Math 5707 Spring 2017 (Darij Grinberg): midterm 3
due: Wed, 3 May 2017, in class or by email (dgrinber@umn.edu) before class

See the website for relevant material.

Results proven in the notes, or in the handwritten notes, or in class, or in previous homework sets can be used without proof; but they should be referenced clearly (e.g., not "by a theorem done in class" but "by the theorem that states that a strongly connected digraph has a Eulerian circuit if and only if each vertex has indegree equal to its outdegree"). If you reference results from the lecture notes, please **mention the date and time** of the version of the notes you are using (as the numbering changes during updates).

As always, proofs need to be provided, and they have to be clear and rigorous. Obvious details can be omitted, but they actually have to be obvious.

This is a midterm, so you are **not allowed to collaborate or contact others** (apart from me) for help with the problems. (Feel free to ask me for clarifications, but I will not give hints towards solving the problems.) Reading up (in books or on the internet) is allowed, but asking for help is not. If you get your solution from a book (or paper, or website), do cite the source[1], and do explain the solution in your own words.

0.1. Exercise 1: the distances between four points in a graph

Exercise 1. Let G be a connected multigraph. Let x, y, z and w be four vertices of G.

Assume that the two largest ones among the three numbers $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ are **not** equal.

Prove that G has a cycle of length $\leq d(x,z) + d(y,w) + d(x,w) + d(y,z)$.

[**Hint:** This is a strengthening of Exercise 6 on midterm #2. Try deriving it by applying the latter exercise to a strategically chosen submultigraph of G.]

0.2. Exercise 2: more examples of chromatic polynomials

See Exercise 4 on midterm #2 for the definition and the properties of the chromatic polynomial.

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[1]You won't be penalized for this.

Exercise 2. (a) Let $n > 1$ be an integer. Prove that the chromatic polynomial of the cycle graph C_n is

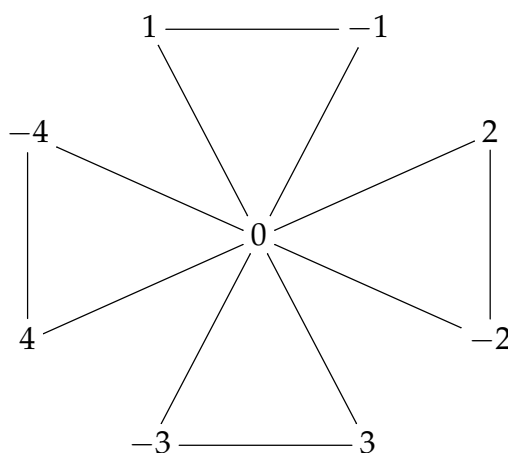$$\chi_{C_n} = (x-1)^n + (-1)^n (x-1).$$

[**Hint:** Induct over n.]

(b) Let $g \in \mathbb{N}$. Let G be the simple graph whose vertices are the $2g+1$ integers $-g, -g+1, \ldots, g-1, g$, and whose edges are

$$\begin{array}{ll} \{0,i\} & \text{for all } i \in \{1,2,\ldots,g\}; \\ \{0,-i\} & \text{for all } i \in \{1,2,\ldots,g\}; \\ \{i,-i\} & \text{for all } i \in \{1,2,\ldots,g\} \end{array}$$

(these are $3g$ edges in total).

Compute the chromatic polynomial χ_G of G.

[Here is how G looks like in the case when $g = 4$:

]

0.3. Exercise 3: unhappy marriages in a lopsided bipartite graph

Exercise 3. Let $(G; X, Y)$ be a bipartite graph such that $|Y| \geq 2|X| - 1$. Prove that there exists an injective map $f : X \to Y$ such that each $x \in X$ satisfies one of the following two statements:

- *Statement 1:* The vertices x and $f(x)$ of G are adjacent.

- *Statement 2:* There exists no $x' \in X$ such that the vertices x and $f(x')$ of G are adjacent.

(The intuition behind Exercise 3 is something along the lines of: If there are significantly more ladies than there are gentlemen, then we can marry all gentlemen off in such a way that each gentleman is either married to a lady he likes, or none of the ladies he likes is married.)

0.4. Exercise 4: crossing numbers of n-matchings

The purpose of the next exercise is to fill in some details (namely, the verification that the signs match) in our proof of one of the basic properties of the Pfaffian (the fact that the square of the Pfaffian is the determinant).

For this section, fix an $n \in \mathbb{N}$. We shall use the following notations:

- We use the Iverson bracket notation (i.e., the truth value of a statement $\mathcal{A}$ will be called $[\mathcal{A}]$).

- We set $[k] = \{1, 2, \ldots, k\}$ for each $k \in \mathbb{N}$. (This does not conflict with the Iverson bracket notation, since k is not a statement.)

- We let S_n denote the set of all the $n!$ permutations of the set $[n]$. (These permutations are the bijective maps $[n] \to [n]$.)

- The 2-element subsets of $[n]$ will be called *edges*. (This makes sense, since they are the edges of the complete graph K_n. We shall not consider any other graphs in this section.)

- The *length* $\ell(\sigma)$ of a permutation $\sigma \in S_n$ is the number of all pairs $(i, j) \in [n]^2$ satisfying $i < j$ and $\sigma(i) > \sigma(j)$. For example, the permutation in S_4 that sends $1, 2, 3, 4$ to $3, 1, 4, 2$ (respectively) has length 3.

- The *sign* of a permutation $\sigma \in S_n$ is the integer $(-1)^{\ell(\sigma)}$. It is denoted by $(-1)^{\sigma}$ (or by $\operatorname{sign} \sigma$).

- A *matching* shall mean a set $M \subseteq \mathcal{P}_2([n])$ (that is, a set M of edges) with the property that each element of $[n]$ belongs to **at most** one element of M (equivalently, with the property that the edges in M are disjoint). In other words, a *matching* is a matching of the complete graph K_n. For instance, if $n \geq 7$, then $\{\{1, 4\}, \{2, 7\}\}$ is a matching (but $\{\{1, 4\}, \{4, 7\}\}$ is not).

- A *perfect matching* shall mean a matching M such that each element of $[n]$ belongs to **exactly** one element of M. In other words, a *perfect matching* is a perfect matching of the complete graph K_n. Note that such a perfect matching exists only if n is even; in this case, there are $(n-1)(n-3)(n-5)\cdots 1$ many such perfect matchings, and each of them has size $n/2$.

- If e and f are two disjoint edges, then we say that e *crosses* f if and only if exactly one element of e lies between[2] the two elements of f. (For example, $\{2,6\}$ crosses $\{3,7\}$, but does not cross $\{3,4\}$ or $\{1,7\}$ or $\{7,8\}$.)

  It is easy to check that the relation "crosses" is symmetric: i.e., if e and f are two disjoint edges, then e crosses f if and only if f crosses e.

  The geometrical meaning of "crossing" is simple: If we draw a regular polygon with n vertices $v_1, v_2, \ldots, v_n$ in the plane, then an edge $\{i,j\}$ crosses an edge $\{k,\ell\}$ if and only if the diagonal $v_i v_j$ crosses the diagonal $v_k v_\ell$.

  Also easy to verify is the following simple fact: If $\{i,j\}$ and $\{k,\ell\}$ are two disjoint edges, then

  $$[\{i,j\} \text{ crosses } \{k,\ell\}] \equiv [i > k] + [i > \ell] + [j > k] + [j > \ell] \mod 2. \quad (1)$$

  (You can use these facts without proof.)

- If M is a matching, then the *crossing number* xing M of M is defined to be the number of all unordered pairs $\{e,f\}$ of two disjoint edges $e \in M$ and $f \in M$ such that e crosses f. (Notice that we are counting unordered pairs; hence, we do not count $\{e,f\}$ and $\{f,e\}$ twice.)

  For example, the crossing number of the matching $\{\{1,5\}, \{2,6\}, \{3,4\}\}$ is 1, since the only two edges crossing in this matching are $\{1,5\}$ and $\{2,6\}$.

Exercise 4. Let $n \in \mathbb{N}$ be even. Let $\sigma \in S_n$ be a permutation.
  (a) Show that the perfect matching

  $$M_\sigma = \{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \{\sigma(5), \sigma(6)\}, \ldots, \{\sigma(n-1), \sigma(n)\}\}$$

satisfies
$$(-1)^{\text{xing}(M_\sigma)} (-1)^k = (-1)^\sigma,$$

where k is the number of $i \in \{1, 2, \ldots, n/2\}$ satisfying $\sigma(2i-1) > \sigma(2i)$.
  (b) Let
  $$M = \{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_{n/2}, b_{n/2}\}\}$$
be any perfect matching. Define a new perfect matching σM by

  $$\sigma M = \{\{\sigma(a_1), \sigma(b_1)\}, \{\sigma(a_2), \sigma(b_2)\}, \ldots, \{\sigma(a_{n/2}), \sigma(b_{n/2})\}\}.$$

  Let p be the number of $i \in \{1, 2, \ldots, n/2\}$ satisfying $a_i > b_i$.
  Let q be the number of $i \in \{1, 2, \ldots, n/2\}$ satisfying $\sigma(a_i) > \sigma(b_i)$.
  Prove that
  $$(-1)^{\text{xing}(\sigma M)} (-1)^p = (-1)^\sigma (-1)^{\text{xing} M} (-1)^q.$$

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[2]The word "between" is to be understood in the sense of inequalities: An integer p is said to lie *between* two integers u and v if $\min\{u,v\} < p < \max\{u,v\}$.

0.5. Exercise 5: pushing sources in orientations 1

See homework set #5 for the concepts of orientations and of acyclic orientations.

> **Definition 0.1.** Let $G = (V, E, \psi)$ be a multigraph.
> Let ϕ be an orientation of G.
> A vertex $v \in V$ is said to be a *source* of ϕ if no arc of the multidigraph (V, E, ϕ) has target v. Exercise 6 **(a)** on homework set #5 shows that if ϕ is acyclic and if $V \neq \varnothing$, then there exists a source of ϕ.
> If v is a source of ϕ, then we can define a new orientation ϕ' of G as follows:
>
> - For each $e \in E$ satisfying $v \in \psi(e)$, we set $\phi'(e) = (u, v)$, where u is chosen such that $\phi(e) = (v, u)$.
>
> - For all other $e \in E$, we set $\phi'(e) = \phi(e)$.
>
> (Roughly speaking, this simply means that ϕ' is obtained by ϕ by reversing the directions of all edges that contain v.) We say that this new orientation ϕ' is obtained from ϕ by *pushing the source v*.

**Example 0.2.** Let $G = (V, E, \psi)$ be the following multigraph:
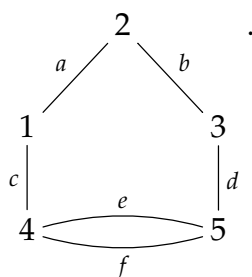
Consider the orientation ϕ of G for which the multidigraph (V, E, ϕ) looks as follows:
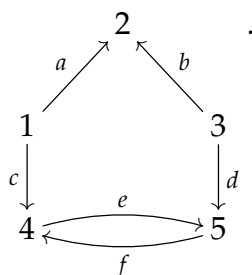

(Formally speaking, this is the orientation ϕ that sends the edges a, b, c, d, e, f to the pairs $(1, 2), (3, 2), (1, 4), (3, 5), (4, 5), (5, 4)$, respectively.)

This orientation ϕ has two sources 1 and 3. We can transform this orientation by pushing the source 1; this results in the following orientation ϕ' (shown here by drawing the multidigraph (V, E, ϕ')):
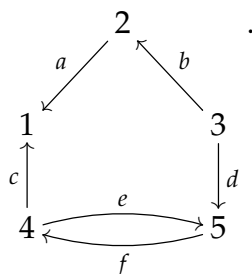
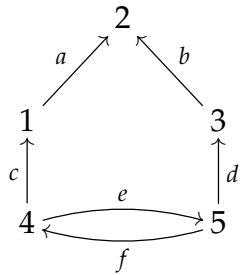

This new orientation ϕ' has a single source, 3. If we push this source, we obtain a new orientation ϕ'', which looks as follows (again, represented by the multidigraph (V, E, ϕ'')):
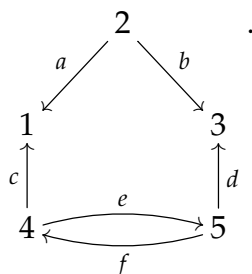
This orientation ϕ'', in turn, has a single source, 2. If we push this source, we

obtain a new orientation ϕ''', which looks as follows (again, represented by the multidigraph (V, E, ϕ''')):

This orientation ϕ''' has no sources, and thus cannot be transformed any further by pushing sources.

The preceding example might have suggested some questions: For example, given an orientation of a multigraph, can we keep pushing sources indefinitely, or will we eventually end up at an orientation that has no more sources? The following is easy to see:

Proposition 0.3. Let ϕ be an acyclic orientation of a multigraph $G = (V, E, \psi)$. Let v be a source of ϕ. Then, the orientation obtained from ϕ by pushing the source v is again acyclic.

This proposition shows that if we start with an acyclic orientation of a multigraph (with at least one vertex), then we can keep pushing sources indefinitely (since the orientation always remains acyclic, and thus there always will be sources to push). The next exercise (specifically, Exercise 5 **(b)**) yields a converse (for connected multigraphs): If we can keep pushing sources indefinitely, then our orientation must have been acyclic.

Exercise 5. Let $G = (V, E, \psi)$ be a connected multigraph. Set $n = |V|$ and $h = |E|$.

Let $(\phi_0, \phi_1, \ldots, \phi_k)$ be a sequence of orientations of G, and let $(v_1, v_2, \ldots, v_k)$ be a sequence of vertices of G such that for each $i \in \{1, 2, \ldots, k\}$, the orientation ϕ_i is obtained from ϕ_{i-1} by pushing the source v_i (in particular, this is saying that v_i is a source of ϕ_{i-1}).

Assume that $k \geq \binom{n+h-1}{n-1}$.

(a) Prove that each vertex of G appears at least once in the sequence $(v_1, v_2, \ldots, v_k)$.

(b) Prove that the orientations $\phi_0, \phi_1, \ldots, \phi_k$ are acyclic.

(At this point, let me remind you that you can freely use the exercises on the previous homework sets.)

0.6. Exercise 6: pushing sources in orientations 2

Exercise 6. Let $G = (V, E, \psi)$ be a tree. Let α and β be two orientations of G.
Prove that β can be obtained from α by repeatedly pushing sources.
(More rigorously: Prove that there exists a sequence $(\phi_0, \phi_1, \ldots, \phi_k)$ of orientations of G, and a sequence $(v_1, v_2, \ldots, v_k)$ of vertices of G such that

- we have $\phi_0 = \alpha$ and $\phi_k = \beta$, and

- for each $i \in \{1, 2, \ldots, k\}$, the orientation ϕ_i is obtained from ϕ_{i-1} by pushing the source v_i (in particular, this is saying that v_i is a source of ϕ_{i-1}).

Notice that k is allowed to be 0.)