 claims, this is a walk from 1 to 6 (since $\mathbf{a}$ is a walk from 1 to 3, while $\mathbf{b}$ is a walk from 3 to 6). Note, however, that $\mathbf{a} * \mathbf{b}$ is not a path, even though $\mathbf{a}$ and $\mathbf{b}$ are paths.

Proof of Proposition 2.8.4. The list $(a_0, a_1, \ldots, a_k)$ is a walk from u to v. Hence, all of $a_0a_1, a_1a_2, \ldots, a_{k-1}a_k$ are edges of G, and we have $a_0 = u$ and $a_k = v$.

The list $(b_0, b_1, \ldots, b_\ell)$ is a walk from v to w. Hence, all of $b_0b_1, b_1b_2, \ldots, b_{\ell-1}b_\ell$ are edges of G, and we have $b_0 = v$ and $b_\ell = w$.

We need to prove three claims:

- *Claim 1:* We have $(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell) = (a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell)$.
- *Claim 2:* We have $(a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell) = (a_0, a_1, \ldots, a_{k-1}, v, b_1, b_2, \ldots, b_\ell)$.
- *Claim 3:* The list $(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell)$ is a walk from u to w.

Proof of Claim 1. The two lists $(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell)$ and $(a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell)$ differ only in their $(k+1)$-st entry, which is a_k for the first list and is b_0 for the second list. But since their $(k+1)$-st entries are also equal (because $a_k = v = b_0$), this shows that these two lists are identical. In other words, $(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell) = (a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell)$. Claim 1 is proven.

Proof of Claim 2. The two lists $(a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell)$ and $(a_0, a_1, \ldots, a_{k-1}, v, b_1, b_2, \ldots, b_\ell)$ differ only in their $(k+1)$-st entry, which is b_0 for the first list and is v for the second list. But since their $(k+1)$-st entries are also equal (because $b_0 = v$), this shows that these two lists are identical. In other words, $(a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell) = (a_0, a_1, \ldots, a_{k-1}, v, b_1, b_2, \ldots, b_\ell)$. Claim 2 is proven.

Proof of Claim 3. Recall that all of $b_0b_1, b_1b_2, \ldots, b_{\ell-1}b_\ell$ are edges of G. Since $b_0 = v = a_k$, this rewrites as follows: All of $a_kb_1, b_1b_2, \ldots, b_{\ell-1}b_\ell$ are edges of G. Combining this with the fact that all of $a_0a_1, a_1a_2, \ldots, a_{k-1}a_k$ are edges of G, we obtain the following: All of $a_0a_1, a_1, a_2, \ldots, a_{k-1}a_k, a_kb_1, b_1b_2, \ldots, b_{\ell-1}b_\ell$ are edges of G. Hence, the list $(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell)$ is a walk (by the definition of "walk"). Furthermore, this list is actually a walk from u to w (since $a_0 = u$ and $b_\ell = w$). This proves Claim 3.

Thus, all three Claims are proven. Together, they yield Proposition 2.8.4. $\square$

The next easy property of walks is that a walk can be reversed (i.e., walked backwards from its ending point to its starting point), and moreover, if the walk is a path, then its reversal is a path too:

Proposition 2.8.6. Let G be a simple graph. Let u and v be two vertices of G. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a walk from u to v. Then:

- **(a)** The list $(a_k, a_{k-1}, \ldots, a_0)$ is a walk from v to u.

  This list shall be denoted by $\operatorname{rev} \mathbf{a}$, and called the *reversal* of $\mathbf{a}$.

- **(b)** If the walk $\mathbf{a}$ is a path, then $\operatorname{rev} \mathbf{a}$ is a path as well.

Example 2.8.7. Let V and E be as in Example 2.4.2 **(c)**. Let G be the graph (V, E). Let $\mathbf{a} = (1, 2, 3, 1, 6)$. Then, $\operatorname{rev} \mathbf{a} = (6, 1, 3, 2, 1)$.

Proof of Proposition 2.8.6. The list $(a_0, a_1, \ldots, a_k)$ is a walk from u to v. Hence, all of $a_0a_1, a_1a_2, \ldots, a_{k-1}a_k$ are edges of G, and we have $a_0 = u$ and $a_k = v$.

(a) All of $a_0a_1, a_1a_2, \ldots, a_{k-1}a_k$ are edges of G. In other words,

$$a_ia_{i+1} \text{ is an edge of } G \qquad \text{for each } i \in \{0, 1, \ldots, k-1\}. \tag{27}$$

Now, $a_{k-j}a_{k-(j+1)}$ is an edge of G for each $j \in \{0, 1, \ldots, k-1\}$ [28]. In other words, all of $a_ka_{k-1}, a_{k-1}a_{k-2}, \ldots, a_1a_0$ are edges of G. In other words, the list

[28] *Proof.* Let $j \in \{0, 1, \ldots, k-1\}$. Then, $k-1-j \in \{0, 1, \ldots, k-1\}$. Hence, (27) (applied to $i = k-1-j$) shows that $a_{k-1-j}a_{(k-1-j)+1}$ is an edge of G. Since $\underbrace{a_{k-1-j}}_{=a_{k-(j+1)}} \underbrace{a_{(k-1-j)+1}}_{=a_{k-j}} = a_{k-(j+1)}a_{k-j} = a_{k-j}a_{k-(j+1)}$, this rewrites as follows: $a_{k-j}a_{k-(j+1)}$ is an edge of G. This completes the proof.

$(a_k, a_{k-1}, \ldots, a_0)$ is a walk. This list is furthermore a walk from v to u (since $a_k = v$ and $a_0 = v$). This proves Proposition 2.8.6 **(a)**.

(b) The vertices of the walk $\mathbf{a}$ are $a_0, a_1, \ldots, a_k$ (since $\mathbf{a} = (a_0, a_1, \ldots, a_k)$), whereas the vertices of the walk $\operatorname{rev} \mathbf{a}$ are $a_k, a_{k-1}, \ldots, a_0$ (since $\operatorname{rev} \mathbf{a} = (a_k, a_{k-1}, \ldots, a_0)$). Hence, the vertices of the walk $\operatorname{rev} \mathbf{a}$ are the same as the vertices of the walk $\mathbf{a}$, listed in a different order.

Now, assume that $\mathbf{a}$ is a path. Thus, the vertices of the walk $\mathbf{a}$ are distinct (by the definition of "path"). In other words, the vertices of the walk $\operatorname{rev} \mathbf{a}$ are distinct (since the vertices of the walk $\operatorname{rev} \mathbf{a}$ are the same as the vertices of the walk $\mathbf{a}$, listed in a different order). Hence, this walk $\operatorname{rev} \mathbf{a}$ is a path (by the definition of "path"). This proves Proposition 2.8.6 **(b)**. $\square$

2.8.3. Reducing a walk to a path

The next result is simple but crucial:

Proposition 2.8.8. Let G be a simple graph. Let u and v be two vertices of G. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a walk from u to v. Assume that $\mathbf{a}$ is not a path. Then, there exists a walk from u to v whose length is smaller than k.

Example 2.8.9. Let G be the graph (V, E), where

$$V = \{1,2,3,4,5,6\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,1\}, \{1,3\}, \{3,5\}, \{5,1\}\}.$$

(This graph can be drawn in such a way as to look like a hexagon with three extra diagonals forming a triangle.) The sequence $(4,3,2,1,3,6)$ of vertices of G is a walk from 4 to 6. But it is not a path, since it passes through the vertex 3 twice (i.e., the vertex 3 appears twice among its vertices). Thus, Proposition 2.8.8 predicts that there exists a walk from 4 to 6 whose length is smaller than 5. And indeed, such a walk can easily be obtained from our walk $(4,3,2,1,3,6)$, just by removing the part between the two appearances of the vertex 3 (including one of these two appearances). The result is the walk $(4,3,6)$, whose length 2 is indeed smaller than 5.

The same idea (removing a "digression" between two appearances of the same vertex) can be used to prove Proposition 2.8.8 in full generality. This is what we shall do in the proof below.

Proof of Proposition 2.8.8. Recall that $(a_0, a_1, \ldots, a_k) = \mathbf{a}$ is a walk from u to v. Hence, $a_0 = u$ and $a_k = v$.

If the vertices of $\mathbf{a}$ were distinct, then the walk $\mathbf{a}$ would be a path (by the definition of "path"). Thus, the vertices of $\mathbf{a}$ are not distinct (since $\mathbf{a}$ is not a path). In other words, $a_0, a_1, \ldots, a_k$ are not pairwise distinct (since the vertices of $\mathbf{a}$ are $a_0, a_1, \ldots, a_k$). In other words, there exist two elements p and q of $\{0, 1, \ldots, k\}$ such that $p < q$ and $a_p = a_q$. Fix such p and q. Define a vertex x of G by $x = a_p = a_q$. (This is possible since $a_p = a_q$.)

Recall that $(a_0, a_1, \ldots, a_k)$ is a walk. Hence, the list $(a_0, a_1, \ldots, a_p)$ is a walk as well. Moreover, this list $(a_0, a_1, \ldots, a_p)$ is a walk from u to x (since $a_0 = u$ and $a_p = x$). Denote this walk by $\mathbf{p}$.

Again, recall that $(a_0, a_1, \ldots, a_k)$ is a walk. Hence, the list $(a_q, a_{q+1}, \ldots, a_k)$ is a walk as well. Moreover, this list $(a_q, a_{q+1}, \ldots, a_k)$ is a walk from x to v (since $a_q = x$ and $a_k = v$). Denote this walk by $\mathbf{q}$.

Altogether, we know that u, x and v are three vertices of G; we know that $\mathbf{p} = (a_0, a_1, \ldots, a_p)$ is a walk from u to x; and we know that $\mathbf{q} = (a_q, a_{q+1}, \ldots, a_k)$ is a walk from x to v. Hence, we can apply Proposition 2.8.4 to x, v, $\mathbf{p}$, p, a_i, $\mathbf{q}$, $k - q$ and a_{q+j} instead of v, w, $\mathbf{a}$, k, a_i, $\mathbf{b}$, ℓ and b_j. We thus conclude that

$$\begin{aligned}(a_0, a_1, \ldots, a_p, a_{q+1}, a_{q+2}, \ldots, a_k) &= (a_0, a_1, \ldots, a_{p-1}, a_q, a_{q+1}, \ldots, a_k) \\ &= (a_0, a_1, \ldots, a_{p-1}, x, a_{q+1}, a_{q+2}, \ldots, a_k)\end{aligned}$$

is a walk from u to v. This walk has length smaller than k (in fact, its length is $(p-1) + 2 + (k-q-1) = \underbrace{p}_{<q} + k - q < q + k - q = k$). Hence, there exists a walk from u to v whose length is smaller than k (namely, the walk we have just constructed). This proves Proposition 2.8.8. $\square$

From Proposition 2.8.8, we obtain the following corollary:

Corollary 2.8.10. Let G be a simple graph. Let u and v be two vertices of G. Let $k \in \mathbb{N}$. Assume that there exists a walk from u to v of length k. Then, there exists a path from u to v of length $\leq k$.

Example 2.8.11. Let V, E and G be as in Example 2.8.2. Consider the walk $\mathbf{a} = (4, 3, 2, 1, 3, 4, 3, 1, 5)$ from 4 to 5, having length 8. Corollary 2.8.10 predicts that there exists a path from 4 to 5 of length ≤ 8. This is, of course, obvious (the path $(4, 5)$ works), but let us try to construct such a path from $\mathbf{a}$ by removing "digressions" as we did in Exercise 2.8.9:

- One way to start is by noticing that the vertex 3 appears three times in our walk, and removing the "digression" that the walk makes between the first and the second appearance of 3. The result is the walk $(4, 3, 4, 3, 1, 5)$. This latter walk is still not a path, since the vertex 4 appears in it twice. Again, remove the "digression" to obtain the shorter walk $(4, 3, 1, 5)$. This new walk is a path and has length $3 \leq 8$. So we have found a path from 4 to 5 of length ≤ 8.

- An alternative way to start is by noticing that the vertex 1 appears twice in our walk, and removing the "digression" that the walk makes between its two appearances. The result of this removal is the walk $(4, 3, 2, 1, 5)$, which is already a path and has length $4 \leq 8$. So we have found a path from 4 to 5 of length ≤ 8.

- Yet another way to start is by noticing (again) that the vertex 3 appears three times in our walk, and removing the "digression" between its first and third appearances. The result is the path $(4, 3, 1, 5)$.

- There are two other possible ways to start; do you see them?

Note that the different ways to remove "digressions" don't always lead to the same path in the end.

Corollary 2.8.10 can be proved in full generality using this method. Our proof below will do exactly that.

Proof of Corollary 2.8.10. The idea behind this proof is the following: Proposition 2.8.8 shows that, whenever we have a walk from u to v that is not a path, we can "shorten it" (i.e., find a shorter walk from u to v). Thus, we can start with any walk from u to v of length k (its existence is guaranteed by the hypothesis), and keep "shortening it" until it becomes a path (which has to happen eventually, because we cannot keep shortening it indefinitely). The result will be a path from u to v of length $\leq k$.

This idea can be formalized to give a rigorous proof. The way to formalize it is fairly standard and straightforward, so I am only going to show it now, but leave it to the reader in similar situations that may appear further below.

Our rigorous proof of Corollary 2.8.10 shall proceed by strong induction over k. So we fix an integer $\ell \in \mathbb{N}$, and we assume that Corollary 2.8.10 holds whenever $k < \ell$. We then must prove that Corollary 2.8.10 holds when $k = \ell$. (This is the induction step. Note that a strong induction needs no induction base, because the induction step includes the case $\ell = 0$ already.)

We have assumed that Corollary 2.8.10 holds whenever $k < \ell$. In other words, the following claim holds:

> *Claim 1:* Let G be a simple graph. Let u and v be two vertices of G. Let $k \in \mathbb{N}$ be such that $k < \ell$. Assume that there exists a walk from u to v of length k. Then, there exists a path from u to v of length $\leq k$.

Our goal is to prove that Corollary 2.8.10 holds when $k = \ell$. In other words, our goal is to prove the following claim:

> *Claim 2:* Let G be a simple graph. Let u and v be two vertices of G. Let $k \in \mathbb{N}$ be such that $k = \ell$. Assume that there exists a walk from u to v of length k. Then, there exists a path from u to v of length $\leq k$.

Proof of Claim 2: We have assumed that there exists a walk from u to v of length k. Fix such a walk, and denote it by $\mathbf{a}$. Write this walk $\mathbf{a}$ in the form $\mathbf{a} = (a_0, a_1, \ldots, a_k)$. (This is possible, since $\mathbf{a}$ has length k.) We are in one of the following two cases:

- *Case 1:* The walk $\mathbf{a}$ is a path.
- *Case 2:* The walk $\mathbf{a}$ is not a path.

Let us consider Case 1 first. In this case, the walk $\mathbf{a}$ is a path. Thus, $\mathbf{a}$ is a path from u to v (since $\mathbf{a}$ is a walk from u to v), and has length $\leq k$ (since $\mathbf{a}$ has length k). Hence, there exists a path from u to v of length $\leq k$ (namely, $\mathbf{a}$). Thus, Claim 2 is proven in Case 1.

Let us now consider Case 2. In this case, the walk $\mathbf{a}$ is not a path. Hence, Proposition 2.8.8 shows that there exists a walk from u to v whose length is smaller than k. Fix such a walk, and denote it by $\mathbf{b}$. Thus, $\mathbf{b}$ is a walk from u to v whose length is smaller than k.

Let r be the length of $\mathbf{b}$. Then, $r < k$ (since the length of $\mathbf{b}$ is smaller than k). Hence, $r < k = \ell$. Now, there exists a walk from u to v of length r (namely, $\mathbf{b}$). Hence, Claim 1 (applied to r instead of k) shows that there exists a path from u to v of length $\leq r$. This path has length $\leq r < k$; hence, it has length $\leq k$. Therefore, there exists a path from u to v of length $\leq k$ (namely, the path that we have just constructed). Hence, Claim 2 is proven in Case 2.

Now, we have proven Claim 2 in both possible cases. Therefore, the proof of Claim 2 is complete. In other words, Corollary 2.8.10 holds when $k = \ell$. This completes the (inductive) proof of Corollary 2.8.10. □

2.8.4. The equivalence relation $\simeq_G$

We can use the concept of paths (and the results proven above) to define a certain equivalence relation on the vertex set of a graph G. First, let us recall what relations and equivalence relations are:

Definition 2.8.12. Let X be a set. A *binary relation* on X shall mean a subset of the Cartesian product $X \times X$. (In other words, it shall mean a set of pairs of elements of X.)

If R is a binary relation on X, and if a and b are two elements of X, then we shall often use the notation aRb as a synonym for $(a, b) \in R$. This notation is called *infix notation*. For some relations R, it is usual to always use this notation (i.e., one never writes $(a, b) \in R$, but always writes aRb instead); in this case, we say that the relation R is *written infix*.

Examples of binary relations on the set $\mathbb{Z}$ are $=$, $\leq$, $<$, $\geq$, $>$ and $\neq$. (These are all written infix: For example, we write $2 < 4$, but we don't write $(2, 4) \in <$.)

Now, what is an equivalence relation?

Definition 2.8.13. Let R be a binary relation on a set X. This relation R shall be written infix throughout this definition.

(a) We say that R is *reflexive* if it has the following property: For each $x \in X$, we have xRx.

(b) We say that R is *symmetric* if it has the following property: For each $x \in X$ and $y \in X$ satisfying xRy, we have yRx.

(c) We say that R is *transitive* if it has the following property: For each $x \in X$, $y \in X$ and $z \in X$ satisfying xRy and yRz, we have xRz.

(d) We say that R is an *equivalence relation* if it is reflexive, symmetric and transitive.

We refer to [LeLeMe16, §10.10], or [Hammac15, Chapter 11] (for a particularly detailed treatment), or [Oggier14, Chapter 9], or [Day16, §3.E] for more about equivalence relations. Let us here just briefly mention a few examples:

- Among the six relations $=$, $\leq$, $<$, $\geq$, $>$ and $\neq$ on the set $\mathbb{Z}$, only the first one ($=$) is an equivalence relation. However, $\leq$ and $\geq$ are reflexive and transitive (but not symmetric), and $<$ and $>$ are transitive. Furthermore, $\neq$ is symmetric.

- For each positive integer m, we can define a binary relation $\overset{m}{\equiv}$ on $\mathbb{Z}$ (written infix) by setting

$$a \overset{m}{\equiv} b \qquad \text{if and only if} \qquad a \equiv b \mod m.$$

  This is an equivalence relation (for each m).

- Here is a fairly general example: If X and Y are two sets, and if $f : X \to Y$ is any map, then we can define a binary relation $\overset{f}{\equiv}$ on X (written infix) by setting

$$a \overset{f}{\equiv} b \qquad \text{if and only if} \qquad f(a) = f(b).$$

  This is an equivalence relation (for each f). The equivalence relation $\overset{m}{\equiv}$ on $\mathbb{Z}$ defined in the previous example is actually a particular case of this $\overset{f}{\equiv}$: Namely, if we define $f : \mathbb{Z} \to \{0, 1, \ldots, m-1\}$ to be the map that sends each $n \in \mathbb{Z}$ to the remainder obtained when n is divided by m, then the relation $\overset{f}{\equiv}$ becomes identical with $\overset{m}{\equiv}$.

Each simple graph G gives rise to an equivalence relation; here is how it is defined:

Definition 2.8.14. Let G be a simple graph. We define a binary relation $\simeq_G$ on the set $\operatorname{V}(G)$ (written infix) as follows:

If u and v are two elements of $\operatorname{V}(G)$, then we set $u \simeq_G v$ if and only if there exists a walk from u to v in G.

Example 2.8.15. Let G be the simple graph defined in Example 2.4.2 **(a)**. Then, there exists a walk from 1 to 4 in G (for example, the walk $(1,2,3,4)$, or the walk $(1,2,3,2,3,4)$, or the walk $(1,6,5,4)$, or many other walks). In other words, $1 \simeq_G 4$.

It is actually easy to see that $u \simeq_G v$ for any two vertices u and v of G. If we want to prove this just using the definition of $\simeq_G$, then we have to construct a walk from u to v for each pair (u, v) of vertices of G. However, there is a simpler way to prove this – see Example 2.8.18 below.

Example 2.8.16. Let G be the simple graph defined in Example 2.7.10 for $n = 8$. Then, we have $1 \simeq_G 4$ and $1 \simeq_G 7$, but we don't have $1 \simeq_G 3$. It is easy to see that the relation $\simeq_G$ for the graph G is exactly the relation $\overset{3}{\equiv}$ (restricted to the set $\{1, 2, \ldots, n\}$).

The binary relation $\simeq_G$ we just defined is of fundamental importance, although many authors do not define a specific symbol for it. As promised, it is an equivalence relation:

Proposition 2.8.17. Let G be a simple graph. Then, the binary relation $\simeq_G$ (defined in Definition 2.8.14) is an equivalence relation.

Example 2.8.18. Let G be the simple graph defined in Example 2.4.2 **(a)**. Then, there exists a walk from 1 to 2 in G (for example, the walk $(1,2)$). Hence, $1 \simeq_G 2$. Similarly, $2 \simeq_G 3$, $3 \simeq_G 4$, $4 \simeq_G 5$ and $5 \simeq_G 6$. But Proposition 2.8.17 shows that $\simeq_G$ is an equivalence relation. Hence, from $1 \simeq_G 2 \simeq_G 3 \simeq_G 4 \simeq_G 5 \simeq_G 6$, we immediately conclude that $u \simeq_G v$ for any two vertices u and v of G.

Proof of Proposition 2.8.17. We are going to show that the relation $\simeq_G$ is reflexive, symmetric and transitive.

Claim 1: The relation $\simeq_G$ is reflexive.

Proof of Claim 1: Let $x \in \operatorname{V}(G)$. Then, (x) is a walk from x to x in G. Hence, there exists a walk from x to x in G (namely, (x)). In other words, $x \simeq_G x$ (because the definition of $\simeq_G$ reveals that $x \simeq_G x$ holds if and only if there exists a walk from x to x in G).

Now, forget that we fixed x. We thus have shown that for each $x \in \operatorname{V}(G)$, we have $x \simeq_G x$. In other words, the relation $\simeq_G$ is reflexive (by the definition of "reflexive"). This proves Claim 1.

Claim 2: The relation $\simeq_G$ is symmetric.

Proof of Claim 2: Let $x \in \operatorname{V}(G)$ and $y \in \operatorname{V}(G)$ be such that $x \simeq_G y$.

We have $x \simeq_G y$. In other words, there exists a walk from x to y in G (by the definition of $\simeq_G$). Fix such a walk, and denote it by $\mathbf{a} = (a_0, a_1, \ldots, a_k)$.

Thus, $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ is a walk from x to y in G. Hence, Proposition 2.8.6 **(a)** (applied to $u = x$ and $v = y$) shows that the list $(a_k, a_{k-1}, \ldots, a_0)$ is a walk from y to x. Thus, there exists a walk from y to x in G (namely, this list). In other words, $y \simeq_G x$ (by the definition of $\simeq_G$).

Now, forget that we fixed x and y. We thus have shown that for each $x \in \operatorname{V}(G)$ and $y \in \operatorname{V}(G)$ satisfying $x \simeq_G y$, we have $y \simeq_G x$. In other words, the relation $\simeq_G$ is symmetric (by the definition of "symmetric"). This proves Claim 2.

Claim 3: The relation $\simeq_G$ is transitive.

Proof of Claim 3: Let $x \in \operatorname{V}(G)$, $y \in \operatorname{V}(G)$ and $z \in \operatorname{V}(G)$ be such that $x \simeq_G y$ and $y \simeq_G z$.

We have $x \simeq_G y$. In other words, there exists a walk from x to y in G (by the definition of $\simeq_G$). Fix such a walk, and denote it by $\mathbf{a} = (a_0, a_1, \ldots, a_k)$.

We have $y \simeq_G z$. In other words, there exists a walk from y to z in G (by the definition of $\simeq_G$). Fix such a walk, and denote it by $\mathbf{b} = (b_0, b_1, \ldots, b_\ell)$.

We know that $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ is a walk from x to y in G, and we also know that $\mathbf{b} = (b_0, b_1, \ldots, b_\ell)$ is a walk from y to z in G. Hence, Proposition 2.8.4 (applied to $u = x$, $v = y$ and $w = z$) shows that

$$\begin{aligned}(a_0, a_1, \ldots, a_k, b_1, b_2, \ldots, b_\ell) &= (a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_\ell) \\ &= (a_0, a_1, \ldots, a_{k-1}, y, b_1, b_2, \ldots, b_\ell)\end{aligned}$$

is a walk from x to z. Thus, there exists a walk from x to z in G (namely, the walk we have just constructed). In other words, $x \simeq_G z$ (by the definition of $\simeq_G$).

Now, forget that we fixed x, y and z. We thus have shown that for each $x \in \operatorname{V}(G)$, $y \in \operatorname{V}(G)$ and $z \in \operatorname{V}(G)$ satisfying $x \simeq_G y$ and $y \simeq_G z$, we have $x \simeq_G z$. In other words, the relation $\simeq_G$ is transitive (by the definition of "transitive"). This proves Claim 3.

Altogether, we now have shown that the relation $\simeq_G$ is reflexive, symmetric and transitive. In other words, this relation $\simeq_G$ is an equivalence relation (by the definition of an "equivalence relation"). This proves Proposition 2.8.17. $\square$

The following proposition gives a few equivalent characterizations for the relation $\simeq_G$ introduced in Definition 2.8.14:

Proposition 2.8.19. Let G be a simple graph. Let u and v be two vertices of G. The following seven statements are equivalent:

- *Statement $\mathcal{S}_0$:* We have $u \simeq_G v$.
- *Statement $\mathcal{S}_1$:* There exists a walk from u to v.
- *Statement $\mathcal{S}_2$:* There exists a walk from v to u.
- *Statement $\mathcal{S}_3$:* There exists a walk from one of the two vertices u and v to the other.

- *Statement $\mathcal{S}_4$:* There exists a path from u to v.
- *Statement $\mathcal{S}_5$:* There exists a path from v to u.
- *Statement $\mathcal{S}_6$:* There exists a path from one of the two vertices u and v to the other.

Proof of Proposition 2.8.19. The definition of the relation $\simeq_G$ shows that $u \simeq_G v$ holds if and only if there exists a walk from u to v in G. In other words, Statement $\mathcal{S}_0$ holds if and only if Statement $\mathcal{S}_1$ holds. This proves the equivalence $\mathcal{S}_0 \iff \mathcal{S}_1$.

The implication $\mathcal{S}_1 \Longrightarrow \mathcal{S}_2$ is easy to prove[29]. The same argument, but with the roles of u and v interchanged, proves the implication $\mathcal{S}_2 \Longrightarrow \mathcal{S}_1$. Combining the two implications $\mathcal{S}_1 \Longrightarrow \mathcal{S}_2$ and $\mathcal{S}_2 \Longrightarrow \mathcal{S}_1$, we obtain the equivalence $\mathcal{S}_1 \iff \mathcal{S}_2$.

The statement $\mathcal{S}_3$ is the disjunction of the two statements $\mathcal{S}_1$ and $\mathcal{S}_2$. In other words, $\mathcal{S}_3 \iff (\mathcal{S}_1 \vee \mathcal{S}_2)$. But from $\mathcal{S}_1 \iff \mathcal{S}_2$, we obtain $(\mathcal{S}_1 \vee \mathcal{S}_2) \iff (\mathcal{S}_2 \vee \mathcal{S}_2) \iff \mathcal{S}_2$. Hence, $\mathcal{S}_3 \iff (\mathcal{S}_1 \vee \mathcal{S}_2) \iff \mathcal{S}_2$.

Combining the three equivalences $\mathcal{S}_0 \iff \mathcal{S}_1$, $\mathcal{S}_1 \iff \mathcal{S}_2$ and $\mathcal{S}_3 \iff \mathcal{S}_2$, we obtain the chain of equivalences

$$\mathcal{S}_0 \iff \mathcal{S}_1 \iff \mathcal{S}_2 \iff \mathcal{S}_3. \tag{28}$$

The implication $\mathcal{S}_4 \Longrightarrow \mathcal{S}_1$ holds for obvious reasons (namely, because every path is a walk). But the implication $\mathcal{S}_1 \Longrightarrow \mathcal{S}_4$ also holds[30]. Combining the two implications $\mathcal{S}_1 \Longrightarrow \mathcal{S}_4$ and $\mathcal{S}_4 \Longrightarrow \mathcal{S}_1$, we obtain the equivalence $\mathcal{S}_1 \iff \mathcal{S}_4$. The same argument (but with the roles of u and v interchanged) proves the equivalence $\mathcal{S}_2 \iff \mathcal{S}_5$. Thus, $\mathcal{S}_1 \iff \mathcal{S}_2 \iff \mathcal{S}_5$.

The statement $\mathcal{S}_6$ is the disjunction of the two statements $\mathcal{S}_4$ and $\mathcal{S}_5$. In other words, $\mathcal{S}_6 \iff (\mathcal{S}_4 \vee \mathcal{S}_5)$. But from $\mathcal{S}_4 \iff \mathcal{S}_1$ and $\mathcal{S}_5 \iff \mathcal{S}_1$, we obtain $(\mathcal{S}_4 \vee \mathcal{S}_5) \iff (\mathcal{S}_1 \vee \mathcal{S}_1) \iff \mathcal{S}_1$. Hence, $\mathcal{S}_6 \iff (\mathcal{S}_4 \vee \mathcal{S}_5) \iff \mathcal{S}_1$.

Because of the three equivalences $\mathcal{S}_1 \iff \mathcal{S}_4$, $\mathcal{S}_1 \iff \mathcal{S}_5$ and $\mathcal{S}_1 \iff \mathcal{S}_6$, we can extend the chain of equivalences (28) to

$$\mathcal{S}_0 \iff \mathcal{S}_1 \iff \mathcal{S}_2 \iff \mathcal{S}_3 \iff \mathcal{S}_4 \iff \mathcal{S}_5 \iff \mathcal{S}_6.$$

Thus, Proposition 2.8.19 is proven. □

[29] *Proof.* Assume that $\mathcal{S}_1$ holds. We must show that $\mathcal{S}_2$ holds.

There exists a walk from u to v (since $\mathcal{S}_1$ holds). Fix such a walk, and denote it by $\mathbf{a} = (a_0, a_1, \ldots, a_k)$.

Thus, $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ is a walk from u to v in G. Hence, Proposition 2.8.6 **(a)** shows that the list $(a_k, a_{k-1}, \ldots, a_0)$ is a walk from v to u. Thus, there exists a walk from v to u (namely, this list). In other words, $\mathcal{S}_2$ holds. This proves the implication $\mathcal{S}_1 \Longrightarrow \mathcal{S}_2$.

[30] *Proof.* Assume that $\mathcal{S}_1$ holds. We must show that $\mathcal{S}_4$ holds.

There exists a walk from u to v (since $\mathcal{S}_1$ holds). Fix such a walk, and denote it by $\mathbf{a}$. Let k be the length of this walk $\mathbf{a}$. Then, exists a walk from u to v of length k (namely, $\mathbf{a}$). Corollary 2.8.10 therefore shows that there exists a path from u to v of length $\leq k$. In particular, there exists a path from u to v. In other words, $\mathcal{S}_4$ holds. This proves the implication $\mathcal{S}_1 \Longrightarrow \mathcal{S}_4$.

2.8.5. Dijkstra's algorithm

TODO 2.8.20. Dijkstra's algorithm. Use tables of paths to formalize it.

TODO 2.8.21. Proof.

2.8.6. Circuits and cycles

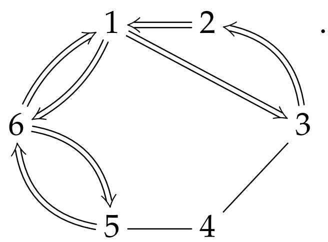
Definition 2.8.22. Let G be a simple graph.

(a) A *circuit* (in G) means a walk $(v_0, v_1, \ldots, v_k)$ in G satisfying $v_k = v_0$.

(b) A *cycle* (in G) means a circuit $(v_0, v_1, \ldots, v_k)$ in G such that $k \geq 3$, and such that the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct.

Example 2.8.23. Let V and E be as in Example 2.4.2 **(c)**. Let G be the graph (V, E). Then:

- The sequence $(1, 3, 2, 1, 6, 5, 6, 1)$ of vertices of G is a circuit (in G), since it is a walk and satisfies $1 = 1$. But it is not a cycle, since the vertices $1, 3, 2, 1, 6, 5, 6$ are not distinct. Let us visualize this circuit by marking all the edges that it uses with double arrows:

  Note that this picture does not determine the circuit uniquely; it could just as well stand for the circuit $(1, 6, 5, 6, 1, 3, 2, 1)$ or for the circuit $(2, 1, 6, 5, 6, 1, 3, 2)$ (or for several other circuits).

- The sequence $(1, 2, 3, 1)$ of vertices of G is a circuit and a cycle (since the vertices $1, 2, 3$ are distinct).

- The sequence $(1, 2, 1)$ of vertices of G is a circuit, but not a cycle (since it fails the $k \geq 3$ condition in Definition 2.8.22).

- The sequence $(1, 3, 4, 5)$ of vertices of G is not a circuit (since $5 \neq 1$).

Example 2.8.24. **(a)** The path graph P_n introduced in Definition 2.7.6 has no cycles (although it has several circuits, such as $(1, 2, 3, 2, 1)$ for $n \geq 3$).

(b) The cycle graph C_n introduced in Definition def.intro.cycle-graph has $2n$ cycles (for $n \geq 3$): Namely, for each vertex $i \in \{1, 2, \ldots, n\}$ of C_n, we can either move forward along the circle (obtaining the cycle $(i, i+1, i+2, \ldots, n, 1, 2, \ldots, i)$) or move backward along the circle (obtaining the cycle $(i, i-1, i-2, \ldots, 1, n, n-1, \ldots, i)$). These $2n$ cycles are all distinct according to our definition of a cycle, although they all use the same edges.

TODO 2.8.25. Explain that various books define cycles differently, e.g. Diestel:

- Several sources define a cycle not as a list of vertices, but rather as a subgraph isomorphic to C_k for some $k \geq 3$. For example, [Dieste16], [BonMur08] and [West01] do this. This definition is closely related to ours, because when $(v_0, v_1, \ldots, v_k)$ is a cycle of G (in our meaning of this word), then the subgraph

$$(\{v_0, v_1, \ldots, v_k\}, \{v_i v_{i+1} \mid i \in \{0, 1, \ldots, k-1\}\})$$

  of G is isomorphic to C_k. But the two definitions are not literally equivalent, because the cycle $(v_0, v_1, \ldots, v_k)$ (in our sense) cannot be uniquely reconstructed from the subgraph defined by it; indeed, any rotated version $(v_i, v_{i+1}, \ldots, v_{k-1}, v_0, v_1, \ldots, v_{i-1})$ of this cycle (as well as its reflected version $(v_k, v_{k-1}, \ldots, v_0)$) yields the same subgraph.

- [Bollob98] defines a cycle as an equivalence class of what we call "cycle" with respect to reflections and rotations. This ends up equivalent to the convention used in [Dieste16], [BonMur08] and [West01].

- I am not sure how Ore defines cycles in [Ore90] and [Ore74]; but he doesn't seem to do much with this notion.

- [Harju14] defines cycles in the same way as we do, but erroneously omits the condition $k \geq 3$.

- The definition of cycles in [BehCha71] is equivalent to ours (though circuits are defined slightly differently).

- ...

I haven't even started discussing the various definitions of "walk", "path" and "circuit". Be careful.

Some authors define "circuit" to mean "closed walk with distinct edges", which too differs from what we do.

TODO 2.8.26. Facts about cycles and circuits:
- Rotating a circuit yields a circuit.
- Rotating a cycle yields a cycle.
- Reversal of a circuit is a circuit.
- Reversal of a cycle is a cycle.
- Edges of a cycle are distinct.
- Each walk with no two identical adjacent edges contains a cycle, unless it is a path.
- If there are two different paths from u to v, then there is a cycle.

2.8.7. Connectedness

Definition 2.8.27. Let $G = (V, E)$ be a simple graph. We say that G is *connected* if G satisfies the following two properties:

- Its vertex set V is nonempty.
- Each two vertices $u \in V$ and $v \in V$ satisfy $u \simeq_G v$.

TODO 2.8.28. Examples. Reuse Example 2.7.10.

Definition 2.8.29. Let $G = (V, E)$ be a simple graph. Let $v \in V$. The *connected component of* v (in G) denotes the subset $\{u \in V \mid u \simeq_G v\}$ of V. We shall denote this connected component of v by $\operatorname{conncomp} v$ (or by $\operatorname{conncomp}_G v$, when G is not clear from the context).

A *connected component of* G means a subset of V that is the connected component of some $v \in V$. (Note that the v is usually not uniquely determined – in fact, several different vertices $v \in V$ often have the same connected component.)

Roughly speaking, if v is a vertex of a simple graph G, then the connected component of v is the set of all vertices that are reachable from v by walks (i.e., that are ending points of walks that start at v).

Example 2.8.30. **(a)** Let $G = (V, E)$ be the simple graph defined in Example 2.4.2 **(a)**. Then, we know (from Example 2.8.15) that $u \simeq_G v$ for any two vertices u and v of G. Hence, for each $v \in V$, we have $\operatorname{conncomp} v = \left\{ u \in V \mid \underbrace{u \simeq_G v}_{\text{this always holds}} \right\} = \{u \in V\} = V$. In other words, the connected component of any vertex v of G is the whole set V. Thus, G has only one connected component – namely, V.

(b) Let $n \in \mathbb{N}$ be such that $n \geq 3$. Let $G = (V, E)$ be the simple graph defined in Example 2.7.10. Then, two vertices u and v of G satisfy $u \simeq_G v$ if and only if they satisfy $u \equiv v \mod 3$. Hence, G has exactly three connected components:

- The connected component $\operatorname{conncomp} 1 = \{u \in V \mid u \simeq_G 1\} = \{u \in V \mid u \equiv 1 \mod 3\} = \{1, 4, 7, \ldots\} \cap V$.
- The connected component $\operatorname{conncomp} 2 = \{u \in V \mid u \simeq_G 2\} = \{u \in V \mid u \equiv 2 \mod 3\} = \{2, 5, 8, \ldots\} \cap V$.
- The connected component $\operatorname{conncomp} 3 = \{u \in V \mid u \simeq_G 3\} = \{u \in V \mid u \equiv 3 \mod 3\} = \{3, 6, 9, \ldots\} \cap V$.

If we have $n < 3$ instead, then some of these connected components are missing (e.g., for $n = 2$, there is no $\operatorname{conncomp} 3$, and G has only 2 connected components).

TODO 2.8.31. Connected iff exactly 1 connected component.
Remark about the graph with 0 vertices.
Remark about K_n and E_n.

TODO 2.8.32. Restate using paths and walks. Equivalence of:
- $u \simeq_G v$.
- There is a path from u to v.
- There is a walk from u to v.
- $\operatorname{conncomp} u = \operatorname{conncomp} v$.
- $u \in \operatorname{conncomp} v$.
- $v \in \operatorname{conncomp} u$.
- $\operatorname{conncomp} u \cap \operatorname{conncomp} v \neq \varnothing$.

TODO 2.8.33. Each $v \in V$ lies in exactly one connected component.

TODO 2.8.34. Connected components are connected and maximal at that.

TODO 2.8.35. A graph is isomorphic to the disjoint union of its connected components.

2.9. Questions to ask about graphs

TODO 2.9.1. Connectedness.

TODO 2.9.2. Hamiltonian paths and cycles: Do they exist? How to find them? (We will have partial results, mainly sufficient conditions.)

TODO 2.9.3. Eulerian walks and circuits. (Theory is really nice here, with easy necessary-and-sufficient conditions, but we'll wait for a better notion of graph.)

TODO 2.9.4. Matchings: how large, how many, structure? (Nice and large theory, still under research; we will see a lot of it but still barely scratch the surface.)

3. Dominating sets

I will next digress to discuss the notion of *dominating sets*. The reason why I am doing this at this point is not that dominating sets are of any particular fundamental importance (there are arguably more crucial notions in graph theory left to consider), but rather that they neatly illustrate the concepts we have seen so far and provide some experience with graph-theoretical proofs, all that while not requiring any complex theory or advanced techniques.

3.1. Definition

Definition 3.1.1. Let G be a simple graph. A subset U of $\operatorname{V}(G)$ is said to be *dominating* (for G) if it has the following property: For every vertex $v \in \operatorname{V}(G) \setminus U$, at least one neighbor of v belongs to U.

A *dominating set* of G means a dominating subset of $\operatorname{V}(G)$.

Example 3.1.2. For this example, let us consider the graph $G = (V, E)$, where

$$V = \{1, 2, 3, 4, 5\} \qquad \text{and}$$
$$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}.$$

(This graph has already been introduced in Example 2.4.2 **(e)**. It can be drawn to look like a pentagon.)

The subset $\{1,3\}$ of V is dominating (for G). (This can be checked directly: We must show that for every vertex $v \in \operatorname{V}(G) \setminus \{1,3\}$, at least one neighbor of v belongs to $\{1,3\}$. There are three vertices $v \in \operatorname{V}(G) \setminus \{1,3\}$, namely 2, 4 and 5. For $v = 2$, the neighbor 1 of v belongs to $\{1,3\}$ (and so does the neighbor 3). For $v = 4$, the neighbor 3 of v belongs to $\{1,3\}$. For $v = 5$, the neighbor 1 of v belongs to $\{1,3\}$.)

The subset $\{1,2\}$ of V is **not** dominating (for G). (Indeed, the vertex $4 \in \operatorname{V}(G) \setminus \{1,2\}$ does **not** have the property that at least one neighbor of 4 belongs to $\{1,2\}$.)

Every subset of V having at least 3 elements is dominating, whereas no subset of V having at most 1 element is dominating. A 2-element subset can be either dominating or not.

(Of course, more complicated graphs exhibit more complex behavior.)

Exercise 3.1.3. Let $n \in \mathbb{N}$. What is the smallest possible size of a dominating set of the cycle graph C_{3n} ?

Clearly, if G is a simple graph, then its vertex set $\operatorname{V}(G)$ is a dominating set. One natural question to ask is how small a dominating set of a graph can be. When the graph G is empty, only the vertex set $\operatorname{V}(G)$ itself is dominating. On the other hand, when G is a complete graph on $n \geq 1$ vertices, every nonempty subset of $\operatorname{V}(G)$ is dominating. Clearly, the more edges a simple graph has, the more dominating sets it has (in the sense that if we add a new edge, then all sets that are dominating remain dominating, and possibly new dominating sets appear). It is furthermore clear that if a vertex of a simple graph G has no neighbors, then it must belong to each dominating set of G (because otherwise, at least one neighbor of this vertex would need to lie in the dominating set; but this is impossible, since it has no neighbors). Such vertices are said to be *isolated*.

Definition 3.1.4. Let G be a simple graph. A vertex v of G is said to be *isolated* if it has no neighbors. (In other words, a vertex v of G is said to be *isolated* if it belongs to no edge of G. In other words, a vertex v of G is said to be *isolated* if its degree $\deg v$ equals 0.)

Proposition 3.1.5. Let $G = (V, E)$ be a simple graph that has no isolated vertices.

(a) There exist two disjoint dominating subsets A and B of V such that $A \cup B = V$.

(b) There exists a dominating subset of V having size $\leq |V| / 2$.

We will see two proofs of this proposition later, in Subsections 3.5 and 3.6.

Again, Proposition 3.1.5 can be neatly restated in terms of people and friendships[31].

Example 3.1.6. **(a)** Let V, E and G be as in Example 3.1.2. Then, Proposition 3.1.5 **(b)** predicts that there exists a dominating subset of V having size $\leq |V| / 2 = 5/2$. Since the size of a finite set is an integer, this shows that there exists a dominating subset of V having size ≤ 2 (because any integer that is $\leq 5/2$ must automatically be ≤ 2). And indeed, such a dominating subset exists (for example, $\{5, 3\}$).

(b) Now, let us instead consider the graph $G = (V, E)$, where

$$V = \{1, 2, 3, 4, 5\} \qquad \text{and}$$
$$E = \mathcal{P}_2(V).$$

[31] Namely:

*Restatement of Proposition 3.1.5 **(a)**:* Given a group of people, each of whom has at least one friend (among the others), it is always possible to subdivide the group into two teams such that each person has a friend in the opposite team.

*Restatement of Proposition 3.1.5 **(b)**:* Given a group of (finitely many) people each of whom has at least one friend (among the others), it is always possible to choose at most $|V| / 2$ people from this group such that everyone who is not chosen has at least one of the chosen ones among his friends.

(As usual, we assume that the group of people is finite, and that the relation of friendship is symmetric.)

(This is the complete graph K_5.) Again, Proposition 3.1.5 **(b)** predicts that there exists a dominating subset of V having size $\leq |V|/2 = 5/2$. Again, this entails that there exists a dominating subset of V having size ≤ 2 (since the size of a finite set is an integer). This is indeed true, but for this particular graph we can even find a dominating subset of V having size 1: for example, the subset $\{1\}$.

3.2. Brouwer's theorem and the Heinrich-Tittmann formula

Next, we state a surprising recent result by Brouwer ([Brouwe09], from 2009) about the number of dominating sets of a graph:

Theorem 3.2.1. Let G be a simple graph. Then, the number of dominating sets of G is odd.

Brouwer (in [Brouwe09]) gives three proofs of this theorem. We are going to give another. Better yet, we shall prove a more precise result which is even more recent (a preprint [HeiTit17] from 2017), due to Heinrich and Tittmann:

Theorem 3.2.2. Let $G = (V, E)$ be a simple graph. Let $n = |V|$. Assume that $n > 0$.

A *detached pair* shall mean a pair (A, B) of two disjoint subsets A and B of V having the property that there exists no edge $ab \in E$ satisfying $a \in A$ and $b \in B$.

Let α be the number of all detached pairs (A, B) for which both numbers $|A|$ and $|B|$ are even and positive.

Let β be the number of all detached pairs (A, B) for which both numbers $|A|$ and $|B|$ are odd.

Then:

(a) The numbers α and β are even.

(b) The number of dominating sets of G is $2^n - 1 + \alpha - \beta$.

At this point, let me stress that the word "pair" always means an ordered pair throughout these notes. In particular, in Theorem 3.2.2, a detached pair (A, B) should be distinguished from (B, A), unless they actually are equal (which only happens when both A and B are the empty set).

Theorem 3.2.2 is a restatement of [HeiTit17, Theorem 8]. The proof we shall give below is shorter than the proof in [HeiTit17], but does not lead us through as many interesting intermediate results.

Let us first see how Theorem 3.2.1 can be derived from Theorem 3.2.2:

Proof of Theorem 3.2.1 using Theorem 3.2.2. Write the graph G in the form $G = (V, E)$. If $|V| = 0$, then Theorem 3.2.1 holds[32]. Hence, for the rest of this proof, we WLOG assume that $|V| = 0$ does not hold.

[32]*Proof.* Assume that $|V| = 0$. Hence, the set V is empty. Thus, the only subset of V is $\varnothing$. This subset $\varnothing$ is dominating (because it is the whole set V). Thus, there exists exactly 1 dominating set of G (namely, $\varnothing$). In other words, the number of dominating sets of G is 1. Therefore, this number is odd. Hence, Theorem 3.2.1 is proven (under the assumption that $|V| = 0$).

Let us use the notations of Theorem 3.2.2. From Theorem 3.2.2 **(a)**, we know that α and β are even. In other words, $\alpha \equiv 0 \mod 2$ and $\beta \equiv 0 \mod 2$. Furthermore, $n = |V| \neq 0$ (since $|V| = 0$ does not hold), so that n is a positive integer. Thus, 2^n is even. In other words, $2^n \equiv 0 \mod 2$. Now, Theorem 3.2.2 **(b)** shows that the number of dominating sets of G is

$$\underbrace{2^n}_{\equiv 0 \mod 2} -1 + \underbrace{\alpha}_{\equiv 0 \mod 2} - \underbrace{\beta}_{\equiv 0 \mod 2} \equiv 0 - 1 + 0 - 0 = -1 \equiv 1 \mod 2.$$

In other words, the number of dominating sets of G is odd. This proves Theorem 3.2.1. $\square$

Thus, it remains to prove Theorem 3.2.2.

3.3. The Iverson bracket

Our proof of Theorem 3.2.2 will rely on a few lemmas. But first, let us introduce a very basic notation, which has nothing to do with graphs specifically but is useful throughout mathematics (particularly combinatorics):

Definition 3.3.1. Let $\mathcal{A}$ be a logical statement. (It is not required to be true.) Then, a number $[\mathcal{A}] \in \{0,1\}$ is defined as follows: We set $[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$. This number $[\mathcal{A}]$ is called the *truth value* of $\mathcal{A}$. (For example, $[1+1=2] = 1$ and $[1+1=3] = 0$. For another example, $[\text{Proposition 2.4.1 holds}] = 1$, because we have proven Proposition 2.4.1.) The notation $[\mathcal{A}]$ for the truth value of $\mathcal{A}$ is known as the *Iverson bracket notation*.

Truth values satisfy certain simple rules:

Proposition 3.3.2. (a) If $\mathcal{A}$ and $\mathcal{B}$ are two equivalent logical statements, then $[\mathcal{A}] = [\mathcal{B}]$.

(b) If $\mathcal{A}$ is any logical statement, then $[\text{not } \mathcal{A}] = 1 - [\mathcal{A}]$.

(c) If $\mathcal{A}$ and $\mathcal{B}$ are two logical statements, then $[\mathcal{A} \wedge \mathcal{B}] = [\mathcal{A}]\,[\mathcal{B}]$.

(d) If $\mathcal{A}$ and $\mathcal{B}$ are two logical statements, then $[\mathcal{A} \vee \mathcal{B}] = [\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}]\,[\mathcal{B}]$.

Proposition 3.3.3. Let P be a finite set. Let Q be a subset of P.

(a) Then, $|Q| = \sum\limits_{p \in P} [p \in Q]$.

(b) For each $p \in P$, let a_p be a number (for example, a real number). Then, $\sum\limits_{p \in P} [p \in Q]\, a_p = \sum\limits_{p \in Q} a_p$.

(c) For each $p \in P$, let a_p be a number (for example, a real number). Let $q \in P$. Then, $\sum\limits_{p \in P} [p = q]\, a_p = a_q$.

Exercise 3.3.4. **(a)** Prove Proposition 3.3.2.

(b) Prove Proposition 3.3.3.

Now, let G be a simple graph.

(c) Prove that $\deg v = \sum\limits_{u \in \operatorname{V}(G)} [uv \in \operatorname{E}(G)]$ for each vertex v of G.

(d) Prove that $2\,|\operatorname{E}(G)| = \sum\limits_{u \in \operatorname{V}(G)} \sum\limits_{v \in \operatorname{V}(G)} [uv \in \operatorname{E}(G)]$.

The following lemma is fundamental to much of combinatorics (if not to say much of mathematics):

Lemma 3.3.5. Let P be a finite set. Then,

$$\sum_{A \subseteq P} (-1)^{|A|} = [P = \varnothing].$$

(The symbol "$\sum\limits_{A \subseteq P}$" means "sum over all subsets A of P". In other words, it means "$\sum\limits_{A \in \mathcal{P}(P)}$".)

Proof of Lemma 3.3.5. If $P = \varnothing$, then Lemma 3.3.5 holds[33]. Hence, for the rest of this proof, we WLOG assume that $P \neq \varnothing$. Thus, there exists at least one element p of P. Pick such a p.

There are two kinds of subsets of P: the ones that contain p, and the ones that do not. Hence, the sum $\sum\limits_{A \subseteq P} (-1)^{|A|}$ can be decomposed as follows:

$$\sum_{A \subseteq P} (-1)^{|A|} = \sum_{\substack{A \subseteq P;\\ p \in A}} (-1)^{|A|} + \sum_{\substack{A \subseteq P;\\ p \notin A}} (-1)^{|A|}. \tag{29}$$

But every subset A of P that contains p has the form $B \cup \{p\}$ for a **unique** subset B of P that does not contain p (namely, for $B = A \setminus \{p\}$). Conversely, of course, if B is a subset of P that does not contain p, then $B \cup \{p\}$ will always be a subset of P that contains p. Hence, there exists a bijection (i.e., a bijective map) from the set of all subsets of P that do not contain p to the set of all subsets of P that do contain p; namely, this bijection sends each subset B of P that does not contain p to $B \cup \{p\}$.

[33] *Proof.* Assume that $P = \varnothing$. Then, there exists only one subset of P, namely $\varnothing$. Hence, the sum $\sum\limits_{A \subseteq P} (-1)^{|A|}$ has only one addend, namely the addend for $A = \varnothing$. Therefore, this sum simplifies to

$$\sum_{A \subseteq P} (-1)^{|A|} = (-1)^{|\varnothing|} = 1$$

(since $|\varnothing| = 0$). Comparing this with $[P = \varnothing] = 1$ (which holds, since $P = \varnothing$ is true), we obtain $\sum\limits_{A \subseteq P} (-1)^{|A|} = [P = \varnothing]$. Hence, we have shown that Lemma 3.3.5 holds if $P = \varnothing$.

Using this bijection, we can rewrite the sum $\sum\limits_{\substack{A\subseteq P;\\ p\in A}} (-1)^{|A|}$ as follows:

$$\sum_{\substack{A\subseteq P;\\ p\in A}} (-1)^{|A|} = \sum_{\substack{B\subseteq P;\\ p\notin B}} \underbrace{(-1)^{|B\cup\{p\}|}}_{\substack{=(-1)^{|B|+1}\\ \text{(since } |B\cup\{p\}|=|B|+1\\ \text{(because } p\notin B))}} = \sum_{\substack{B\subseteq P;\\ p\notin B}} \underbrace{(-1)^{|B|+1}}_{=-(-1)^{|B|}} = \sum_{\substack{B\subseteq P;\\ p\notin B}} \left(-(-1)^{|B|}\right)$$
$$= -\sum_{\substack{B\subseteq P;\\ p\notin B}} (-1)^{|B|} = -\sum_{\substack{A\subseteq P;\\ p\notin A}} (-1)^{|A|}$$

(here, we have renamed the summation index B as A). Thus, (29) becomes

$$\sum_{A\subseteq P} (-1)^{|A|} = \underbrace{\sum_{\substack{A\subseteq P;\\ p\in A}} (-1)^{|A|}}_{=-\sum\limits_{\substack{A\subseteq P;\\ p\notin A}} (-1)^{|A|}} + \sum_{\substack{A\subseteq P;\\ p\notin A}} (-1)^{|A|} = -\sum_{\substack{A\subseteq P;\\ p\notin A}} (-1)^{|A|} + \sum_{\substack{A\subseteq P;\\ p\notin A}} (-1)^{|A|} = 0.$$

Comparing this with $[P=\varnothing] = 0$ (which holds, since $P = \varnothing$ is false (because we assumed $P \neq \varnothing$)), we obtain $\sum\limits_{A\subseteq P} (-1)^{|A|} = [P=\varnothing]$. Hence, Lemma 3.3.5 is proven. □

A different proof of Lemma 3.3.5 can be obtained using the binomial formula and the combinatorial interpretation of binomial coefficients. The main idea of this latter proof is to observe that for each $k \in \{0,1,\ldots,|P|\}$, exactly $\dbinom{|P|}{k}$ among the subsets of P have size k, and therefore $\sum\limits_{A\subseteq P} (-1)^{|A|}$ can be rewritten as $\sum\limits_{k=0}^{|P|} \dbinom{|P|}{k} (-1)^k$, which in turn can be simplified (using the binomial formula) to $(1-1)^{|P|} = 0^{|P|} = [|P|=0] = [P=\varnothing]$. We leave the details of this alternative proof to the interested reader.

The following exercise demonstrates an application of Lemma 3.3.5 to (elementary) number theory:

Exercise 3.3.6. Let $\mathbb{N}_+$ denote the set $\{1,2,3,\ldots\}$. An integer n is said to be *squarefree* if it is not divisible by any perfect square apart from 1. (Note that if n is a positive integer, then n is squarefree if and only if n is a product of **distinct** prime numbers.) We define a map $\mu : \mathbb{N}_+ \to \{-1,0,1\}$ by setting

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \text{if } n \text{ is squarefree;} \\ 0, & \text{if not} \end{cases} \qquad \text{for each } n \in \mathbb{N}_+,$$

where $\omega(n)$ denotes the number of distinct prime divisors of n. (For example, $\mu(6) = (-1)^2 = 1$, $\mu(30) = (-1)^3 = -1$, $\mu(1) = (-1)^0 = 1$, and $\mu(12) = 0$. The map μ is called the *(number-theoretical) Möbius function*.)

Prove that each $n \in \mathbb{N}_+$ satisfies

$$\sum_{d \mid n} \mu(d) = [n = 1],$$

where the sum on the left hand side should be understood as a sum over all positive divisors of n.

[**Hint:** Apply Lemma 3.3.5 with P being the set of all prime factors of n. Relate the subsets of P to the squarefree divisors of n.]

Exercise 3.3.7. Let A be a finite set. Let $A_1, A_2, \ldots, A_m$ be some subsets of A. Prove that

$$\begin{aligned} &\left|\left\{U \subseteq A \mid U \cap A_i \neq \varnothing \text{ for all } i \in \{1, 2, \ldots, m\}\right\}\right| \\ &\equiv \left|\left\{J \subseteq \{1, 2, \ldots, m\} \mid \bigcup_{j \in J} A_j = A\right\}\right| \mod 2. \end{aligned}$$

[**Hint:** Set $I = \{1, 2, \ldots, m\}$. Compute the double sum

$$\sum_{U \subseteq A} \sum_{J \subseteq I} (-1)^{|J|} (-1)^{|U|} \left[U \subseteq A \setminus \bigcup_{j \in J} A_j\right]$$

in two different ways (once by interchanging the summations, and once again by rewriting $\left[U \subseteq A \setminus \bigcup_{j \in J} A_j\right]$ as $[J \subseteq \{i \in I \mid U \cap A_i = \varnothing\}]$). Notice that powers of -1 can be discarded when working modulo 2.]

3.4. Proving the Heinrich-Tittmann formula

In this section, we shall finally prove Theorem 3.2.2. Instead of presenting the proof as a monolithic piece of work, I shall distribute most of it into a series of easy lemmas (some of which are of independent interest).

Throughout this section, we consider a simple graph $G = (V, E)$. The notion of a "detached pair" is to be understood as in Theorem 3.2.2.

Lemma 3.4.1. Let B be a subset of V. Then,

$$\sum_{\substack{A \subseteq V; \\ (A,B) \text{ is a detached pair}}} (-1)^{|A|} = [B \text{ is dominating}].$$

Proof of Lemma 3.4.1. Let B' be the set of all vertices $v \in V \setminus B$ such that at least one neighbor of v belongs to B. Then, for each subset A of V, the following equivalence holds:

$$((A, B) \text{ is a detached pair}) \iff \left(A \subseteq V \setminus \left(B \cup B'\right)\right) \tag{30}$$

[34].

On the other hand, the following equivalence holds:

$$\left(V \setminus \left(B \cup B'\right) = \varnothing\right) \iff (B \text{ is dominating}) \tag{31}$$

[35].

[34] *Proof of* (30). We shall prove the $\Longrightarrow$ and $\Longleftarrow$ directions of the equivalence (30) separately:

$\Longrightarrow$: Assume that (A, B) is a detached pair. We must show that $A \subseteq V \setminus (B \cup B')$.

We know that (A, B) is a detached pair. By the definition of a "detached pair", this means that A and B are two disjoint subsets of V having the property that there exists no edge $ab \in E$ satisfying $a \in A$ and $b \in B$.

We have $A \subseteq V \setminus B$ (since A and B are disjoint).

We claim that $A \cap B' = \varnothing$. Indeed, assume the contrary. Thus, the set $A \cap B'$ is nonempty. Hence, there exists some $v \in A \cap B'$. Fix such a v. From $v \in A \cap B' \subseteq B'$, we conclude that v is a vertex in $V \setminus B$ such that at least one neighbor of v belongs to B (by the definition of B'). In particular, at least one neighbor of v belongs to B. Let w be such a neighbor. Then, $vw \in E$ (since w is a neighbor of v). Notice also that $v \in A \cap B' \subseteq A$ and $w \in B$. Now, recall that there exists no edge $ab \in E$ satisfying $a \in A$ and $b \in B$. This contradicts the fact that vw is such an edge (since $vw \in E$, $v \in A$ and $w \in B$). This contradiction shows that our assumption was wrong. Hence, $A \cap B' = \varnothing$. Thus, $A \subseteq V \setminus B'$.

Combining $A \subseteq V \setminus B$ with $A \subseteq V \setminus B'$, we find

$$A \subseteq (V \setminus B) \cap \left(V \setminus B'\right) = V \setminus \left(B \cup B'\right).$$

Hence, the $\Longrightarrow$ direction of the equivalence (30) is proven.

$\Longleftarrow$: Assume that $A \subseteq V \setminus (B \cup B')$. We must then show that (A, B) is a detached pair.

First of all, we have $A \subseteq V \setminus (B \cup B') \subseteq V \setminus B$ (since $B \cup B' \supseteq B$). Thus, the sets A and B are disjoint.

Next, I claim that there exists no edge $ab \in E$ satisfying $a \in A$ and $b \in B$. Indeed, assume the contrary. Thus, there exists an edge $ab \in E$ satisfying $a \in A$ and $b \in B$. Consider such an edge. Then, b is a neighbor of a (since $ab \in E$) and belongs to B. Hence, at least one neighbor of a belongs to B (namely, the neighbor b). The element a is a vertex in $V \setminus B$ (since $a \in A \subseteq V \setminus B$) such that at least one neighbor of a belongs to B. In other words, a belongs to B' (by the definition of B'). Thus, $a \in B' \subseteq B \cup B'$. But from $a \in A \subseteq V \setminus (B \cup B')$, we obtain $a \notin B \cup B'$. This contradicts $a \in B \cup B'$. This contradiction shows that our assumption was wrong. Hence, we have shown that there exists no edge $ab \in E$ satisfying $a \in A$ and $b \in B$. Since the subsets A and B of V are disjoint, this shows that (A, B) is a detached pair (by the definition of a "detached pair"). This proves the $\Longleftarrow$ direction of the equivalence (30).

Hence, both directions of (30) are proven.

[35] *Proof of* (31). Again, we are going to separately prove the $\Longrightarrow$ and $\Longleftarrow$ directions of the equivalence:

$\Longrightarrow$: Assume that $V \setminus (B \cup B') = \varnothing$. We must show that B is dominating.

Let $v \in \operatorname{V}(G) \setminus B$. Thus, $v \in \operatorname{V}(G)$ and $v \notin B$. We have $v \in \operatorname{V}(G) = V \subseteq B \cup B'$ (since $V \setminus (B \cup B') = \varnothing$). Combined with $v \notin B$, we obtain $v \in (B \cup B') \setminus B \subseteq B'$. According to the definition of B', this means that v is a vertex in $V \setminus B$ such that at least one neighbor of v belongs to B. In particular, we thus have shown that at least one neighbor of v belongs to B.

Now, we can use the equivalence (30) to rewrite the summation sign $\sum\limits_{\substack{A\subseteq V;\\ (A,B) \text{ is a detached pair}}}$ as $\sum\limits_{\substack{A\subseteq V;\\ A\subseteq V\setminus(B\cup B')}}$. This latter summation sign can be further simplified to $\sum\limits_{A\subseteq V\setminus(B\cup B')}$ (since every subset A of $V\setminus(B\cup B')$ is clearly a subset of V as well). Hence, we can replace the summation sign $\sum\limits_{\substack{A\subseteq V;\\ (A,B) \text{ is a detached pair}}}$ by $\sum\limits_{A\subseteq V\setminus(B\cup B')}$. In particular,

$$\begin{aligned}
&\sum_{\substack{A\subseteq V;\\ (A,B) \text{ is a detached pair}}} (-1)^{|A|} \\
&= \sum_{A\subseteq V\setminus(B\cup B')} (-1)^{|A|} \\
&= \left[V\setminus\left(B\cup B'\right)=\varnothing\right] \qquad \left(\text{by Lemma 3.3.5, applied to } P = V\setminus\left(B\cup B'\right)\right) \\
&= [B \text{ is dominating}]
\end{aligned}$$

(by the equivalence (31)). This proves Lemma 3.4.1. $\square$

Lemma 3.4.1 has the following consequence:

Corollary 3.4.2. Let B be a subset of V. Then,

$$\sum_{\substack{A\subseteq V;\\ A\neq\varnothing;\\ (A,B) \text{ is a detached pair}}} (-1)^{|A|} = [B \text{ is dominating}] - 1.$$

Proof of Corollary 3.4.2. The pair $(\varnothing, B)$ is a detached pair[36]. Hence, the sum

Now, forget that we fixed v. We thus have proven that for every vertex $v \in \operatorname{V}(G)\setminus B$, at least one neighbor of v belongs to B. In other words, the set B is dominating (because this is how we defined "dominating sets"). This proves the $\Longrightarrow$ direction of the equivalence (31).

$\Longleftarrow$: Assume that B is dominating. We must prove that $V\setminus(B\cup B') = \varnothing$.

We know that B is dominating. In other words, for every vertex $v \in \operatorname{V}(G)\setminus B$, at least one neighbor of v belongs to B (because this is what "dominating" means). In other words, for every $v \in V\setminus B$, at least one neighbor of v belongs to B (since $\operatorname{V}(G) = V$).

Now, let $v \in V\setminus B$ be arbitrary. As we have just seen, we then know that at least one neighbor of v belongs to B.

Let $v \in V\setminus B$. Thus, v is a vertex in $V\setminus B$ such that at least one neighbor of v belongs to B. This means that $v \in B'$ (by the definition of B').

Now, forget that we fixed v. We thus have shown that $v \in B'$ for each $v \in V\setminus B$. In other words, $V\setminus B \subseteq B'$. But now, $V\setminus(B\cup B') = (V\setminus B)\setminus B' = \varnothing$ (since $V\setminus B\subseteq B'$). Thus, the $\Longleftarrow$ direction of the equivalence (31) is proven.

[36] *Proof.* The sets $\varnothing$ and B are two disjoint subsets of V (disjoint because $\varnothing\cap B = \varnothing$) having the property that there exists no edge $ab\in E$ satisfying $a\in\varnothing$ and $b\in B$ (this is vacuously true, since there exists no $a\in\varnothing$). In other words, $(\varnothing, B)$ is a detached pair (by the definition of a "detached pair").

$\sum\limits_{\substack{A\subseteq V;\\(A,B)\text{ is a detached pair}}} (-1)^{|A|}$ has an addend for $A = \varnothing$. If we split off this addend from this sum, we obtain

$$\sum_{\substack{A\subseteq V;\\(A,B)\text{ is a detached pair}}} (-1)^{|A|} = (-1)^{|\varnothing|} + \sum_{\substack{A\subseteq V;\\(A,B)\text{ is a detached pair;}\\A\neq\varnothing}} (-1)^{|A|}.$$

Hence,

$$\sum_{\substack{A\subseteq V;\\(A,B)\text{ is a detached pair;}\\A\neq\varnothing}} (-1)^{|A|} = \underbrace{\sum_{\substack{A\subseteq V;\\(A,B)\text{ is a detached pair}}} (-1)^{|A|}}_{\substack{=[B\text{ is dominating}]\\(\text{by Lemma 3.4.1})}} - \underbrace{(-1)^{|\varnothing|}}_{=(-1)^0=1}$$
$$= [B\text{ is dominating}] - 1.$$

Thus,

$$\sum_{\substack{A\subseteq V;\\A\neq\varnothing;\\(A,B)\text{ is a detached pair}}} (-1)^{|A|} = \sum_{\substack{A\subseteq V;\\(A,B)\text{ is a detached pair;}\\A\neq\varnothing}} (-1)^{|A|} = [B\text{ is dominating}] - 1.$$

This proves Corollary 3.4.2. $\square$

Lemma 3.4.3. Let A and B be two subsets of V. Then, (A,B) is a detached pair if and only if (B,A) is a detached pair.

Proof of Lemma 3.4.3. We have the following chain of logical equivalences:

$$\begin{aligned}
&((B,A)\ \text{ is a detached pair})\\
\Longleftrightarrow\ &(B\text{ and }A\text{ are two disjoint subsets of }V\text{ having the property that}\\
&\qquad\qquad\text{there exists no edge }ab\in E\text{ satisfying }a\in B\text{ and }b\in A)\\
&\qquad(\text{by the definition of “detached pair”})\\
\Longleftrightarrow\ &(B\text{ and }A\text{ are two disjoint subsets of }V\text{ having the property that}\\
&\qquad\qquad\text{there exists no edge }ba\in E\text{ satisfying }b\in B\text{ and }a\in A)\\
&\qquad(\text{here, we renamed }a\text{ and }b\text{ as }b\text{ and }a\text{, respectively})\\
\Longleftrightarrow\ &(A\text{ and }B\text{ are two disjoint subsets of }V\text{ having the property that}\\
&\qquad\qquad\text{there exists no edge }ba\in E\text{ satisfying }a\in A\text{ and }b\in B)\\
&\qquad(\text{since }B\text{ and }A\text{ are disjoint if and only if }A\text{ and }B\text{ are disjoint})\\
\Longleftrightarrow\ &(A\text{ and }B\text{ are two disjoint subsets of }V\text{ having the property that}\\
&\qquad\qquad\text{there exists no edge }ab\in E\text{ satisfying }a\in A\text{ and }b\in B)\\
&\qquad(\text{since }ba=ab)\\
\Longleftrightarrow\ &((A,B)\ \text{ is a detached pair})\\
&\qquad(\text{by the definition of “detached pair”}).
\end{aligned}$$

This proves Lemma 3.4.3. $\square$

Next comes another general lemma about cardinalities of sets:

Lemma 3.4.4. Let S be a finite set. Let $\sigma : S \to S$ be a map such that $\sigma \circ \sigma = \mathrm{id}_S$. (Such a map is called an *involution* on S.)

Let $F = \{i \in S \mid \sigma(i) = i\}$. (The elements of F are known as the *fixed points* of σ.) Then, $|F| \equiv |S| \mod 2$.

Proof of Lemma 3.4.4. In a nutshell, the proof of Lemma 3.4.4 is very simple: We have $\sigma \circ \sigma = \mathrm{id}_S$; in other words, we have $\sigma(\sigma(i)) = i$ for each $i \in S$. Each $i \in S \setminus F$ satisfies $\sigma(i) \neq i$. Thus, we can assign to each element $i \in S \setminus F$ the two-element subset $\{i, \sigma(i)\} \subseteq S$. This assignment has the property that the two-element subset assigned to $\sigma(i)$ is the same as the one assigned to i (since $\left\{\sigma(i), \underbrace{\sigma(\sigma(i))}_{=i}\right\} = \{\sigma(i), i\} = \{i, \sigma(i)\}$). Thus, each two-element subset that gets assigned at all is assigned to exactly two elements of $S \setminus F$ (namely, the subset assigned to i is assigned to i and $\sigma(i)$, and to no other elements of $S \setminus F$). As a consequence, the elements of $S \setminus F$ are paired up (each pair consisting of two elements to which the same subset is assigned). Correspondingly, $|S \setminus F|$ is even. Thus, $|S| - |F| = |S \setminus F|$ is even, so that $|F| \equiv |S| \mod 2$.

If you found this insufficiently rigorous (or unclear), here is a *rigorous version of this proof:* For each $i \in S \setminus F$, we have $\{i, \sigma(i)\} \in \mathcal{P}_2(S)$ [37]. In other words, for each $i \in S \setminus F$, there exists a unique $K \in \mathcal{P}_2(S)$ satisfying $\{i, \sigma(i)\} = K$ (namely, $K = \{i, \sigma(i)\}$).

But

$$\begin{aligned} |S \setminus F| &= (\text{the number of } i \in S \setminus F) \\ &= \sum_{K \in \mathcal{P}_2(S)} (\text{the number of } i \in S \setminus F \text{ satisfying } \{i, \sigma(i)\} = K) \end{aligned} \tag{32}$$

(because for each $i \in S \setminus F$, there exists a unique $K \in \mathcal{P}_2(S)$ satisfying $\{i, \sigma(i)\} = K$).

On the other hand, for each $K \in \mathcal{P}_2(S)$, we have

$$(\text{the number of } i \in S \setminus F \text{ satisfying } \{i, \sigma(i)\} = K) \equiv 0 \mod 2 \tag{33}$$

[38].

[37] *Proof.* Let $i \in S \setminus F$. Thus, $i \in S$ and $i \notin F$. If we had $\sigma(i) = i$, then we would have $i \in F$ (by the definition of F), which would contradict $i \notin F$. Hence, we do not have $\sigma(i) = i$. Thus, we have $\sigma(i) \neq i$. Hence, $\{i, \sigma(i)\}$ is a 2-element set. Since $\{i, \sigma(i)\} \subseteq S$ (because both i and $\sigma(i)$ are elements of S), this shows that $\{i, \sigma(i)\}$ is a 2-element subset of S. In other words, $\{i, \sigma(i)\} \in \mathcal{P}_2(S)$, qed.

[38] *Proof of* (33). Let $K \in \mathcal{P}_2(S)$. If there exists no $i \in S \setminus F$ satisfying $\{i, \sigma(i)\} = K$, then (33) is

Now, (32) becomes

$$\begin{aligned}|S \setminus F| &= \sum_{K \in \mathcal{P}_2(S)} \underbrace{(\text{the number of } i \in S \setminus F \text{ satisfying } \{i, \sigma(i)\} = K)}_{\substack{\equiv 0 \mod 2 \\ (\text{by } (33))}} \\ &\equiv \sum_{K \in \mathcal{P}_2(S)} 0 = 0 \mod 2.\end{aligned}$$

Since $|S \setminus F| = |S| - |F|$ (because $F \subseteq S$), this rewrites as $|S| - |F| \equiv 0 \mod 2$. In other words, $|S| \equiv |F| \mod 2$. This proves Lemma 3.4.4. $\square$

Here is a particular case of Lemma 3.4.4:

Corollary 3.4.5. Let S be a finite set. Let $\sigma : S \to S$ be a map such that $\sigma \circ \sigma = \mathrm{id}_S$. Assume that each $i \in S$ satisfies $\sigma(i) \neq i$. Then, $|S|$ is even.

Proof of Corollary 3.4.5. Let $F = \{i \in S \mid \sigma(i) = i\}$. Lemma 3.4.4 yields $|F| \equiv |S| \mod 2$.

But each $i \in S$ satisfies $\sigma(i) \neq i$. In other words, no $i \in S$ satisfies $\sigma(i) = i$. In other words, $\{i \in S \mid \sigma(i) = i\} = \varnothing$. Thus, $F = \{i \in S \mid \sigma(i) = i\} = \varnothing$. Hence, $|F| = |\varnothing| = 0$. But from $|F| \equiv |S| \mod 2$, we obtain $|S| \equiv |F| = 0 \mod 2$. In other words, $|S|$ is even. Hence, Corollary 3.4.5 is proven. $\square$

Now, it is time to get rid of part **(a)** of Theorem 3.2.2:

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true (because in this case, we have $(\text{the number of } i \in S \setminus F \text{ satisfying } \{i, \sigma(i)\} = K) = 0 \equiv 0 \mod 2$). Hence, we WLOG assume that there does exist some $i \in S \setminus F$ satisfying $\{i, \sigma(i)\} = K$. Fix such an i, and denote it by j. Thus, j is an element of $S \setminus F$ and satisfies $\{j, \sigma(j)\} = K$.

Set $k = \sigma(j)$. Thus, $\sigma(k) = \sigma(\sigma(j)) = \underbrace{(\sigma \circ \sigma)}_{=\mathrm{id}_S}(j) = \mathrm{id}_S(j) = j$ and therefore $\left\{\underbrace{k}_{=\sigma(j)}, \underbrace{\sigma(k)}_{=j}\right\} = \{\sigma(j), j\} = \{j, \sigma(j)\} = K$. Furthermore, from $j \in S \setminus F$, we obtain $j \in S$ and $j \notin F$. If we had $\sigma(j) = j$, then we would have $j \in F$ (by the definition of F), which would contradict $j \notin F$. Hence, we do not have $\sigma(j) = j$. Thus, we have $\sigma(j) \neq j$. Hence, $j \neq \sigma(j) = k$. In other words, j and k are distinct.

If $i \in S \setminus F$ satisfies $\{i, \sigma(i)\} = K$, then i must be either j or k (because $i \in \{i, \sigma(i)\} = K = \left\{j, \underbrace{\sigma(j)}_{=k}\right\} = \{j, k\}$).

If we had $k \in F$, then we would have $\sigma(k) = k$ (by the definition of F), which would contradict $\sigma(k) = j \neq k$. Hence, we do not have $k \in F$. Thus, $k \notin F$. Combining this with $k \in S$, we find $k \in S \setminus F$. Hence, j and k are two elements of $S \setminus F$. Furthermore, j and k are two $i \in S \setminus F$ satisfying $\{i, \sigma(i)\} = K$ (since j is an element of $S \setminus F$ satisfying $\{j, \sigma(j)\} = K$, and since k is an element of $S \setminus F$ satisfying $\{k, \sigma(k)\} = K$). Furthermore, j and k are the **only** such i (because we have shown that if $i \in S \setminus F$ satisfies $\{i, \sigma(i)\} = K$, then i must be either j or k). Therefore, we conclude that there are exactly two $i \in S \setminus F$ that satisfy $\{i, \sigma(i)\} = K$ (namely, j and k) (because j and k are distinct). Hence, $(\text{the number of } i \in S \setminus F \text{ satisfying } \{i, \sigma(i)\} = K) = 2 \equiv 0 \mod 2$. This completes the proof of (33).

Lemma 3.4.6. Let α and β be defined as in Theorem 3.2.2. Then, α and β are even.

Proof of Lemma 3.4.6. Let us first show that β is even.

Indeed, let S be the set of all detached pairs (A, B) for which both numbers $|A|$ and $|B|$ are odd. Then, the definition of β can be rewritten as $\beta = |S|$. Clearly, S is a finite set.

For every $(A, B) \in S$, we have $(B, A) \in S$ [39]. Hence, we can define a map $\sigma : S \to S$ by setting

$$\sigma(A, B) = (B, A) \qquad \text{for all } (A, B) \in S.$$

Consider this σ. It is easy to see that $\sigma \circ \sigma = \mathrm{id}_S$ [40].

Each $i \in S$ satisfies $\sigma(i) \neq i$ [41]. Hence, Corollary 3.4.5 shows that $|S|$ is even. In other words, β is even (since $\beta = |S|$).

The same argument (with the obvious changes[42]) shows that α is even. This completes the proof of Lemma 3.4.6. $\square$

We will spend the rest of this section manipulating sums in various artful ways.

[39] *Proof.* Let $(A, B) \in S$. Thus, (A, B) is a detached pair for which both numbers $|A|$ and $|B|$ are odd (by the definition of S). Hence, A and B are two subsets of V (since (A, B) is a detached pair). But Lemma 3.4.3 shows that (A, B) is a detached pair if and only if (B, A) is a detached pair. Hence, (B, A) is a detached pair (since (A, B) is a detached pair). Furthermore, both numbers $|B|$ and $|A|$ are odd. Thus, (B, A) is a detached pair for which both numbers $|B|$ and $|A|$ are odd. In other words, $(B, A) \in S$ (by the definition of S), qed.

[40] *Proof.* For every $(A, B) \in S$, we have

$$(\sigma \circ \sigma)(A, B) = \sigma\left(\underbrace{\sigma(A, B)}_{=(B,A)}\right) = \sigma(B, A) = (A, B)$$

(by the definition of σ). In other words, the map $\sigma \circ \sigma$ sends each $(A, B) \in S$ to itself. Hence, $\sigma \circ \sigma = \mathrm{id}_S$.

[41] Proof. Let $i \in S$. We must prove that $\sigma(i) \neq i$.

We have $i \in S$. In other words, i is a detached pair (A, B) for which both numbers $|A|$ and $|B|$ are odd (by the definition of S). Consider this (A, B). The number $|B|$ is odd and is a nonnegative integer. Hence, this number $|B|$ is positive (because any odd nonnegative integer is positive). Thus, the set B is nonempty. In other words, $B \neq \varnothing$.

But (A, B) is a detached pair. In other words, A and B are two disjoint subsets of V having the property that there exists no edge $ab \in E$ satisfying $a \in A$ and $b \in B$ (by the definition of a "detached pair"). Now, $A \cap B = \varnothing$ (since A and B are disjoint). If we had $A = B$, then we would have $\underbrace{A}_{=B} \cap B = B \cap B = B \neq \varnothing$, which would contradict $A \cap B = \varnothing$. Hence, we cannot have $A = B$. Thus, $A \neq B$.

If we had $(A, B) = (B, A)$, then we would have $A = B$, which would contradict $A \neq B$. Hence, we cannot have $(A, B) = (B, A)$. Therefore, $(A, B) \neq (B, A)$.

But $i = (A, B)$, and thus $\sigma(i) = \sigma(A, B) = (B, A)$ (by the definition of σ). Now, $i = (A, B) \neq (B, A) = \sigma(i)$. In other words, $\sigma(i) \neq i$, qed.

[42] For example, "odd" has to be replaced by "even and positive".

These manipulations will culminate in a proof of Theorem 3.2.2 **(b)**; they also provide examples of techniques that are useful throughout mathematics.

Lemma 3.4.7. We have

$$\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} \left((-1)^{|A|} + (-1)^{|B|} \right) = 2 \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|A|}.$$

Proof of Lemma 3.4.7. First of all,

$$\begin{aligned} & \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} \left((-1)^{|A|} + (-1)^{|B|} \right) \\ &= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|A|} + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|B|}. \end{aligned} \tag{34}$$

Let us now rewrite the second sum on the right hand side of this equation:

$$\begin{aligned} & \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|B|} \\ &= \sum_{\substack{(B,A) \text{ is a detached pair;} \\ B \neq \varnothing;\ A \neq \varnothing}} (-1)^{|A|} \\ & \qquad \left(\text{here, we have renamed the summation index } (A,B) \text{ as } (B,A)\right) \\ &= \sum_{\substack{(B,A) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|A|} = \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|A|} \end{aligned}$$

(because the condition "(B,A) is a detached pair" under the summation sign is equivalent to the condition "(A,B) is a detached pair" (by Lemma 3.4.3)). Hence,

(34) becomes

$$\begin{aligned}
&\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} \left((-1)^{|A|} + (-1)^{|B|}\right) \\
&= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|} + \underbrace{\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|B|}}_{= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|}} \\
&= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|} + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|} \\
&= 2 \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|}.
\end{aligned}$$

This proves Lemma 3.4.7. □

Lemma 3.4.8. Let $n = |V|$. Assume that $n > 0$. Let δ be the number of dominating sets of G. Then,

$$\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|} = \delta - (2^n - 1).$$

Proof of Lemma 3.4.8. We first observe that the set V has $2^{|V|}$ subsets. In other words, the set V has 2^n subsets (since $|V| = n$). Thus, the sum $\sum_{B\subseteq V} 1$ has 2^n addends. Since each of these addends equals 1, we conclude that this sum equals $2^n \cdot 1 = 2^n$. In other words,

$$\sum_{B\subseteq V} 1 = 2^n. \tag{35}$$

On the other hand, each subset B of V is either dominating or not. Hence,

$$\begin{aligned}
&\sum_{B\subseteq V} [B \text{ is dominating}] \\
&= \sum_{\substack{B\subseteq V; \\ B \text{ is dominating}}} \underbrace{[B \text{ is dominating}]}_{\substack{=1 \\ (\text{since } B \text{ is dominating})}} + \sum_{\substack{B\subseteq V; \\ B \text{ is not dominating}}} \underbrace{[B \text{ is dominating}]}_{\substack{=0 \\ (\text{since } B \text{ is not dominating})}} \\
&= \sum_{\substack{B\subseteq V; \\ B \text{ is dominating}}} 1 + \underbrace{\sum_{\substack{B\subseteq V; \\ B \text{ is not dominating}}} 0}_{=0} = \sum_{\substack{B\subseteq V; \\ B \text{ is dominating}}} 1.
\end{aligned}$$

But the number of dominating sets of G is δ. In other words, the number of dominating subsets $B \subseteq V$ is δ. Hence, the sum $\sum\limits_{\substack{B \subseteq V; \\ B \text{ is dominating}}} 1$ has δ addends. Since each of these addends equals 1, we thus see that this sum equals $\delta \cdot 1 = \delta$. In other words,

$$\sum_{\substack{B \subseteq V; \\ B \text{ is dominating}}} 1 = \delta.$$

Hence,

$$\sum_{B \subseteq V} [B \text{ is dominating}] = \sum_{\substack{B \subseteq V; \\ B \text{ is dominating}}} 1 = \delta. \tag{36}$$

The subset $\varnothing$ of V is not dominating[43]. Hence, $[\varnothing \text{ is dominating}] = 0$.

Each detached pair (A, B) consists of two subsets A and B of V (by the definition of a "detached pair"). Hence,

$$\begin{aligned}
&\sum_{\substack{(A,B) \text{ is a detached pair}; \\ A \neq \varnothing;\ B \neq \varnothing}} (-1)^{|A|} \\
&= \sum_{\substack{B \subseteq V; \\ B \neq \varnothing}} \underbrace{\sum_{\substack{A \subseteq V; \\ A \neq \varnothing; \\ (A,B) \text{ is a detached pair}}} (-1)^{|A|}}_{\substack{= [B \text{ is dominating}] - 1 \\ \text{(by Corollary 3.4.2)}}} \\
&= \sum_{\substack{B \subseteq V; \\ B \neq \varnothing}} \left([B \text{ is dominating}] - 1 \right).
\end{aligned} \tag{37}$$

But the sum $\sum\limits_{B \subseteq V} \left([B \text{ is dominating}] - 1 \right)$ has an addend for $B = \varnothing$. If we split off this addend, we find

$$\begin{aligned}
&\sum_{B \subseteq V} \left([B \text{ is dominating}] - 1 \right) \\
&= \left(\underbrace{[\varnothing \text{ is dominating}]}_{=0} - 1 \right) + \sum_{\substack{B \subseteq V; \\ B \neq \varnothing}} \left([B \text{ is dominating}] - 1 \right) \\
&= -1 + \sum_{\substack{B \subseteq V; \\ B \neq \varnothing}} \left([B \text{ is dominating}] - 1 \right).
\end{aligned}$$

[43] *Proof.* Assume the contrary. Thus, the subset $\varnothing$ of V is dominating.

The set V is nonempty (since $|V| = n > 0$). Thus, there exists some $q \in V$. Consider this q. But the set $\varnothing$ is dominating. In other words, for every vertex $v \in \operatorname{V}(G) \setminus \varnothing$, at least one neighbor of v belongs to $\varnothing$. Applying this to $v = q$, we conclude that at least one neighbor of q belongs to $\varnothing$ (since $q \in V = \operatorname{V}(G) = \operatorname{V}(G) \setminus \varnothing$). Hence, at least some object belongs to $\varnothing$. In other words, $\varnothing$ is nonempty. This is absurd. This contradiction shows that our assumption is false, qed.

Thus,

$$\sum_{\substack{B\subseteq V;\\ B\neq\varnothing}} \left(\left[B \text{ is dominating}\right]-1\right) = \underbrace{\sum_{B\subseteq V} \left(\left[B \text{ is dominating}\right]-1\right)}_{=\sum_{B\subseteq V}\left[B \text{ is dominating}\right]-\sum_{B\subseteq V}1} +1$$
$$= \underbrace{\sum_{B\subseteq V} \left[B \text{ is dominating}\right]}_{\substack{=\delta\\ \text{(by (36))}}} - \underbrace{\sum_{B\subseteq V} 1}_{\substack{=2^n\\ \text{(by (35))}}} +1 = \delta - 2^n + 1.$$

Thus, (37) becomes

$$\sum_{\substack{(A,B) \text{ is a detached pair;}\\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|}$$
$$= \sum_{\substack{B\subseteq V;\\ B\neq\varnothing}} \left(\left[B \text{ is dominating}\right]-1\right) = \delta - 2^n + 1 = \delta - \left(2^n - 1\right).$$

This proves Lemma 3.4.8. $\square$

Let us now finally step to the proof of Theorem 3.2.2:

Proof of Theorem 3.2.2. **(a)** Theorem 3.2.2 **(a)** is precisely Lemma 3.4.6, which has already been proven.

(b) Let δ be the number of dominating sets of G.

We shall compute the sum $\sum\limits_{\substack{(A,B) \text{ is a detached pair;}\\ A\neq\varnothing;\ B\neq\varnothing}} \left((-1)^{|A|} + (-1)^{|B|}\right)$ in two different ways, and then compare the results.

The first way relies on Lemma 3.4.7 and on Lemma 3.4.8: From Lemma 3.4.7, we obtain

$$\sum_{\substack{(A,B) \text{ is a detached pair;}\\ A\neq\varnothing;\ B\neq\varnothing}} \left((-1)^{|A|} + (-1)^{|B|}\right) = 2 \underbrace{\sum_{\substack{(A,B) \text{ is a detached pair;}\\ A\neq\varnothing;\ B\neq\varnothing}} (-1)^{|A|}}_{\substack{=\delta-(2^n-1)\\ \text{(by Lemma 3.4.8)}}}$$
$$= 2\left(\delta - \left(2^n - 1\right)\right). \tag{38}$$

Now, let us try another way. We observe that each detached pair (A, B) satisfies one and only one of the following four conditions:

1. The number $|A|$ is even, and the number $|B|$ is even.

2. The number $|A|$ is even, and the number $|B|$ is odd.

3. The number $|A|$ is odd, and the number $|B|$ is even.

4. The number $|A|$ is odd, and the number $|B|$ is odd.

Accordingly, the sum on the left hand side of (38) can be split into four smaller sums:

$$\begin{aligned}
&\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing}} \left((-1)^{|A|} + (-1)^{|B|}\right) \\
&= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}} \left(\underbrace{(-1)^{|A|}}_{\substack{=1 \\ (\text{since } |A| \text{ is even})}} + \underbrace{(-1)^{|B|}}_{\substack{=1 \\ (\text{since } |B| \text{ is even})}} \right) \\
&\quad + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is odd}}} \left(\underbrace{(-1)^{|A|}}_{\substack{=1 \\ (\text{since } |A| \text{ is even})}} + \underbrace{(-1)^{|B|}}_{\substack{=-1 \\ (\text{since } |B| \text{ is odd})}} \right) \\
&\quad + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is even}}} \left(\underbrace{(-1)^{|A|}}_{\substack{=-1 \\ (\text{since } |A| \text{ is odd})}} + \underbrace{(-1)^{|B|}}_{\substack{=1 \\ (\text{since } |B| \text{ is even})}} \right) \\
&\quad + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is odd}}} \left(\underbrace{(-1)^{|A|}}_{\substack{=-1 \\ (\text{since } |A| \text{ is odd})}} + \underbrace{(-1)^{|B|}}_{\substack{=-1 \\ (\text{since } |B| \text{ is odd})}} \right) \\
&= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}} \underbrace{(1+1)}_{=2} + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is odd}}} \underbrace{(1+(-1))}_{=0} \\
&\quad + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is even}}} \underbrace{((-1)+1)}_{=0} + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is odd}}} \underbrace{((-1)+(-1))}_{=-2} \\
&= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}} 2 + \underbrace{\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is odd}}} 0}_{=0} \\
&\quad + \underbrace{\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is even}}} 0}_{=0} + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is odd}}} (-2)
\end{aligned}$$

$$= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}} 2 + \sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is odd}}} (-2)\,. \tag{39}$$

We shall now analyze the two sums on the right hand side of (39) more closely. Let us begin with the first sum. For any subset A of V, the condition "$A \neq \varnothing$" is equivalent to the condition "$|A|$ is positive" (because clearly, the finite set A is distinct from $\varnothing$ if and only if its size $|A|$ is positive). Likewise, for any subset B of V, the condition "$B \neq \varnothing$" is equivalent to the condition "$|B|$ is positive". Combining the preceding two sentences, we conclude that the conditions "$A \neq \varnothing$" and "$B \neq \varnothing$" under the summation sign $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}}$ are equivalent to the conditions "$|A|$ is positive" and "$|B|$ is positive", respectively. Hence, we can rewrite this summation sign as $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ |A| \text{ is positive; } |B| \text{ is positive;} \\ |A| \text{ is even; } |B| \text{ is even}}}$. Thus,

$$\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}} 2 = \sum_{\substack{(A,B) \text{ is a detached pair;} \\ |A| \text{ is positive; } |B| \text{ is positive;} \\ |A| \text{ is even; } |B| \text{ is even}}} 2 = \sum_{\substack{(A,B) \text{ is a detached pair;} \\ |A| \text{ is even and positive;} \\ |B| \text{ is even and positive}}} 2$$
$$= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are even and positive}}} 2. \tag{40}$$

But the sum $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are even and positive}}} 2$ has exactly α terms (since the number of all detached pairs (A, B) for which both numbers $|A|$ and $|B|$ are even and positive is α), and thus equals $\alpha \cdot 2$ (since each of its α terms equals 2). Thus, $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are even and positive}}} 2 = \alpha \cdot 2 = 2\alpha$. Hence, (40) becomes

$$\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}} 2 = \sum_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are even and positive}}} 2 = 2\alpha. \tag{41}$$

Let us now attack the second sum. We have

$$\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A\neq\varnothing;\ B\neq\varnothing; \\ |A| \text{ is odd; } |B| \text{ is odd}}} (-2) = \sum_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd and positive}}} (-2)\,. \tag{42}$$

(Indeed, this can be proven in the same way as we proved (40), except that we must replace every word "even" by "odd", and every appearance of 2 by -2.)

But if (A,B) is a detached pair, then the condition "both numbers $|A|$ and $|B|$ are odd and positive" is equivalent to the simpler condition "both numbers $|A|$ and $|B|$ are odd"[44]. Hence, the summation sign $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd and positive}}} (-2)$ can be rewritten as $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd}}} (-2)$. Thus,

$$\sum_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd and positive}}} (-2) = \sum_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd}}} (-2).$$

But the sum $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd}}} (-2)$ has exactly β terms (since the number of all detached pairs (A,B) for which both numbers $|A|$ and $|B|$ are odd is β), and thus equals $\beta \cdot (-2)$ (since each of its β terms equals -2). Thus, $\sum\limits_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd}}} (-2) = \beta \cdot (-2) = -2\beta$. Hence, (42) becomes

$$\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing; \\ |A| \text{ is odd; } |B| \text{ is odd}}} (-2) = \sum_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd and positive}}} (-2)$$
$$= \sum_{\substack{(A,B) \text{ is a detached pair;} \\ \text{both numbers } |A| \text{ and } |B| \text{ are odd}}} (-2) = -2\beta. \qquad (43)$$

Now, (39) becomes

$$\begin{aligned}
&\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing}} \left((-1)^{|A|} + (-1)^{|B|} \right) \\
&= \underbrace{\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing; \\ |A| \text{ is even; } |B| \text{ is even}}} 2}_{\substack{=2\alpha \\ \text{(by (41))}}} + \underbrace{\sum_{\substack{(A,B) \text{ is a detached pair;} \\ A \neq \varnothing;\ B \neq \varnothing; \\ |A| \text{ is odd; } |B| \text{ is odd}}} (-2)}_{\substack{=-2\beta \\ \text{(by (43))}}} \\
&= 2\alpha + (-2\beta) = 2(\alpha - \beta).
\end{aligned}$$

[44] *Proof.* Let (A,B) be a detached pair. Thus, A and B are two subsets of V (by the definition of "detached pair"), and thus are two finite sets. Hence, $|A|$ and $|B|$ are nonnegative integers. Therefore, if $|A|$ and $|B|$ are odd, then $|A|$ and $|B|$ are automatically positive (because if a nonnegative integer is odd, then it is automatically positive). Therefore, the condition "both numbers $|A|$ and $|B|$ are odd and positive" is equivalent to the simpler condition "both numbers $|A|$ and $|B|$ are odd". Qed.

Comparing this with (38), we find

$$2\left(\delta - \left(2^n - 1\right)\right) = 2\left(\alpha - \beta\right).$$

Cancelling 2 from this equality, we obtain $\delta - (2^n - 1) = \alpha - \beta$. Solving this equation for δ, we find $\delta = 2^n - 1 + \alpha - \beta$.

Now, recall that δ is the number of dominating sets of G. Hence, the number of dominating sets of G is $\delta = 2^n - 1 + \alpha - \beta$. This proves Theorem 3.2.2 **(b)**. $\square$

3.5. Distances in graphs; first proof of Proposition 3.1.5

TODO 3.5.1. Write this proof up. Note that it has several steps: First, splitting into connected components, and then (WLOG assuming G to be connected) the actual argument using distances.

3.6. Independent sets; second proof of Proposition 3.1.5

Now, let us prove Proposition 3.1.5 using the concept of *independent sets*. This concept (which is actually more fundamental than that of dominating sets) is easily defined:

Definition 3.6.1. Let $G = (V, E)$ be a simple graph. A subset S of V is said to be *independent* (with respect to G) if and only if no two distinct elements of S are adjacent.

An *independent set* of G means a subset of V that is independent (with respect to G).

Independent sets are often called *stable sets*.

If $G = (V, E)$ is a simple graph, then each subset S of V that has size $|S| \leq 1$ is automatically independent (since it has no two distinct elements). A subset S of V that has size $|S| = 2$ is independent if and only if its two elements are non-adjacent. A subset S of V that has size 3 is the same as an anti-triangle of G, as defined in Exercise 2.4.6 above. Here is a concrete example:

Example 3.6.2. Let V, E and G be as in Example 2.4.2 **(c)**. Then, the subset $\{2, 4, 6\}$ of V is independent (since no two distinct elements of this subset are adjacent), whereas the subset $\{1, 3, 5\}$ is not independent (since the two distinct elements 1 and 3 of this subset are adjacent).

TODO 3.6.3. explain that independent sets are induced subgraphs isomorphic to the empty graph.

The following property of independent set is almost trivial:

Lemma 3.6.4. Let $G = (V, E)$ be a simple graph. Let S be an independent set of G. Let $v \in V$. Assume that no neighbor of v belongs to S. Then, $S \cup \{v\}$ also is an independent set of G.

Proof of Lemma 3.6.4. We have $\operatorname{V}(G) = V$ (since $G = (V, E)$). Furthermore, recall that S is an independent set of G. In other words, the subset S of V is independent. In other words, no two distinct elements of S are adjacent (by the definition of "independent").

Clearly, $S \cup \{v\}$ is a subset of V (since $S \subseteq V$ and $v \in V$).

Assume (for the sake of contradiction) that some two distinct elements of $S \cup \{v\}$ are adjacent. Fix two such elements, and denote them by p and q. Hence, p and q are two distinct elements of $S \cup \{v\}$, and are adjacent. We have $p \neq q$ (since p and q are distinct). Hence, at least one of the elements p and q must be distinct from v (because otherwise, both p and q would be equal to v, which would yield that $p = q$, but this would contradict $p \neq q$). In other words, p is distinct from v, or q is distinct from v (or both). Hence, we WLOG assume that p is distinct from v (since otherwise, we can simply switch p with q). Then, $p \neq v$. Combining this with $p \in S \cup \{v\}$, we obtain $p \in (S \cup \{v\}) \setminus \{v\} \subseteq S$. Hence, $q = v$ [45]. Now, p is a neighbor of q (since p and q are adjacent). In other words, p is a neighbor of v (since $q = v$). Hence, at least one neighbor of v belongs to S (namely, the neighbor p). This contradicts the fact that no neighbor of v belongs to S.

This contradiction proves that our assumption (that some two distinct elements of $S \cup \{v\}$ are adjacent) was false. Hence, no two distinct elements of $S \cup \{v\}$ are adjacent. In other words, the subset $S \cup \{v\}$ of V is independent (by the definition of "independent"). In other words, $S \cup \{v\}$ is an independent set of G. This proves Lemma 3.6.4. $\square$

We shall now consider two specific classes of independent sets: the *maximum* and the *maximal* ones. These are not the same, and the contrast between them is rather important to understand. Let us define the two classes:

Definition 3.6.5. Let $G = (V, E)$ be a simple graph.

(a) A *maximum independent set* of G means a subset of V that is independent and has the highest possible size among all independent sets. In other words, a *maximum independent set* of G means a subset S of V that is independent and that has the property that every independent subset T of V satisfies $|T| \leq |S|$.

(b) A *maximal independent set* of G means a subset of V that is independent and that cannot be written as a proper subset of any independent subset of V. In other words, a *maximal independent set* of G means a subset of V that

[45] *Proof.* Assume the contrary. Thus, $q \neq v$. Combining this with $q \in S \cup \{v\}$, we obtain $q \in (S \cup \{v\}) \setminus \{v\} \subseteq S$. Now, p and q are two distinct elements of S (since $p \in S$, $q \in S$ and $p \neq q$), and are adjacent. Hence, there are two distinct elements of S that are adjacent (namely, p and q). This contradicts the fact that no two distinct elements of S are adjacent. This contradiction proves that our assumption was wrong, qed.

is independent and that has the property that no independent subset T of V satisfies $S \subsetneq T$.

Thus, roughly speaking, a maximum independent set is an independent set having maximum size (among the independent sets), whereas a maximal independent set is an independent set that cannot be extended to a larger independent set by adding new elements.

Notice that a simple graph often will have several maximum independent sets (after all, the requirement to have maximum size does not uniquely determine the set; it merely uniquely determines the size), and also several maximal independent sets.

It is easy to see that each maximum independent set of a simple graph G is automatically a maximal independent set of G as well. However, the opposite is not true, as the following example shows:

Example 3.6.6. Let V, E and G be as in Example 2.4.2 **(c)**. Then, the subset $\{2,4,6\}$ of V is independent, and is a maximum independent set of G (indeed, it is easy to check that each independent set of G has size ≤ 3). The subset $\{1,5\}$ is also independent, and is a **maximal** independent set of G (indeed, it is easy to check that no independent subset T of V satisfies $\{1,5\} \subsetneq T$, because each of the remaining vertices $2,3,4,6$ of G is connected to at least one of 1 and 5 and therefore cannot be in an independent set together with 1 and 5), but not a **maximum** independent set of G (since the independent set $\{2,4,6\}$ has greater size).

You might wonder how to compute a maximum independent set or a maximal independent set of a simple graph $G = (V,E)$. Obviously, one way to do so is to catalogue all subsets of V, then check which of them are independent, and then check which of them are maximum independent sets and which are maximal independent sets. This algorithm, while theoretically doable, is massively inefficient, since the number of subsets of V grows exponentially with $|V|$. Are there fast algorithms? This is a more interesting question, and the answers for **maximum** and for **maximal** independent sets are completely different:

- There are, most likely, no quick algorithms to find a **maximum** independent set of a simple graph $G = (V,E)$. Indeed, this problem is NP-hard. There are better algorithms than checking all subsets of V, but no algorithm of polynomial time in $|V|$ is possible (unless $P = NP$).
- Finding a **maximal** independent set of a simple graph $G = (V,E)$ is easy: Start with the empty set (which is clearly independent), and keep adding new elements to it while keeping the subset independent[46], until it is no longer possible. Once it is no longer possible, your independent subset is a maximal

[46]It is easy to check whether adding a given new vertex v to a given independent set S keeps the set independent (in fact, we just need to check that v is not adjacent to any of the elements of S).

independent set. This is a polynomial-time algorithm. We shall formalize this algorithm (and prove its correctness) further below (in Proposition 3.6.9).

Now, let us show a simple yet crucial fact about maximal independent sets:

Lemma 3.6.7. Let $G = (V, E)$ be a simple graph. Let S be an independent subset of V. Then, S is a maximal independent set of G if and only if S is dominating.

Proof of Lemma 3.6.7. We have $\operatorname{V}(G) = V$ (since $G = (V, E)$). Furthermore, recall that the subset S of V is independent. In other words, no two distinct elements of S are adjacent (by the definition of "independent").

Now, let us make the following two claims:

Claim 1: If S is a maximal independent set of G, then S is dominating.

Claim 2: If S is dominating, then S is a maximal independent set of G.

Proof of Claim 1: Assume that S is a maximal independent set of G. We must prove that S is dominating.

We know that S is a maximal independent set of G. In other words, S is a subset of V that is independent and that cannot be written as a proper subset of any independent subset of V.

Let $v \in \operatorname{V}(G) \setminus S$ be a vertex. Thus, $v \in \operatorname{V}(G)$ and $v \notin S$. From $S \subseteq V$ and $v \in \operatorname{V}(G) = V$, we conclude that $S \cup \{v\}$ is a subset of V. This subset $S \cup \{v\}$ of V is not independent[47]. Hence, at least one neighbor of v belongs to S [48].

Now, forget that we have fixed v. We thus have shown that for every vertex $v \in \operatorname{V}(G) \setminus S$, at least one neighbor of v belongs to S. In other words, the subset S of V is dominating (by the definition of "dominating"). This proves Claim 1.

Proof of Claim 2: Assume that S is dominating. We must prove that S is a maximal independent set of G.

Assume (for the sake of contradiction) that S can be written as a proper subset of some independent subset of V. Let I be this independent subset. Thus, S is a proper subset of I. In particular, $S \subseteq I$. But $I \setminus S \neq \varnothing$ (since S is a **proper** subset of I). Hence, there exists some $i \in I \setminus S$. Consider such an i. Combining $i \in I \setminus S \subseteq I \subseteq V = \operatorname{V}(G)$ with $i \notin S$ (since $i \in I \setminus S$), we obtain $i \in \operatorname{V}(G) \setminus S$.

The subset I of V is independent. In other words, no two distinct elements of I are adjacent (by the definition of "independent").

[47] *Proof.* Assume the contrary. Thus, the subset $S \cup \{v\}$ of V is independent. But S is a proper subset of $S \cup \{v\}$ (since $v \notin S$). Hence, S can be written as a proper subset of some independent subset of V (namely, of the independent subset $S \cup \{v\}$). This contradicts the fact that S cannot be written as a proper subset of any independent subset of V. This contradiction proves that our assumption was wrong, qed.

[48] *Proof.* Assume the contrary. Thus, no neighbor of v belongs to S. Hence, Lemma 3.6.4 shows that $S \cup \{v\}$ also is an independent set of G (since $v \in \operatorname{V}(G) = V$). In other words, the subset $S \cup \{v\}$ of V is independent. This contradicts the fact that the subset $S \cup \{v\}$ of V is not independent. This contradiction proves that our assumption was wrong, qed.

But the subset S of V is dominating. In other words, for every vertex $v \in \operatorname{V}(G) \setminus S$, at least one neighbor of v belongs to S (by the definition of "dominating"). Applying this to $v = i$, we find that at least one neighbor of i belongs to S (since $i \in \operatorname{V}(G) \setminus S$). Let j be this neighbor. Thus, $j \in S$. Also, the vertices i and j are adjacent (since j is a neighbor of i). Hence, the vertices i and j are distinct. Also, i and j are elements of I (since $i \in I$ and $j \in S \subseteq I$). Thus, there exist two distinct elements of I that are adjacent (namely, i and j). This contradicts the fact that no two distinct elements of I are adjacent. This contradiction proves that our assumption (that S can be written as a proper subset of some independent subset of V) is wrong. Hence, S cannot be written as a proper subset of any independent subset of V. Thus, S is a subset of V that is independent and that cannot be written as a proper subset of any independent subset of V. In other words, S is a maximal independent set of G (by the definition of maximal independent sets). This proves Claim 2.

Combining Claim 1 and Claim 2, we conclude that S is a maximal independent set of G if and only if S is dominating. This proves Lemma 3.6.7. □

Let us now rigorously state the algorithm for finding a maximal independent set:

Algorithm 3.6.8. Input: a simple graph $G = (V, E)$.
Output: a maximal independent set S of G.

1. Define a subset S of V by $S = \varnothing$.
2. **While** there exists some $v \in V \setminus S$ such that no neighbor of v belongs to S, **do** the following:
   - Choose one such v, and add this v to S. (Clearly, the new S is still a subset of V.)
3. Output the subset S of V.

(Note that in each iteration of the while-loop in Step 2 of Algorithm 3.6.8, we only add one v to S, even if there exist several candidates. Then, we go back to the beginning of the while-loop, and we check again whether there exists some $v \in V \setminus S$ such that no neighbor of v belongs to S.)

Proposition 3.6.9. Algorithm 3.6.8 always terminates, and the subset S that it outputs is indeed a maximal independent set of G.

We shall show the proof of Proposition 3.6.9 in all detail due to it being a neat introductory example of reasoning about algorithms; nevertheless, the proof is almost evident.

Proof of Proposition 3.6.9. We will prove the following claims:

Claim 1: During each iteration of the while-loop in Algorithm 3.6.8, the size $|S|$ increases by 1.

Claim 2: Algorithm 3.6.8 always terminates.

Claim 3: Consider one single iteration of the while-loop in Algorithm 3.6.8. If the subset S is independent before this iteration, then S is also independent after the iteration.

Claim 4: The subset S stays independent throughout the execution of Algorithm 3.6.8.

Claim 5: The subset S that is outputted by Algorithm 3.6.8 is a maximal independent set of G.

Clearly, Claim 2 and Claim 5 combined will yield Proposition 3.6.9 (once they are proven). The other three claims are merely auxiliary results.

Let us now prove the five claims:

Proof of Claim 1. Consider one single iteration of the while-loop in Algorithm 3.6.8. Let S_{old} be the value of S at the beginning of this iteration, and let S_{new} be the value of S at the end of this iteration. Thus, S_{new} is obtained from S_{old} by adding the element v that was chosen during this particular iteration of the while-loop (because what happens to S during an iteration of the while-loop is that the element v is being added to S). In other words, we have $S_{\text{new}} = S_{\text{old}} \cup \{v\}$. But remember how the element v was chosen at the beginning of the iteration: It was chosen to be an element of $V \setminus S$ such that no neighbor of v belongs to S. Thus, v is an element of $V \setminus S_{\text{old}}$ such that no neighbor of v belongs to S_{old}. In particular, v is an element of $V \setminus S_{\text{old}}$. Hence, $v \notin S_{\text{old}}$. But from $S_{\text{new}} = S_{\text{old}} \cup \{v\}$, we obtain $|S_{\text{new}}| = |S_{\text{old}} \cup \{v\}| = |S_{\text{old}}| + 1$ (since $v \notin S_{\text{old}}$). In other words, the size $|S|$ has increased by 1 during our iteration.

We thus have shown that during each iteration of the while-loop in Algorithm 3.6.8, the size $|S|$ increases by 1. This proves Claim 1.

Proof of Claim 2. Let us show that the while-loop in Algorithm 3.6.8 (more precisely, in its Step 2) cannot go on forever.

Throughout the execution of the algorithm, S remains a subset of V. Hence, $|S|$ remains a nonnegative integer smaller or equal to $|V|$ throughout the execution of the algorithm. Therefore, $|S|$ cannot keep increasing by 1 forever (since it would eventually surpass $|V|$ this way). But Claim 1 shows that $|S|$ increases by 1 during each iteration of the while-loop in Algorithm 3.6.8. Therefore, the while-loop in Algorithm 3.6.8 cannot go on forever (since $|S|$ cannot keep increasing by 1 forever). Hence, Algorithm 3.6.8 must eventually terminate (since the only part of Algorithm 3.6.8 that could possibly go on forever is the while-loop). This proves Claim 2.

Proof of Claim 3. Let $\mathcal{I}$ be a single iteration of the while-loop in Algorithm 3.6.8. We must show that if the subset S is independent before the iteration $\mathcal{I}$, then S is also independent after the iteration $\mathcal{I}$.

Let S_{old} be the value of S at the beginning of this iteration $\mathcal{I}$, and let S_{new} be the value of S at the end of this iteration $\mathcal{I}$.

Assume that the subset S_{old} of V is independent. In other words, S_{old} is an independent set of G. Now, S_{new} is obtained from S_{old} by adding the element v that was chosen during this particular iteration of the while-loop (because what happens to S during an iteration of the while-loop is that the element v is being added to S). In other words, we have $S_{\text{new}} = S_{\text{old}} \cup \{v\}$.

The element v was chosen to be an element of $V \setminus S$ such that no neighbor of v belongs to S at the beginning of the iteration $\mathcal{I}$. Hence, v is an element of $V \setminus S_{\text{old}}$ such that no neighbor of v belongs to S_{old}. Thus, we have $v \in V \setminus S_{\text{old}} \subseteq V$, and no neighbor of v belongs to S_{old}. Hence, Lemma 3.6.4 (applied to S_{old} instead of S) shows that $S_{\text{old}} \cup \{v\}$ also is an independent set of G. In other words, S_{new} is an independent set of G (since $S_{\text{new}} = S_{\text{old}} \cup \{v\}$). In other words, the subset S_{new} of V is independent.

Now, forget our assumption that the subset S_{old} of V is independent. We thus have shown that if the subset S_{old} of V is independent, then the subset S_{new} of V is independent. In other words, if the subset S is independent before the iteration $\mathcal{I}$, then S is also independent after the iteration $\mathcal{I}$. This proves Claim 3.

Proof of Claim 4. The empty set $\varnothing$ is clearly independent. Hence, the subset S of V is independent when it is first defined in Algorithm 3.6.8 (because it is first defined to be the empty set $\varnothing$). Then, throughout the while-loop in Algorithm 3.6.8, it remains independent (because of Claim 3). Therefore, S stays independent throughout Algorithm 3.6.8. This proves Claim 4.

Proof of Claim 5. Consider the subset S that is outputted by Algorithm 3.6.8. This subset S is independent (by Claim 4), i.e., is an independent set of G.

Notice that $G = (V, E)$, hence $\operatorname{V}(G) = V$. We have $S \subseteq V = \operatorname{V}(G)$.

Furthermore, looking back at Algorithm 3.6.8, we see that the subset S gets outputted immediately after the while-loop is escaped. Thus, the condition of the while-loop cannot be satisfied for this subset S (because otherwise, the while-loop would not be escaped at this point). In other words, there exists **no** $v \in V \setminus S$ such that no neighbor of v belongs to S. In other words, for every $v \in V \setminus S$, at least one neighbor of v belongs to S. In other words, for every $v \in \operatorname{V}(G) \setminus S$, at least one neighbor of v belongs to S (because $V = \operatorname{V}(G)$). In other words, the subset S of $\operatorname{V}(G)$ is dominating (by the definition of "dominating"). But Lemma 3.6.7 shows that S is a maximal independent set of G if and only if S is dominating. Thus, S is a maximal independent set of G (since S is dominating). This proves Claim 5.

Now, all five claims are proven. Proposition 3.6.9 follows from Claim 2 and Claim 5. □

Lemma 3.6.10. Let $G = (V, E)$ be a simple graph that has no isolated vertices. Let S be an independent set of G. Then, the subset $V \setminus S$ of V is dominating.

Proof of Lemma 3.6.10. We have $G = (V, E)$, thus $\operatorname{V}(G) = V$. But S is an independent set of G, hence an independent subset of V. In other words, no two distinct elements of S are adjacent (by the definition of "independent").

Let $v \in \operatorname{V}(G) \setminus (V \setminus S)$ be a vertex. Thus, $v \in \underbrace{\operatorname{V}(G)}_{=V} \setminus (V \setminus S) = V \setminus (V \setminus S) = S$ (since $S \subseteq V$). Furthermore, the vertex v of G has a neighbor[49]. Fix such a neighbor, and denote it by u. Thus, u and v are adjacent (since u is a neighbor of v). We have $u \notin S$ [50]. Hence, $u \in V \setminus S$ (since $u \in V$ and $u \notin S$). Recall also that u is a neighbor of v. Thus, at least one neighbor of v belongs to $V \setminus S$ (namely, the neighbor u).

Now, forget that we fixed v. We thus have shown that for every vertex $v \in \operatorname{V}(G) \setminus (V \setminus S)$, at least one neighbor of v belongs to $V \setminus S$. In other words, the subset $V \setminus S$ of V is dominating (by the definition of "dominating"). This proves Lemma 3.6.10. □

Proof of Proposition 3.1.5. **(a)** There exists a maximal independent set of G [51]. Consider such a set, and denote it by S. Lemma 3.6.7 shows that S is a maximal independent set of G if and only if S is dominating. Thus, S is dominating (since S is a maximal independent set of G). But Lemma 3.6.10 shows that the subset $V \setminus S$ of V is dominating. Clearly, the sets S and $V \setminus S$ are disjoint subsets of V. Furthermore, $S \cup (V \setminus S) = V$ (since $S \subseteq V$). Hence, there exist two disjoint dominating subsets A and B of V such that $A \cup B = V$ (namely, $A = S$ and $B = V \setminus S$). This proves Proposition 3.1.5 **(a)**.

(b) Proposition 3.1.5 **(a)** shows that there exist two disjoint dominating subsets A and B of V such that $A \cup B = V$. Consider such A and B. We WLOG assume that $|A| \geq |B|$ (since otherwise, we can simply switch A with B). But since the sets A and B are disjoint, we have $|A \cup B| = |A| + |B|$. Now, from $V = A \cup B$, we obtain

$$|V| = |A \cup B| = \underbrace{|A|}_{\geq |B|} + |B| \geq |B| + |B| = 2\,|B|,$$

[49] *Proof.* Assume the contrary. Thus, the vertex v has no neighbors. Hence, the vertex v of G is isolated (because the vertex v of G is isolated if and only if v has no neighbors). Therefore, the graph G has at least one isolated vertex. This contradicts the fact that G has no isolated vertices. This contradiction proves that our assumption was false, qed.

[50] *Proof.* Assume the contrary. Thus, $u \in S$. Now, u and v are two elements of S (since $u \in S$ and $v \in S$) and are distinct (since u and v are adjacent). Hence, some two distinct elements of S are adjacent (namely, u and v). This contradicts the fact that no two distinct elements of S are adjacent. This contradiction proves that our assumption was false, qed.

[51] *First proof.* Such a set can be obtained by Algorithm 3.6.8 (because Proposition 3.6.9 shows that this algorithm outputs a maximal independent set of G).

Second proof. We have the following two observations:

- There exists at least one independent set of G. (In fact, the subset $\varnothing$ of V is clearly independent.)
- There are only finitely many independent sets of G. (In fact, each independent set of G is a subset of V, and there are only finitely many subsets of V.)

From these two observations, we can conclude that there exists an independent set of G having maximum size. In other words, there exists a maximum independent set of G. But since each maximum independent set of G is a maximal independent set of G (this is easy to show), we can thus conclude that there exists a maximal independent set of G.

so that $|B| \leq |V|/2$. In other words, the subset B of V has size $\leq |V|/2$. Hence, there exists a dominating subset of V having size $\leq |V|/2$ (namely, the subset B). This proves Proposition 3.1.5 **(b)**. □

4. Hamiltonian paths

TODO 4.0.1. Continue from here...

[...]
[to be continued]

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