University of Minnesota

Math 5707: Graph Theory, Spring 2017 Midterm 3

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1 Exercise 1

1.1 Problem

Let G be a connected multigraph. Let x, y, z and w be four vertices of G.

Assume that the two largest ones among the three numbers $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ are **not** equal.

Prove that G has a cycle of length $\leq d(x,z) + d(y,w) + d(x,w) + d(y,z)$.

1.2 Solution

Proof. For brevity, define the following:

$$\begin{aligned} a &= d(x,y) + d(z,w) \\ b &= d(x,z) + d(y,w) \\ c &= d(x,w) + d(y,z) . \end{aligned}$$

We are then asked to show that G has a cycle of length $\leq b + c$. For any unordered pair of $\{u,v\}$ of vertices of G, let us choose (and fix) a shortest path $p_{u,v}$ from u to v.

Among the values a, b, and c, pick the two smallest. Let us call these two smallest values b' and c'. We can find a permutation (x', y', z', w') of (x, y, z, w) such that $b' = d(x', z') + d(y', w')$ and $c' = d(x', w') + d(y', z')$.

Form a new multigraph G' from G by restricting the edges and vertices to those lying on (one of) the chosen shortest paths $p_{x',z'}, p_{y',w'}, p_{x',w'}, p_{y',z'}$. (For example, if a and b are the two smallest among the values a, b, and c, then we restrict the vertices to those lying on the paths $p_{x,y}, p_{z,w}, p_{x,z}, p_{y,w}$.) Notice that the new multigraph G' is still connected.

Now, the two chosen sums of distances $d(x', z') + d(y', w')$ and $d(x', w') + d(y', z')$ are the same in G and in G', since the chosen shortest paths $p_{x',z'}, p_{y',w'}, p_{x',w'}, p_{y',z'}$ are retained. The third sum (the largest in G) cannot be smaller in G', since G' was obtained from G by removing vertices and edges (so no new paths can have arisen, but old paths might have disappeared). Hence, it is still the largest in G'. Thus, the two largest sums in G' are still unequal (as the smaller of them is the same as in G, while the larger one is the same or larger). Therefore, G' is not a tree (by Midterm 2, Exercise 6). Since G' is connected, this shows that G' contains a cycle.

But G' has at most $b' + c' = a + b + c - \max\{a, b, c\}$ vertices, which is $\leq b + c$. Since the vertices on a cycle are distinct, the cycle cannot have length greater than $b + c$. □

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2 Exercise 2

2.1 Problem

(a) Let $n > 1$ be an integer. Prove that the chromatic polynomial of the cycle graph C_n is

$$\chi_{C_n} = (x-1)^n + (-1)^n (x-1).$$

(b) Let $g \in \mathbb{N}$. Let G be the simple graph whose vertices are the $2g+1$ integers $-g, -g+1, \ldots, g-1, g$, and whose edges are

$$\begin{aligned} &\{0, i\} && \text{for all } i \in \{1, 2, \ldots, g\}; \\ &\{0, -i\} && \text{for all } i \in \{1, 2, \ldots, g\}; \\ &\{i, -i\} && \text{for all } i \in \{1, 2, \ldots, g\} \end{aligned}$$

(these are $3g$ edges in total).

Compute the chromatic polynomial χ_G of G.

2.2 Solution

Proof of part (a): Let $V = \operatorname{V}(C_n)$, and $E = \operatorname{E}(C_n)$. First show that for $F \subseteq E$, we have

$$\operatorname{conn}(V, F) = \begin{cases} n - |F|, & \text{if } |F| < n; \\ 1, & \text{if } |F| = n. \end{cases} \tag{1}$$

[*First proof of* (1)*:* We prove (1) by induction over $|F|$:

Base: $|F| = 0$. In this case, each vertex is isolated, so $\operatorname{conn}(V, F) = n - |F| = n$.

Step: Suppose $\operatorname{conn}(V, F) = n - |F|$. Consider the graph $(V, F \cup \{e\})$, where $e \in E \backslash F$. There are two cases to consider:

(1) The two endpoints of e are in the same connected component of (V, F). In this case, $|F|$ must be $n-1$ (since adding e creates a cycle, but the only cycle that could possibly be added is the one that contains all n vertices of C_n), so $\operatorname{conn}(V, F \cup \{e\}) = 1$.

(2) The two endpoints of e are not in the same connected component of (V, F). In this case, adding e joins two connected components, so $\operatorname{conn}(V, F \cup \{e\}) = \operatorname{conn}(V, F) - 1 = n - |F| - 1 = n - |F \cup \{e\}|$.

]

[*Second proof of* (1)*:* If F is a proper subset of E, then the graph (V, F) has no cycles (since the only cycle it could have is the full n-vertex cycle, but this would require F to be the whole set E), and thus is a forest. Therefore, in this case, we have $\operatorname{conn}(V, F) = n - |F|$ (since the number of connected components of a forest always equals its number of vertices minus its number of edges[1]). Remains to handle the case $F = E$; but this is clear.]

Since $\operatorname{conn}(V, F)$ depends only on $|F|$ (by (1)), we can sum over $|F|$ rather than all $F \subseteq E$. For each $k \in \{0, 1, \ldots, n\}$, there are $\binom{n}{k}$ possible subsets F of E having size $|F| = k$, so we get

$$\begin{aligned} \chi_{C_n} &= \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)} = \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k x^{n-k} + (-1)^n x^1 \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k} - (-1)^n x^{n-n} + (-1)^n x \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k} + (-1)^n (x - 1) \\ &= (x-1)^n + (-1)^n (x - 1), \end{aligned}$$

with the final equality using the binomial theorem. □

[1]For a proof, see Corollary 20 in the handwritten Lecture 9.

Proof of part (b): Let $V = \operatorname{V}(G)$, and $E = \operatorname{E}(G)$. A *lobe* shall mean a subgraph of G comprising the vertices i, $-i$, and 0 and the edges $\{0, i\}$, $\{0, -i\}$, and $\{i, -i\}$. The *hub* of a subgraph (V, F) (with $F \subseteq E$) shall refer to the connected component containing the vertex 0 in (V, F).

Consider how the vertices in a lobe can become part of a connected component distinct from the hub in a graph (V, F). As long as at least 2 of the edges $\{0, i\}$, $\{0, -i\}$, and $\{i, -i\}$ are in F, each of i and $-i$ remains connected to the hub. If only one of the edges $\{0, i\}$, $\{0, -i\}$, and $\{i, -i\}$ is in F, then one connected component separates from the hub (either one isolated vertex or both vertices depending on which edge remains). If none of the edges are in F, then each of i and $-i$ becomes a connected component. Thus the number of connected components in (V, F) can be determined by the number of lobes with 0, 1, 2, and 3 edges removed in (V, F).

For a subset F of E, define four nonnegative integers $\ell_0, \ell_1, \ell_2, \ell_3$ as follows:

$$\text{for } i = 0, 1, 2, 3, \quad \text{set } \ell_i = \#(\text{lobes with } i \text{ edges removed in } (V, F)).$$

We can express the size of F as $|F| = 3\ell_0 + 2\ell_1 + \ell_2$. There are $\binom{g}{\ell_0, \ell_1, \ell_2, \ell_3}$ choices[2] of how many edges to remove from each lobe. For each such choice, we have $3^{\ell_1} 3^{\ell_2}$ choices of edges to remove. Finally, for each choice of lobes and edges, there are $1 + \ell_2 + 2\ell_3$ connected components. We can now express χ_G as the following cumbersome sum:

$$\begin{aligned}
\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)} &= \sum_{\substack{\ell_0, \ell_1, \ell_2, \ell_3, \\ \ell_0 + \ell_1 + \ell_2 + \ell_3 = g}} \binom{g}{\ell_0, \ell_1, \ell_2, \ell_3} 3^{\ell_1} 3^{\ell_2} (-1)^{3\ell_0 + 2\ell_1 + \ell_2} x^{1 + \ell_2 + 2\ell_3} \\
&= \sum_{\substack{\ell_0, \ell_1, \ell_2, \ell_3, \\ \ell_0 + \ell_1 + \ell_2 + \ell_3 = g}} \binom{g}{\ell_0, \ell_1, \ell_2, \ell_3} x [(-1)^3]^{\ell_0} [3(-1)^2]^{\ell_1} [3x(-1)]^{\ell_2} [x^2]^{\ell_3} \\
&= x \sum_{\substack{\ell_0, \ell_1, \ell_2, \ell_3, \\ \ell_0 + \ell_1 + \ell_2 + \ell_3 = g}} \binom{g}{\ell_0, \ell_1, \ell_2, \ell_3} (-1)^{\ell_0} 3^{\ell_1} (-3x)^{\ell_2} (x^2)^{\ell_3} \\
&= x(-1 + 3 - 3x + x^2)^g = x(x-1)^g (x-2)^g,
\end{aligned}$$

with the last line using the multinomial theorem. □

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3 Exercise 3

3.1 Problem

Let $(G; X, Y)$ be a bipartite graph such that $|Y| \geq 2|X| - 1$. Prove that there exists an injective map $f : X \to Y$ such that each $x \in X$ satisfies one of the following two

[2] Here, $\binom{g}{\ell_0, \ell_1, \ell_2, \ell_3}$ denotes a multinomial coefficient; it is defined as $\frac{g!}{\ell_0! \ell_1! \ell_2! \ell_3!}$.

statements:

- *Statement 1:* The vertices x and $f(x)$ of G are adjacent.
- *Statement 2:* There exists no $x' \in X$ such that the vertices x and $f(x')$ of G are adjacent.

3.2 Solution

Proof. Induction on $|X|$.

Base: $|X| \le 1$. In this case, the proof is easy: If $|X| = 1$, then mapping the single vertex in X to any of the ≥ 1 vertices in Y satisfies the proposition. If $|X| = 0$, then everything is obvious.

Step: Suppose there is a map f satisfying the proposition for every bipartite graph $(H; A, B)$ with $|A| < |X|$ and $|B| \ge 2|A| - 1$. Assume without loss of generality that there are fewer than $|X|$ isolated vertices in Y. (If there are at least $|X|$ isolated vertices in Y, then we can choose any injective map from X to these isolated vertices.) Let Y' be the set of non-isolated vertices in Y. Choose the smallest nonempty $S \subseteq Y'$ such that $|\mathrm{N}(S)| \le |S|$. (If there is no such S, then for every $S \subseteq Y'$, $|\mathrm{N}(S)| > |S|$ and we have a Y'-complete matching which can be used to define a map in which each vertex satisfies Statement 1.) For this S, it cannot be that $|\mathrm{N}(S)| < |S|$, since each subset $P \subset S$ of size $|S| - 1$ has at least $|S|$ neighbors (and since the vertices in S are non-isolated). Thus $|\mathrm{N}(S)| = |S|$. Now we have for each $P \subseteq S$, $|\mathrm{N}(P)| \ge |P|$ and $|\mathrm{N}(S)| = |S|$, so there is a perfect matching M of $\mathrm{N}(S)$ to S.

Now, consider the bipartite graph $(G \backslash (\mathrm{N}(S) \cup S); X \backslash \mathrm{N}(S), Y \backslash S)$. Since $|\mathrm{N}(S)| = |S|$, we still have $|Y \backslash S| \ge 2|X \backslash \mathrm{N}(S)| - 1$. Hence by the induction hypothesis, there is a map $f : X \backslash \mathrm{N}(S) \to Y \backslash S$ satisfying the conditions of the exercise. If we extend this map to f' by setting $f'(v)$ to the vertex to which v is matched in the perfect matching M for each $v \in \mathrm{N}(S)$, then f' satisfies the proposition:

- For each $v \in \mathrm{N}(S)$, v is adjacent to $f'(v)$.
- For each $v \in X \backslash \mathrm{N}(S)$, either v is adjacent to $f'(v)$, or no other vertex is mapped to any of its neighbors. (In fact, for v not adjacent to $f'(v)$, the induction hypothesis guarantees that no other vertex in $X \backslash \mathrm{N}(S)$ maps to a neighbor of v, and since each vertex in $\mathrm{N}(S)$ is mapped to a vertex in S, it cannot be mapped to a neighbor of v either.)

□

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$ be even. Let $\sigma \in S_n$ be a permutation.

(a) Show that the perfect matching

$$M_\sigma = \{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \{\sigma(5), \sigma(6)\}, \ldots, \{\sigma(n-1), \sigma(n)\}\}$$

satisfies

$$(-1)^{\operatorname{xing}(M_\sigma)} (-1)^k = (-1)^\sigma,$$

where k is the number of $i \in \{1, 2, \ldots, n/2\}$ satisfying $\sigma(2i-1) > \sigma(2i)$.

(b) Let

$$M = \left\{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \left\{a_{n/2}, b_{n/2}\right\}\right\}$$

be any perfect matching. Define a new perfect matching σM by

$$\sigma M = \left\{\{\sigma(a_1), \sigma(b_1)\}, \{\sigma(a_2), \sigma(b_2)\}, \ldots, \left\{\sigma\left(a_{n/2}\right), \sigma\left(b_{n/2}\right)\right\}\right\}.$$

Let p be the number of $i \in \{1, 2, \ldots, n/2\}$ satisfying $a_i > b_i$.

Let q be the number of $i \in \{1, 2, \ldots, n/2\}$ satisfying $\sigma(a_i) > \sigma(b_i)$.

Prove that

$$(-1)^{\operatorname{xing}(\sigma M)} (-1)^p = (-1)^\sigma (-1)^{\operatorname{xing} M} (-1)^q.$$

4.2 Solution

Recall the following basic fact: If $\{i, j\}$ and $\{k, \ell\}$ are two disjoint edges, then

$$[\{i, j\} \text{ crosses } \{k, \ell\}] \equiv [i > k] + [i > \ell] + [j > k] + [j > \ell] \mod 2. \tag{2}$$

Proof of part (a): It suffices to show that $\operatorname{xing} M_\sigma + k \equiv \ell(\sigma) \mod 2$. Here begins a lot of sum manipulation. Below I just split the sum in the definition of $\operatorname{xing} M_\sigma$, and change the indices:

$$
\begin{aligned}
\operatorname{xing} M_\sigma &= \sum_{1\le i<j\le n/2} \underbrace{[\{\sigma(2i-1),\sigma(2i)\} \text{ crosses } \{\sigma(2j-1),\sigma(2j)\}]}_{\substack{\equiv[\sigma(2i-1)>\sigma(2j-1)]+[\sigma(2i-1)>\sigma(2j)]+[\sigma(2i)>\sigma(2j-1)]+[\sigma(2i)>\sigma(2j)] \mod 2\\ \text{(by (2))}}} \\
&\equiv \sum_{1\le i<j\le n/2} \left([\sigma(2i-1)>\sigma(2j-1)]+[\sigma(2i-1)>\sigma(2j)]\right. \\
&\qquad \left. +[\sigma(2i)>\sigma(2j-1)]+[\sigma(2i)>\sigma(2j)]\right) \\
&= \sum_{1\le i<j\le n/2} [\sigma(2i-1)>\sigma(2j-1)] + \sum_{1\le i<j\le n/2} [\sigma(2i-1)>\sigma(2j)] \\
&\qquad + \sum_{1\le i<j\le n/2} [\sigma(2i)>\sigma(2j-1)] + \sum_{1\le i<j\le n/2} [\sigma(2i)>\sigma(2j)] \\
&= \sum_{\substack{1\le i<j\le n,\\ i \text{ odd},\ j \text{ odd}}} [\sigma(i)>\sigma(j)] + \sum_{\substack{1\le i<j\le n,\\ i \text{ odd},\ j \text{ even},\\ j>i+1}} [\sigma(i)>\sigma(j)] \\
&\qquad + \sum_{\substack{1\le i<j\le n,\\ i \text{ even},\ j \text{ odd}}} [\sigma(i)>\sigma(j)] + \sum_{\substack{1\le i<j\le n,\\ i \text{ even},\ j \text{ even}}} [\sigma(i)>\sigma(j)] \mod 2.
\end{aligned}
$$

Now

$$k = \sum_{1\le i\le n/2} [\sigma(2i-1)>\sigma(2i)] = \sum_{\substack{1\le i\le n,\\ i \text{ odd}}} [\sigma(i)>\sigma(i+1)] = \sum_{\substack{1\le i<j\le n,\\ i \text{ odd},\ j \text{ even},\\ j=i+1}} [\sigma(i)>\sigma(j)].$$

This can be combined with the second of the four sums in the last expression above to get

$$
\begin{aligned}
\operatorname{xing} M_\sigma + k &\equiv \sum_{\substack{1\le i<j\le n,\\ i \text{ odd},\ j \text{ odd}}} [\sigma(i)>\sigma(j)] + \sum_{\substack{1\le i<j\le n,\\ i \text{ odd},\ j \text{ even}}} [\sigma(i)>\sigma(j)] \\
&\qquad + \sum_{\substack{1\le i<j\le n,\\ i \text{ even},\ j \text{ odd}}} [\sigma(i)>\sigma(j)] + \sum_{\substack{1\le i<j\le n,\\ i \text{ even},\ j \text{ even}}} [\sigma(i)>\sigma(j)] \\
&= \sum_{1\le i<j\le n} [\sigma(i)>\sigma(j)] = \ell(\sigma) \mod 2.
\end{aligned}
$$

□

Proof of part (b): This is equivalent to showing that $\ell(\sigma) \equiv \operatorname{xing} M + p + \operatorname{xing}(\sigma M) + q$ mod 2. Rename each a_i and b_i as follows: for $i = 1, 2, \ldots, n/2$, let $c_{2i-1} = a_i$ and $c_{2i} = b_i$. We can use the result of part (a) by noting that p, $\operatorname{xing} M$, and c_i here correspond respectively to k, $\operatorname{xing}(M_\sigma)$, and $\sigma(i)$ in part (a), so we get

$$\operatorname{xing} M + p \equiv \sum_{1\le i<j\le n} [c_i > c_j] \mod 2. \tag{3}$$

Similarly, q, $\text{xing}(\sigma M)$, and $\sigma(c_i)$ here correspond to k, $\text{xing}(M_\sigma)$, and $\sigma(i)$ in part (a), so we get

$$\text{xing}(\sigma M) + q \equiv \sum_{1 \le i < j \le n} [\sigma(c_i) > \sigma(c_j)] \mod 2. \tag{4}$$

Adding (3) and (4), we get

$$\text{xing}\, M + p + \text{xing}(\sigma M) + q \equiv \sum_{1 \le i < j \le n} ([c_i > c_j] + [\sigma(c_i) > \sigma(c_j)]) \mod 2. \tag{5}$$

Now, we can express $\ell(\sigma)$ directly from the definition as below:

$$\ell(\sigma) = \sum_{1 \le i, j \le n} [c_i < c_j] \cdot [\sigma(c_i) > \sigma(c_j)]. \tag{6}$$

Now consider the addends in the sum in (5) that are nonzero modulo 2. These addends correspond to pairs of indices $1 \le i < j \le n$ for which exactly one of the following two statements is true:

(a) $i < j$ and $c_i < c_j$ and $\sigma(c_i) > \sigma(c_j)$

(b) $i < j$ and $c_i > c_j$ and $\sigma(c_i) < \sigma(c_j)$

Similarly consider the addends in the sum in (6) that are nonzero modulo 2. These addends correspond to pairs of indices $1 \le i, j \le n$ for which exactly of the following two statements is true:

(c) $i < j$ and $c_i < c_j$ and $\sigma(c_i) > \sigma(c_j)$

(d) $i > j$ and $c_i < c_j$ and $\sigma(c_i) > \sigma(c_j)$

Note that condition (a) is the same as condition (c), and if we reverse the names of i and j in condition (d), it becomes the same as condition (b). Therefore, $\ell(\sigma) \equiv \text{xing}\, M + p + \text{xing}(\sigma M) + q \mod 2.$ □

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5 Exercise 5

5.1 Problem

Let $G = (V, E, \psi)$ be a connected multigraph. Set $n = |V|$ and $h = |E|$.

Let $(\phi_0, \phi_1, \ldots, \phi_k)$ be a sequence of orientations of G, and let $(v_1, v_2, \ldots, v_k)$ be a sequence of vertices of G such that for each $i \in \{1, 2, \ldots, k\}$, the orientation ϕ_i is obtained from ϕ_{i-1} by pushing the source v_i (in particular, this is saying that v_i is a source of ϕ_{i-1}).

Assume that $k \geq \binom{n+h-1}{n-1}$.

(a) Prove that each vertex of G appears at least once in the sequence $(v_1, v_2, \ldots, v_k)$.

(b) Prove that the orientations $\phi_0, \phi_1, \ldots, \phi_k$ are acyclic.

Proof of part (a): Form the multidigraph G^{dir}. (This is the multidigraph obtained from the multigraph G by replacing each edge by two arcs going in opposite directions.)

For each orientation ϕ_i of G, define a configuration f_i on G^{dir} by setting

$$f_i(v) = \deg^+_{(V,E,\phi_i)} v = \#(\text{arcs with source } v \text{ in orientation } \phi_i \text{ of } G)$$

for each $v \in V$. Then, a vertex is active in configuration f_i on G^{dir} if and only if it is a source in orientation ϕ_i of G. (If $v \in V$ is a source, then $\deg^+_{(V,E,\phi_i)} v = \deg_G v = \deg^+_{G^{dir}} v$, thus $f_i(v) = \deg^+_{(V,E,\phi_i)} v = \deg^+_{G^{dir}} v$, so it is active. If it is not a source, then $\deg^+_{(V,E,\phi_i)} v < \deg_G v = \deg^+_{G^{dir}} v$, so that $f_i(v) = \deg^+_{(V,E,\phi_i)} v < \deg^+_{G^{dir}} v$, and it is not active.)

Also, the operation of pushing a source v on orientation ϕ_i results in the legal firing of vertex v in configuration f_i: In fact, if ϕ_{i+1} is the orientation obtained from ϕ_i by pushing the source v, and if f_{i+1} is the configuration corresponding to this ϕ_{i+1}, then each vertex $w \in V$ satisfies

$$f_{i+1}(w) = \left\{ \begin{array}{ll} f_i(w) - \deg^+_{G^{dir}} w, & \text{for } w = v \\ f_i(w) + \#(\text{arcs } v \to w), & \text{for } w \neq v \end{array} \right\} = (f_i - \Delta v)(w),$$

and therefore $f_{i+1} = f_i - \Delta v$. Thus, pushing a sequence of sources in G results in a legal sequence of chip-firings on the same sequence of vertices in G^{dir}. Since G^{dir} is strongly connected, we can apply the result of HW5 Exercise 1 (b), which states that any legal sequence of length $\geq \binom{n+h-1}{n-1}$ must contain each vertex of V. Therefore, every vertex appears in the sequence $(v_1, v_2, \ldots, v_k)$. $\square$

Proof of part (b): Suppose an orientation ϕ_i is not acyclic. Then there is a cycle $c = (u_0, e_1, u_1, \ldots, e_j, u_j = u_0)$. For each $i \in \{0, 1, \ldots, j-1\}$, the vertex u_i is not a source in ϕ_i since the arc e_i has target u_i (where $e_0 := e_j$). Hence $v_{i+1} \notin \{u_0, u_1, \ldots, u_{j-1}\}$. Then each edge on the cycle c maintains the same orientation in ϕ_{i+1} that it had in ϕ_i, so ϕ_{i+1} also contains the same cycle. By induction, any vertex on a cycle will remain on a cycle after any sequence of source-pushing operations.

But as justified in part (a), we can apply the result of HW5 Exercise 1 (a), which says we can perform an arbitrarily long sequence of source-pushing operations starting at orientation ϕ_i. In particular, we can perform a sequence of $\geq \binom{n+h-1}{n-1}$ operations. Then each vertex (including the vertices on c) must appear in this sequence, a contradiction. Therefore, each orientation ϕ_i must be acyclic. $\square$

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6 Exercise 6

6.1 Problem

Let $G = (V, E, \psi)$ be a tree. Let α and β be two orientations of G.

Prove that β can be obtained from α by repeatedly pushing sources.

Proof. First, let us prepare with some general facts:

> *Definition.* Let $G = (V, E, \psi)$ be a multigraph. Let ϕ be an orientation of G. A vertex $v \in V$ is said to be a *sink* of ϕ if no arc of the multidigraph (V, E, ϕ) has source v. If v is a sink of ϕ, then we can define a new orientation ϕ' of G as follows:
>
> - For each $e \in E$ satisfying $v \in \psi(e)$, we set $\phi'(e) = (v, u)$, where u is chosen such that $\phi(e) = (u, v)$.
> - For all other $e \in E$, we set $\phi'(e) = \phi(e)$.
>
> (Roughly speaking, this simply means that ϕ' is obtained by ϕ by reversing the directions of all edges that contain v.) We say that this new orientation ϕ' is obtained from ϕ by *pulling the sink* v.
>
> *Lemma.* Let $G = (V, E, \psi)$ be a multigraph. Let ϕ be an acyclic orientation of G. Let v be a sink of ϕ. Then, the orientation obtained from ϕ by pulling the sink v can also be obtained from ϕ by repeatedly pushing sources.

For a proof of this Lemma, see Exercise 0.1 **(b)** in Darij Grinberg, *An exercise on source and sink mutations of acyclic quivers*. (This exercise differs from the Lemma in that sources and sinks have their roles swapped; but this is easily achieved by reversing all arcs.)

We shall now solve the exercise by induction on $|V|$.

Base: $|V| = 1$. In this case, the tree G has no edges, so that the orientations α and β must be equal already. Hence, we are done in this case.

Step: $|V| > 1$. Suppose any orientation can be obtained from an arbitrary orientation by repeatedly pushing sources on a tree with fewer than $|V| - 1$ vertices.

Pick any leaf ℓ of G. Since it is a leaf, ℓ has exactly one neighbor u, and there is a unique edge between ℓ and u. The graph $G \setminus \{\ell\}$ is again a tree.

Define orientations β' and α' for the tree $G \setminus \{\ell\}$ as restrictions of the orientations β and α. (In other words, we set $\beta'(e) = \beta(e)$ and $\alpha'(e) = \alpha(e)$ for each edge e of $G \setminus \{\ell\}$.) By the induction hypothesis, β' can be obtained from α' on the graph $G \setminus \{\ell\}$ by pushing some sequence of sources. Let s be this sequence.

Our goal is to obtain β from α by pushing some sequence of sources on the graph G. In order to achieve this, we proceed as follows:

1. We start with the orientation α.

2. We push the sequence s, with one little change: Every time we need to push u, we potentially have to push ℓ first (because if ℓ is a source, then the unique edge between ℓ and u is oriented towards u, thus preventing us from pushing u).

3. Now, we have obtained an orientation γ of G whose restriction to $G \setminus \{\ell\}$ is β'. Thus, this orientation differs from β at most in the edge between ℓ and u.

   If this edge is oriented equally in β and in γ, then we have arrived at β, and thus we are done in this case.

4. It remains to deal with the case when the edge between ℓ and u is oriented differently in β and in γ. If ℓ is a source of γ, then we can simply push ℓ in γ and thus obtain β; so we are done again.

   It remains to handle the case when ℓ is a sink of γ. In this case, we want to pull the sink ℓ. In order to do this, we use the Lemma: The orientation γ is acyclic (since the multigraph G is a tree, thus has no cycles), and therefore the Lemma can be applied to $\phi = \gamma$ and $v = \ell$. We thus conclude that we can pull the sink ℓ by repeatedly pushing sources; this allows us to reach β.

$\square$