Math 5707 Spring 2017 (Darij Grinberg): midterm 2
Solution sketches.

See the website for relevant material.

If u and v are two vertices of a simple graph (or multigraph) G, then $d_G(u,v)$ (often abbreviated as $d(u,v)$ when G is clear from the context) means the distance from u to v in G (that is, the minimum length of a path from u to v if such a path exists; otherwise, the symbol ∞).

Results proven in the notes, or in the handwritten notes, or in class, or in previous homework sets can be used without proof; but they should be referenced clearly (e.g., not "by a theorem done in class" but "by the theorem that states that a strongly connected digraph has a Eulerian circuit if and only if each vertex has indegree equal to its outdegree"). If you reference results from the lecture notes, please **mention the date and time** of the version of the notes you are using (as the numbering changes during updates).

As always, proofs need to be provided, and they have to be clear and rigorous. Obvious details can be omitted, but they actually have to be obvious.

Contents

0.1. Reminder: Hall's Marriage Theorem

Recall Hall's Marriage Theorem (or, rather, its "hard" direction):

Theorem 0.1. Let $(G; X, Y)$ be a bipartite graph. (Recall that this means that G is a graph and X and Y are two subsets of $\operatorname{V}(G)$ such that

- each vertex of G lies in exactly one of the two sets X and Y;
- each edge of G has exactly one endpoint in X and exactly one endpoint in Y.

)

Assume that every subset S of X satisfies $|N_G(S)| \geq |S|$. (Here, as usual, $N_G(S)$ denotes the set $\{v \in \operatorname{V}(G) \mid \text{at least one neighbor of } v \text{ belongs to } S\}$.)

Then, G has an X-complete matching.

0.2. Exercise 1: assigning to each vertex an edge avoiding it

Exercise 1. Let $G = (V, E)$ be a simple graph such that $|E| \geq |V|$. Show that there exists an injective map $f : V \to E$ such that each $v \in V$ satisfies $v \notin f(v)$.

(In other words, show that we can assign to each vertex v of G an edge that does not contain v, in such a way that edges assigned to distinct vertices are distinct.)

Solution to Exercise 1 (sketched). Define a simple graph H as follows:

- The vertices of the simple graph H are the elements of $V \cup E$. (We assume WLOG that the sets V and E are disjoint; if they aren't, then rename the elements of V as $1, 2, \ldots, n$, which ensures that they are.)
- The edges of H are the 2-element sets $\{v, e\}$ with $v \in V$ and $e \in E$ satisfying $v \notin e$.

Then, $(H; V, E)$ is a bipartite graph.

It is sufficient to show that the graph H has a V-complete matching[1]. In order to do so, we apply Theorem 0.1 to H, V and E instead of G, X and Y. We thus need to check that every subset S of V satisfies $|N_H(S)| \geq |S|$.

So let us fix a subset S of V. We must prove that $|N_H(S)| \geq |S|$.

- If $|S| = 0$, then this is obvious.
- If $|S| = 1$, then this is easy to see:

  Indeed, assume that $|S| = 1$. Thus, $S = \{v\}$ for a single vertex v of G. Consider this v. Thus, the set $N_H(S)$ consists of all edges of G that do not pass through v. If each edge of G would pass through v, then G would have at most $|V| - 1$ edges, so that we would have $|E| \leq |V| - 1 < |V|$, which would contradict $|E| \geq |V|$. Thus, there is at least one edge of G that does not pass through v. In other words, the set $N_H(S)$ is nonempty (since the set $N_H(S)$ consists of all edges of G that do not pass through v). Hence, $|N_H(S)| \geq 1 = |S|$. Thus, $|N_H(S)| \geq |S|$ holds in this case as well.
- If $|S| = 2$, then this is also easy to check:

  Indeed, assume that $|S| = 2$. Thus, $S = \{v, w\}$ for two distinct vertices v and w of G. Consider these v and w. Thus, the set $N_H(S)$ consists of all edges

[1] Indeed, this will solve the exercise, because if we have found such a matching M, then we can define an injective map $f : V \to E$ by sending each $v \in V$ to the M-partner of v.

of G that do not pass through both v and w. Hence, this set $N_H(S)$ contains either all edges of G or all but one edges of G (since at most one edge can pass through both v and w). Therefore, $|N_H(S)| \geq |E| - 1 \geq |V| - 1$ (since $|E| \geq |V|$).

Recall that we must prove that $|N_H(S)| \geq |S|$. Indeed, assume the contrary. Thus, $|N_H(S)| < |S|$. Hence, $2 = |S| > |N_H(S)| \geq |V| - 1$, so that $|V| < 2 + 1 = 3$, thus $|V| \leq 2$. Hence, the graph G has at most 2 vertices, and therefore at most 1 edge. In other words, $|E| \leq 1$. But $S \subseteq V$ and thus $|S| \leq |V| \leq |E| \leq 1$. This contradicts the fact that $|S| = 2$. This contradiction proves that our assumption was wrong. Hence, $|N_H(S)| \geq |S|$ holds in this case as well.

- If $|S| > 2$, then this holds for simple reasons:

  Indeed, assume that $|S| > 2$. Thus, there is no edge of G passing through each vertex in S. Hence, each edge of G belongs to $N_H(S)$ (since the set $N_H(S)$ consists of all edges of G that do not pass through each vertex in S). In other words, $N_H(S) = E$. Hence, $|N_H(S)| = |E| \geq |V| \geq |S|$ (since $V \supseteq S$). Hence, $|N_H(S)| \geq |S|$ holds in this case as well.

Hence, $|N_H(S)| \geq |S|$ is always proven, and so the exercise is solved. □

0.3. Exercise 2: assigning to each vertex an edge containing it

> **Exercise 2.** Let $G = (V, E)$ be a **connected** simple graph such that $|E| \geq |V|$. Show that there exists an injective map $f : V \to E$ such that each $v \in V$ satisfies $v \in f(v)$.
>
> (In other words, show that we can assign to each vertex v of G an edge that contains v, in such a way that edges assigned to distinct vertices are distinct.)

Solution to Exercise 2 (sketched). Unlike Exercise 1, this exercise is not about applying Hall's marriage theorem (although maybe it can be solved in this way as well). Instead, I solve it using spanning trees:

Corollary 20 from lecture 9 (handwritten notes) shows that if G is a forest, then $|E| = |V| - b_0(G)$ (where $b_0(G)$ is the number of connected components of G). Hence, if G was a forest, then we would have

$$|E| = |V| - \underbrace{b_0(G)}_{\substack{=1 \\ \text{(since } G \text{ is connected)}}} = |V| - 1 < |V|,$$

which would contradict $|E| \geq |V|$. Hence, G cannot be a forest. Consequently, G must have a cycle. Fix such a cycle, and fix any edge e on this cycle.

Let G' be the simple graph obtained from G by removing the edge e. Then, G' is still connected (since the edge we removed belonged to a cycle, and thus could be avoided by going around the cycle). Hence, G' has a spanning tree (since any connected graph has a spanning tree). Fix such a spanning tree, and denote it by T.

Pick any endpoint u of the edge e. Now, define a map $f : V \to E$ as follows:

- Set $f(u) = e$.
- For any $v \in V$ distinct from u, we define $f(v)$ as follows: There is a unique path from v to u in the tree T. This path has length > 0 (since $v \neq u$), and thus has a well-defined first edge. Let $f(v)$ be this first edge.

It is clear that this map f is well-defined and has the property that each $v \in V$ satisfies $v \in f(v)$. It thus remains to check that this map f is injective. This is easy[2]. □

0.4. Exercise 3: a "transitivity" property for arc-disjoint paths

0.4.1. Statement

Exercise 3. Let $D = (V, A)$ be a digraph. Let $k \in \mathbb{N}$. Let u, v and w be three vertices of D. Assume that there exist k arc-disjoint paths from u to v. Assume furthermore that there exist k arc-disjoint paths from v to w.

Prove that there exist k arc-disjoint paths from u to w.

[**Note:** If $u = w$, then the trivial path (u) counts as being arc-disjoint from itself (so in this case, there exist arbitrarily many arc-disjoint paths from u to w).]

0.4.2. First solution

To prepare for the solution of this exercise, let us recall Menger's theorem in its directed arc-disjoint version (see Exercise 1 on Homework set 4):

[2] *Proof.* Assume the contrary. Thus, there exist two distinct vertices v_1 and v_2 in V such that $f(v_1) = f(v_2)$. Consider these v_1 and v_2. Notice that $v_1 \in f(v_1)$ (since each $v \in V$ satisfies $v \in f(v)$) and $v_2 \in f(v_2)$ (similarly).

At least one of v_1 and v_2 is distinct from u (since v_1 and v_2 are distinct). Thus, we can WLOG assume that $v_1 \neq u$. Assume this. From $v_1 \neq u$, we conclude that $f(v_1)$ is the first edge of the path from v_1 to u in the tree T (by the definition of f). In particular, $f(v_1)$ is an edge of the tree T, and thus is distinct from e (since e is not an edge of the tree T). Thus, $f(v_1) \neq e$, so that $f(v_2) = f(v_1) \neq e = f(u)$. Hence, $v_2 \neq u$. Therefore, $f(v_2)$ is the first edge of the path from v_2 to u in the tree T (by the definition of f). Hence, the unique path from v_2 to u in the tree T uses the edge $f(v_2)$. As a consequence, this path uses the vertex v_1 (because $v_1 \in f(v_1) = f(v_2)$). Since $v_1 \neq v_2$, we thus have $d_T(v_2, u) > d_T(v_1, u)$.

But recall that $f(v_1)$ is the first edge of the path from v_1 to u in the tree T. Hence, the unique path from v_1 to u in the tree T uses the edge $f(v_1)$. As a consequence, this path uses the vertex v_2 (because $v_2 \in f(v_2) = f(v_1)$). Since $v_2 \neq v_1$, we thus have $d_T(v_1, u) > d_T(v_2, u)$. This contradicts $d_T(v_2, u) > d_T(v_1, u)$.

Theorem 0.2 (Menger's theorem, DA (directed arc-disjoint version))**.** Let $D = (V, A, \phi)$ be a multidigraph. Let s and t be two distinct vertices of D.

An *s-t-path* in D means a path from s to t in D.

Several paths in D are said to be *arc-disjoint* if no two have an arc in common.

A subset C of A is said to be an *s-t-cut* if it has the form

$$C = \{a \in A \mid \text{the source of } a \text{ belongs to } U\text{, but the target of } a \text{ does not}\}$$

for some subset U of V satisfying $s \in U$ and $t \notin U$.

The maximum number of arc-disjoint s-t-paths equals the minimum size of an s-t-cut.

A corollary of Theorem 0.2 is the following fact:

Corollary 0.3. Let $D = (V, A, \phi)$ be a multidigraph. Let s and t be two vertices of D. Let $k \in \mathbb{N}$. Then, there exist k arc-disjoint paths from s to t if and only if there exists no s-t-cut of size $< k$.

Proof of Corollary 0.3 (sketched). If $s = t$, then it is clear that Corollary 0.3 holds[3]. Thus, we WLOG assume that $s \neq t$. Hence, Theorem 0.2 shows that the maximum number of arc-disjoint s-t-paths equals the minimum size of an s-t-cut. Denote these two equal numbers by m. Thus:

- The number m is the maximum number of arc-disjoint s-t-paths. Hence, there exist m arc-disjoint s-t-paths. Therefore, there exist k arc-disjoint s-t-paths whenever $k \leq m$ (indeed, just throw away $m - k$ of the m arc-disjoint s-t-paths whose existence we have observed in the previous sentence), but not when $k > m$ (since m is the **maximum** number of arc-disjoint s-t-paths). Thus, we have the following logical equivalence:

$$(\text{there exist } k \text{ arc-disjoint } s\text{-}t\text{-paths}) \iff (k \leq m). \tag{1}$$

- The number m is the minimum size of an s-t-cut. Hence, there exists no s-t-cut of any size $< m$, but there exists an s-t-cut of size m. Thus, there exists no s-t-cut of size $< k$ if and only if $k \leq m$. In other words, we have the following logical equivalence:

$$(\text{there exists no } s\text{-}t\text{-cut of size } < k) \iff (k \leq m). \tag{2}$$

Comparing the logical equivalences (1) and (2), we obtain the equivalence

$$(\text{there exist } k \text{ arc-disjoint } s\text{-}t\text{-paths}) \iff (\text{there exists no } s\text{-}t\text{-cut of size } < k).$$

This proves Corollary 0.3. □

[3] *Proof.* Assume that $s = t$. Then, there exist k arc-disjoint paths from s to t (indeed, the path (s) of length 0 is arc-disjoint from itself, and so we can pick it k times), and there exists no s-t-cut of size k (indeed, there exists no s-t-cut of any size, because there exists no subset U of V satisfying $s \in U$ and $t \notin U$). Hence, Corollary 0.3 holds in this case.

First solution to Exercise 3 (sketched). Corollary 0.3 (applied to $s = u$ and $t = v$) shows that there exist k arc-disjoint paths from u to v if and only if there exists no u-v-cut of size $< k$. Hence, there exists no u-v-cut of size $< k$ (since there exist k arc-disjoint paths from u to v). Similarly, there exists no v-w-cut of size $< k$.

Now, we claim that there exists no u-w-cut of size $< k$.

Indeed, assume the contrary. Thus, there exists an u-w-cut of size $< k$. Fix such a u-w-cut, and write it in the form

$$C = \{a \in A \mid \text{the source of } a \text{ belongs to } U, \text{ but the target of } a \text{ does not}\} \qquad (3)$$

for some subset U of V satisfying $u \in U$ and $w \notin U$. Then, $|C| < k$ (since the u-w-cut C has size $< k$).

Now, we have either $v \in U$ or $v \notin U$. But each of these two cases leads to a contradiction:

- If $v \notin U$, then C is an u-v-cut (since it has the form (3) for the subset U of V, which satisfies $u \in U$ and $v \notin U$), and thus there exists a u-v-cut of size $< k$; but this contradicts the fact that there exists no u-v-cut of size $< k$.
- If $v \in U$, then C is a v-w-cut (since it has the form (3) for the subset U of V, which satisfies $v \in U$ and $w \notin U$), and thus there exists a v-w-cut of size $< k$; but this contradicts the fact that there exists no v-w-cut of size $< k$.

Thus, we always get a contradiction. Hence, our assumption was wrong.

We thus have shown that there exists no u-w-cut of size $< k$.

But Corollary 0.3 (applied to $s = u$ and $t = w$) shows that there exist k arc-disjoint paths from u to w if and only if there exists no u-w-cut of size $< k$. Hence, there exist k arc-disjoint paths from u to w (since there exists no u-w-cut of size $< k$). This solves Exercise 3. □

0.4.3. An extension of the stable marriage problem

I am going to outline a second solution to Exercise 3 as well. That solution will rely on a slight generalization of the stable marriage problem.

I assume that you are familiar with the basic theory of the stable marriage problem (see [LeLeMe16, Section 6.4]), specifically with the algorithm that is called the "Mating Ritual" in [LeLeMe16, Section 6.4][4].

Now, let me formulate a more general version of the stable marriage problem, which I shall call the *contracted stable marriage problem*:

Contracted stable marriage problem.

[4] This algorithm is also called the "deferred-acceptance algorithm" in `http://www.math.jhu.edu/~eriehl/pechakucha.pdf`. That said, this name is also used for some variations of this algorithm.

Suppose that we have a population of k men and k women (for some $k \in \mathbb{N}$). Assume furthermore that a finite set C of "contracts" is given. Each contract involves exactly one man and exactly one woman.[5] Assume that, for each pair (m, w) consisting of a man m and a woman w, there is **at least** one contract that involves m and w. (Hence, there are at least k^2 contracts, but there can be more.)

Suppose that each person has a preference list of all the contracts that involve him/her; i.e., he/she ranks all contracts that involve him/her in the order of preferability. (No ties are allowed.)

A *matching* shall mean a subset K of C such that each man is involved in exactly one contract in K, and such that each woman is involved in exactly one contract in K. Thus, visually speaking, a matching is a way to marry off all k men and all k women to each other (in the classical meaning of the word – i.e., heterosexual and monogamous) by having them sign some of the contracts in C (of course, each person signs exactly one contract).

If p is a person and K is a matching, then the unique contract $c \in K$ that involves p will be called the *K-marriage contract of p*.

If K is a matching and $c \in C$ is a contract, then the contract c is said to be *rogue* (for K) if

- this contract c is not in K,
- the man involved in c prefers c to his K-marriage contract, and
- the woman involved in c prefers c to her K-marriage contract.

Thus, roughly speaking, a rogue contract is a contract c that has not been signed in the matching K, but that would make both persons involved in c happier than whatever contracts they did sign in K.

A matching K is called *stable* if there exist no rogue contracts for K.

The *contracted stable marriage problem* asks us to find a stable matching.

Notice that this problem generalizes the stable marriage problem discussed in [LeLeMe16, Section 6.4]; indeed, the latter problem is the particular case when each pair (m, w) consisting of a man m and a woman w is involved in precisely one contract. Roughly speaking, the contracted stable marriage problem extends the latter by allowing some couples to marry in several distinct ways, some of which may be more or less preferable to one of the partners. [6]

The contracted stable marriage problem can be solved by a modification of the "Mating Ritual" analyzed in [LeLeMe16, Section 6.4]. Namely:

[5] Think of the contracts as marriage contracts prepared "just in case". The names of the spouses-to-be have already been written in, but the contracts have not been signed, and in particular there can be mutually exclusive contracts for the same man or woman.

[6] Generally, passing from the classical stable marriage problem to the contracted stable marriage problem is akin to generalizing theorems from simple graphs to multigraphs.

- each man should keep a preference list of contracts that involve him (instead of a list of women), and he should serenade with a contract in hand (i.e., each day he picks up his most preferable **contract**, and then he proposes to the woman involved with this specific contract);
- every afternoon, each woman should dismiss all contracts that her suitors are currently proposing her, except for the best (from her perspective) of these contracts (as opposed to dismissing the suitors themselves);
- a man whose contract gets dismissed crosses off this contract (rather than the woman who dismissed him) from his list, so he may possibly return to her later with another contract;
- when the ritual terminates, the women marry their current suitors using the contracts that these suitors are currently proposing.

The analysis of this algorithm is similar to the one made in [LeLeMe16, Section 6.4.2 and Section 6.4.3]; some modifications need to be made (e.g., Invariant P should be replaced by "For every contract c involving a woman w and a man m, if c is crossed off m's list, then w has a suitor offering her a contract d that she prefers over c."). Thus, we obtain the following result (generalizing [LeLeMe16, Theorem 6.4.4]):

Theorem 0.4. The modified "Mating Ritual" produces a stable matching for the contracted stable marriage problem.

0.4.4. Second solution

I shall now sketch a second solution to Exercise 3, suggested by Alexander Postnikov. But first, let me give the motivation behind this solution:

- The following appears to be a reasonable approach to solving Exercise 3:

  Fix k arc-disjoint paths $p_1, p_2, \ldots, p_k$ from u to v. (These exist by assumption.)

  Fix k arc-disjoint paths $q_1, q_2, \ldots, q_k$ from v to w. (These exist by assumption.)

  Now, we are looking for k arc-disjoint paths from u to w. The most obvious thing one could try is to take the k walks $t_1, t_2, \ldots, t_k$, where each t_i is the concatenation of p_i with q_i.
- However, this does not always work. The t_i are walks, but not necessarily paths. Fortunately, we know how to deal with this: Just keep removing cycles from the t_i until no cycles remain.
- Sadly, we are still not done. The t_i are paths now, but are not necessarily arc-disjoint. Indeed, it could happen that (for example) p_2 and q_5 have a common arc; but then t_2 and t_5 would not be arc-disjoint. The common arc might disappear when we remove cycles, but it does not have to; it might also stay.

- We could now try being more strategical: Let us look for a bijection $\sigma : \{1,2,\ldots,k\} \to \{1,2,\ldots,k\}$ such that if we define t_i as the concatenation of p_i with $q_{\sigma(i)}$ (rather than with q_i), then the resulting paths t_i (after removing cycles) will be arc-disjoint. However, how do we find such a bijection? Does it always exist?

- Let a *common arc* mean an arc which belongs to one of the paths $p_1, p_2, \ldots, p_k$ and also belongs to one of the paths $q_1, q_2, \ldots, q_k$. As we have seen, common arcs are the source of our headache, and we should try to make sure that each common arc survives at most once in the resulting paths $t_1, t_2, \ldots, t_k$ (after the cycles are removed). How do we achieve this?

- Here is one approach that sounds hopeful: If a path p_i has an arc in common with the path $q_{\sigma(i)}$, then the concatenation t_i of these two paths will contain this common arc twice, and therefore, after removing cycles, it will only contain it once; thus, this particular common arc will no longer make troubles. Hence, it appears reasonable to want p_i to have an arc in common with $q_{\sigma(i)}$ for as many i as possible.

- It also appears reasonable to ensure that if p_i has an arc in common with $q_{\sigma(i)}$, then this arc appears as early as possible in p_i, and as late as possible in $q_{\sigma(i)}$. Indeed, this makes sure that the path produced by removing cycles in the concatenation t_i will be as small as possible, and therefore (if we are lucky) we will get rid of other common arcs as well.

- Now we are trying to solve several optimization problems at once – we want to have our common arcs appear as early as possible in p_i and as late as possible in $q_{\sigma(i)}$. Such problems are not always solvable. Usually, there is a tradeoff, and we have to settle for a compromise.

- The stable marriage problem is a prototypical example of the search for such a compromise. We can thus try to apply it here. The first approximation is the following:

  Model the k paths $p_1, p_2, \ldots, p_k$ by k men labelled $1, 2, \ldots, k$.

  Model the k paths $q_1, q_2, \ldots, q_k$ by k women labelled $1, 2, \ldots, k$.

  Man i would be happy to marry woman j if and only if the paths p_i and q_j have an arc in common; the earlier this arc appears in p_i, the happier he would be marrying j. As a last resort, he is also willing to marry woman j if the paths p_i and q_j have no arcs in common, but he would be less happy this way.

  Likewise, woman j would be happy to marry man i if and only if the paths p_i and q_j have an arc in common; the later this arc appears in q_j, the happier she would be marrying i. As a last resort, she is also willing to marry man i if the paths p_i and q_j have no arcs in common, but she would be less happy this way.

Everyone ranks their potential spouses by these preferences, and we seek a stable matching. When man i and woman j marry, we set $\sigma(i) = j$, and thus a bijection $\sigma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ is defined.

- However, this still is not completely correct! The problem is that a path p_i can have more than one arc in common with a path q_j. This may mess up the preferences of both man i and woman j, since they would have to decide whether to take the first or the last intersections into account. In a sense, there seem to be several ways in which man i can marry woman j – one for each arc that the two paths have in common. So we are looking at an instance of the contracted stable marriage problem.

The following solution is what comes out if you follow this strategy. It is fairly long and technical, neat as the idea may be.

Second solution of Exercise 3 (sketched). We have assumed that there exist k arc-disjoint paths from u to v. Fix such k paths, and denote them by $p_1, p_2, \ldots, p_k$.

We have assumed that there exist k arc-disjoint paths from v to w. Fix such k paths, and denote them by $q_1, q_2, \ldots, q_k$.

If p is any path in D, and if a is any arc of p, then we shall let $[a]\, p$ denote the positive integer h such that a is the h-th arc of p. (Of course, this h is uniquely determined, since a path never uses an arc more than once.)

Now, we fabricate a population of

- k men labelled $1, 2, \ldots, k$, and
- k women labelled $1, 2, \ldots, k$,

as well as a set C of contracts (in the sense of the contracted stable marriage problem) defined as follows:

- For each arc a that appears in one of the paths $p_1, p_2, \ldots, p_k$ and also appears in one of the paths $q_1, q_2, \ldots, q_k$, we define a contract c_a as follows: Let i be such that a appears in p_i. (This i is unique, because a cannot appear in more than one of the paths $p_1, p_2, \ldots, p_k$; this is because these paths are arc-disjoint.) Let j be such that a appears in q_j. (This j is unique for a similar reason.) The contract c_a shall involve man i and woman j.
- For each $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, k\}$, we define a contract $d_{i,j}$ involving man i and woman j. This contract will be called a "dummy contract".

The set C shall consist of all contracts c_a and all contracts $d_{i,j}$. The dummy contracts guarantee that for each pair (m, w) consisting of a man m and a woman w, there is at least one contract that involves m and w.

Each of the k men and each of the k women shall have a preference list of all the contracts that involve him/her; namely, we define these preference lists as follows:

- For each $i \in \{1, 2, \ldots, k\}$, the preference list of **man** i shall consist of all the contracts c_a that involve him, as well as all the dummy contracts of the form $d_{i,j}$. The contracts c_a shall appear in the order of **increasing** $[a]\, p_i$ (that is, man i prefers those contracts c_a whose number $[a]\, p_i$ is smaller), whereas the dummy contracts shall appear in arbitrary order; the dummy contracts must appear below all the contracts c_a (that is, man i prefers any of the c_a to any of the dummy contracts).

- For each $j \in \{1, 2, \ldots, k\}$, the preference list of **woman** j shall consist of all the contracts c_a that involve her, as well as all the dummy contracts of the form $d_{i,j}$. The contracts c_a shall appear in the order of **decreasing** $[a]\, q_j$ (that is, woman j prefers those contracts c_a whose number $[a]\, q_j$ is larger), whereas the dummy contracts shall appear in arbitrary order; the dummy contracts must appear below all the contracts c_a (that is, woman j prefers any of the c_a to any of the dummy contracts).

Consider the contracted stable marriage problem corresponding to this data. Theorem 0.4 shows that the modified "Mating Ritual" produces a stable matching. Hence, a stable matching exists. Fix such a stable matching, and denote it by K. For each $i \in \{1, 2, \ldots, k\}$, we let m_i denote the K-marriage contract of man i.

For each $i \in \{1, 2, \ldots, k\}$, we define a path s_i from u to w as follows:

- Let the woman involved in the contract m_i (that is, the woman married off to man i in the stable matching K) be woman j.

- If the contract m_i is **not** a dummy contract, then $m_i = c_a$ for some arc a. In this case:

  - Consider this arc a.
  - Let r_i be the walk consisting of the first $[a]\, p_i$ arcs of the path p_i (that is, all the arcs up to the point where it uses the arc a, including that arc a) and of the last[7] $\ell(q_j) - [a]\, q_j$ arcs of the path q_j (that is, all the arcs that come after the point where it uses the arc a). This is a walk from u to w.

  Otherwise:

  - Let r_i be the walk consisting of all arcs of the path p_i and of all arcs of the path q_j. (In other words, let r_i be the concatenation of the paths p_i and q_j.) This is a walk from u to w.

  In either case, we have defined a walk r_i from u to w.

- We obtain a path s_i from u to w by successively removing cycles from r_i until no cycles remain. (The result of this process may depend on the choices made, but this does not matter to us, as any result is good.)

[7]Here, $\ell(x)$ denotes the length of any walk x.

We have thus defined k paths $s_1, s_2, \ldots, s_k$ from u to w.

Let us make some observations:

> *Observation 1:* Fix $i \in \{1, 2, \ldots, k\}$. Let $j \in \{1, 2, \ldots, k\}$ be such that woman j is the K-partner of man i (that is, the contract m_i involves man i and woman j).
>
> Assume that the contract m_i has the form $m_i = c_a$ for some arc a.
>
> Let b be an arc of the path s_i.
>
> Then, either b is an arc of p_i satisfying $[b]\, p_i \leq [a]\, p_i$, or b is an arc of q_j satisfying $[b]\, q_j > [a]\, q_j$.

Proof of Observation 1. The path s_i was obtained by removing cycles from r_i. Thus, each arc of s_i is an arc of r_i. Hence, b is an arc of r_i (since b is an arc of s_i).

But r_i was defined as the walk consisting of the first $[a]\, p_i$ arcs of the path p_i (that is, all the arcs up to the point where it uses the arc a, including that arc a) and of the last $\ell(q_j) - [a]\, q_j$ arcs of the path q_j (that is, all the arcs that come after the point where it uses the arc a). Therefore, each arc of r_i is either one of the first $[a]\, p_i$ arcs of the path p_i, or one of the last $\ell(q_j) - [a]\, q_j$ arcs of the path q_j. In particular, this must hold for the arc b (since b is an arc of r_i). In other words, either b is an arc of p_i satisfying $[b]\, p_i \leq [a]\, p_i$, or b is an arc of q_j satisfying $[b]\, q_j > [a]\, q_j$. This finishes the proof of Observation 1. □

> *Observation 2:* Fix $i \in \{1, 2, \ldots, k\}$. Let $j \in \{1, 2, \ldots, k\}$ be such that woman j is the K-partner of man i (that is, the contract m_i involves man i and woman j).
>
> Assume that the contract m_i is a dummy contract.
>
> Let b be an arc of the path s_i.
>
> Then, either b is an arc of p_i, or b is an arc of q_j.

Proof of Observation 2. Analogous to the proof of Observation 1. □

> *Observation 3:* Fix $i \in \{1, 2, \ldots, k\}$. Let $j \in \{1, 2, \ldots, k\}$ be such that woman j is the K-partner of man i (that is, the contract m_i involves man i and woman j).
>
> Let b be an arc of the path s_i. Assume that the contract c_b exists.
>
> Then, one of the following two statements holds:
>
> - *Statement O3.1:* The arc b belongs to the path p_i, and man i weakly prefers[8] the contract c_b over his K-marriage contract.

[8]We say that a person p "weakly prefers" a contract κ_1 to a contract κ_2 if we have either $\kappa_1 = \kappa_2$ or the person p prefers κ_1 to κ_2.

- *Statement O3.2:* The arc b belongs to the path q_j, and woman j prefers the contract c_b over her K-marriage contract.

Proof of Observation 3. The contract m_i is either of the form $m_i = c_a$ for some arc a, or a dummy contract.

In the former case, Observation 3 follows from Observation 1.

In the latter case, Observation 3 follows from Observation 2 (using the fact that everyone prefers any contract of the form c_a over any dummy contract). □

Observation 4: Let i and x be two distinct elements of $\{1, 2, \ldots, k\}$ such that the paths s_i and s_x have an arc in common. Let b be an arc common to these two paths s_i and s_x.

Choose $j \in \{1, 2, \ldots, k\}$ such that woman j is the K-partner of man i (that is, the contract m_i involves man i and woman j).

Choose $y \in \{1, 2, \ldots, k\}$ such that woman y is the K-partner of man x (that is, the contract m_x involves man x and woman y).

Then, one of the following two statements holds:

- *Statement O4.1:* The arc b belongs to the path p_i and to the path q_y.
- *Statement O4.2:* The arc b belongs to the path p_x and to the path q_j.

Proof of Observation 4. The women j and y are married (in K) to the two distinct men i and x. Therefore, these two women must too be distinct. In other words, j and y are distinct.

Either b is an arc of p_i, or b is an arc of q_j [9]. Similarly, either b is an arc of p_x, or b is an arc of q_y. Thus, we are in one of the following four cases:

- *Case 1:* The arc b is an arc of p_i and is an arc of p_x.
- *Case 2:* The arc b is an arc of p_i and is an arc of q_y.
- *Case 3:* The arc b is an arc of q_j and is an arc of p_x.
- *Case 4:* The arc b is an arc of q_j and is an arc of q_y.

However, the paths $p_1, p_2, \ldots, p_k$ are arc-disjoint. Hence, the paths p_i and p_x have no arcs in common (since i and x are distinct). Hence, b cannot be an arc of p_i and an arc of p_x at the same time. Therefore, Case 1 is impossible. Similarly, Case 4 is impossible (here, we use the fact that j and y are distinct). Thus, only Case 2 and Case 3 remain to be discussed. But clearly, Statement O4.1 holds in Case 2, whereas Statement O4.2 holds in Case 3. Thus, one of the two Statements always holds. This proves Observation 4. □

[9] In fact, the contract m_i either is a contract of the form $m_i = c_a$ for some arc a, or is a dummy contract. In the former case, the claim follows from Observation 1; in the latter case, the claim follows from Observation 2.

We are now going to show that the k paths $s_1, s_2, \ldots, s_k$ are arc-disjoint.

Indeed, assume the contrary. Thus, there exist two distinct elements i and x of $\{1, 2, \ldots, k\}$ such that the paths s_i and s_x have an arc in common. Consider these i and x.

The paths s_i and s_x have an arc in common. Fix such an arc, and denote it by b.

Choose $j \in \{1, 2, \ldots, k\}$ such that woman j is the K-partner of man i (that is, the contract m_i involves man i and woman j).

Choose $y \in \{1, 2, \ldots, k\}$ such that woman y is the K-partner of man x (that is, the contract m_x involves man x and woman y).

Observation 4 shows that one of the following two statements holds:

- *Statement O4.1:* The arc b belongs to the path p_i and to the path q_y.
- *Statement O4.2:* The arc b belongs to the path p_x and to the path q_j.

We WLOG assume that Statement O4.1 holds (because otherwise, we can simply switch i and j with x and y).

However, the paths $p_1, p_2, \ldots, p_k$ are arc-disjoint. Hence, the paths p_i and p_x have no arcs in common (since i and x are distinct). Hence, the arc b cannot belong to p_i and to p_x at the same time. Thus, b does not belong to p_x (since p belongs to p_i). Similarly, b does not belong to q_j.

The arc b belongs to both paths p_i and q_y. Hence, the contract c_b exists (and involves man i and woman y). Thus, Observation 3 (applied to x and y instead of i and j) shows that one of the following two statements holds:

- *Statement X3.1:* The arc b belongs to the path p_x, and man x weakly prefers the contract c_b over his K-marriage contract.
- *Statement X3.2:* The arc b belongs to the path q_y, and woman y prefers the contract c_b over her K-marriage contract.

But Statement X3.1 cannot hold, since b does not belong to p_x. Hence, Statement X3.2 must hold. In particular, woman y prefers the contract c_b over her K-marriage contract. This shows that c_b is not her K-marriage contract. Therefore, the contract c_b is not in K.

Observation 3 shows that one of the following two statements holds:

- *Statement I3.1:* The arc b belongs to the path p_i, and man i weakly prefers the contract c_b over his K-marriage contract.
- *Statement I3.2:* The arc b belongs to the path q_j, and woman j prefers the contract c_b over her K-marriage contract.

But the arc b does not belong to q_j. Therefore, Statement I3.2 cannot hold. Thus, Statement I3.1 must hold (since one of Statement I3.1 and Statement I3.2 holds). Hence, man i weakly prefers the contract c_b over his K-marriage contract. Since c_b

is not his K-marriage contract (because the contract c_b is not in K), we can remove the word "weakly" from this sentence. Thus, man i prefers the contract c_b over his K-marriage contract. Recall that the same can be said about woman y. Hence, the contract c_b is rogue (for K). This contradicts the fact that there exist no rogue contract for K (since K is a stable matching). This contradiction proves that our assumption was false.

Hence, we have shown that the k paths $s_1, s_2, \ldots, s_k$ are arc-disjoint. Thus, there exist k arc-disjoint paths from u to w (namely, $s_1, s_2, \ldots, s_k$). □

0.5. Exercise 4: the chromatic polynomial

Exercise 4. Let $G = (V, E)$ be a simple graph. Define a polynomial χ_G in a single indeterminate x (with integer coefficients) by

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

(Here, as usual, $\operatorname{conn} H$ denotes the number of connected components of any graph H.) This polynomial χ_G is called the *chromatic polynomial* of G.

Fix $k \in \mathbb{N}$. Recall that a *k-coloring* of G means a map $f : V \to \{1, 2, \ldots, k\}$. (The image $f(v)$ of a vertex $v \in V$ under this map is called the *color* of v under this k-coloring f.) A k-coloring f of G is said to be *proper* if each edge $\{u, v\}$ of G satisfies $f(u) \neq f(v)$. (In other words, a k-coloring f of G is proper if and only if no two adjacent vertices share the same color.)

Prove that the number of proper k-colorings of G is $\chi_G(k)$.

[**Hint:** Show that $k^{\operatorname{conn}(V,F)}$ also counts certain k-colorings (I like to call them "F-improper colorings" – what could that mean?). Then, analyze how often (and with what signs) a given k-coloring of G appears in the sum $\sum_{F \subseteq E} (-1)^{|F|} k^{\operatorname{conn}(V,F)}$.

]

Note that most graph-theoretical literature defines the chromatic polynomial differently than I do in Exercise 4. Use the literature at your own peril! Most authors define χ_G as the polynomial whose value at each $k \in \mathbb{N}$ is the number of proper k-colorings. This may be more intuitive, but it leaves a question unanswered: Why is there such a polynomial in the first place? Exercise 4 answers this question.

Exercise 4 is [Grinbe16, Theorem 3.4]. However, the proof given in [Grinbe16] is a long detour, seeing that the purpose of [Grinbe16] is to generalize the result in several directions. We shall give a more direct proof that uses the same idea. (The idea goes back to Hassler Whitney in 1930 [Whitne32, §6], although he worded the argument differently and in far less modern language.)

We are going to use the Iverson bracket notation (as in [Grinbe17, §3.3]). We first recall an important result ([Grinbe17, Lemma 3.3.5]):

Lemma 0.5. Let P be a finite set. Then,

$$\sum_{A \subseteq P} (-1)^{|A|} = [P = \varnothing].$$

(The symbol "$\sum_{A \subseteq P}$" means "sum over all subsets A of P". In other words, it means "$\sum_{A \in \mathcal{P}(P)}$".)

Next, we introduce a specific notation related to colorings:

Definition 0.6. Let $G = (V, E)$ be a simple graph. Let $k \in \mathbb{N}$. Let $f : V \to \{1, 2, \ldots, k\}$ be a k-coloring. We let E_f denote the set of all edges $\{u, v\}$ of G satisfying $f(u) = f(v)$. (In other words, E_f is the set of all edges $\{u, v\}$ of G whose two endpoints u and v have the same color.) This set E_f is a subset of E.

Notice the following simple fact:

Proposition 0.7. Let $G = (V, E)$ be a simple graph. Let $k \in \mathbb{N}$. Let $f : V \to \{1, 2, \ldots, k\}$ be a k-coloring. Then, the k-coloring f is proper if and only if $E_f = \varnothing$.

Proof of Proposition 0.7. We have the following chain of equivalences:

$$\begin{aligned}
& (\text{the } k\text{-coloring } f \text{ is proper}) \\
\iff & (\text{each edge } \{u, v\} \text{ of } G \text{ satisfies } f(u) \neq f(v)) \\
& \quad (\text{by the definition of "proper"}) \\
\iff & (\text{no edge } \{u, v\} \text{ of } G \text{ satisfies } f(u) = f(v)) \\
\iff & \left(\underbrace{\text{the set of all edges } \{u, v\} \text{ of } G \text{ satisfying } f(u) = f(v)}_{\substack{= E_f \\ (\text{by the definition of } E_f)}} \text{ is empty} \right) \\
\iff & (E_f \text{ is empty}) \iff (E_f = \varnothing).
\end{aligned}$$

This proves Proposition 0.7. □

Lemma 0.8. Let $G = (V, E)$ be a simple graph. Let B be a subset of E. Then, the number of all k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $B \subseteq E_f$ is $k^{\operatorname{conn}(V, B)}$.

Proof of Lemma 0.8 (sketched). We shall say that two vertices u and v of a graph H are *connected* in H if these vertices u and v belong to the same connected component of H. (In other words, two vertices u and v of a graph H are connected in H if and only if there exists a walk from u to v in H.)

Fix any k-coloring $f : V \to \{1, 2, \ldots, k\}$. We are first going to restate the condition $B \subseteq E_f$ in more familiar terms. Indeed, we have the following chain of equivalences:[10]

$$\begin{aligned}
& \left(B \subseteq E_f\right) \\
\iff & \left(\text{each } \{u, v\} \in B \text{ satisfies } \{u, v\} \in E_f\right) \\
\iff & \left(\text{each } \{u, v\} \in B \text{ satisfies } f(u) = f(v)\right) \\
\iff & \left(\text{every two vertices } p \text{ and } q \text{ that are connected in } (V, B) \text{ satisfy } f(p) = f(q)\right) \\
\iff & \left(\begin{array}{c} \text{whenever } C \text{ is a connected component of the graph } (V, B), \\ \text{all the vertices in } C \text{ have the same color (in } f) \end{array}\right).
\end{aligned} \tag{4}$$

Here, the second and third equivalence signs hold for the following reasons:

- The second equivalence sign holds because for each given $\{u, v\} \in B$, we have the equivalence

  $$\begin{aligned}
  \left(\{u, v\} \in E_f\right) & \iff \left(\{u, v\} \text{ is an edge of } G \text{ satisfying } f(u) = f(v)\right) \\
  & \qquad \left(\text{by the definition of } E_f\right) \\
  & \iff \left(f(u) = f(v)\right)
  \end{aligned}$$

  (because $\{u, v\}$ is always an edge of G (since $\{u, v\} \in B \subseteq E$)).

- The third equivalence sign holds for the following reasons:
  - If each $\{u, v\} \in B$ satisfies $f(u) = f(v)$, then every two vertices p and q that are connected in (V, B) satisfy $f(p) = f(q)$ [11].
  - If every two vertices p and q that are connected in (V, B) satisfy $f(p) = f(q)$, then each $\{u, v\} \in B$ satisfies $f(u) = f(v)$ [12].

Now, forget that we fixed f. We thus have shown that for each k-coloring $f : V \to \{1, 2, \ldots, k\}$, the equivalence (4) holds. Therefore, a k-coloring $f : V \to \{1, 2, \ldots, k\}$

[10] See below for justifications for the equivalence signs.

[11] *Proof.* Assume that each $\{u, v\} \in B$ satisfies $f(u) = f(v)$. We must then show that every two vertices p and q that are connected in (V, B) satisfy $f(p) = f(q)$.

Fix any two vertices p and q that are connected in (V, B). Thus, there exists a walk from p to q in (V, B). Fix such a walk, and denote it by $(w_0, w_1, \ldots, w_k)$. Thus, $w_0 = p$ and $w_k = q$. For every $i \in \{1, 2, \ldots, k\}$, we have $\{w_{i-1}, w_i\} \in B$ (since the vertices w_{i-1} and w_i are consecutive vertices on the walk $(w_0, w_1, \ldots, w_k)$, and thus are adjacent in the graph (V, B)) and therefore $f(w_{i-1}) = f(w_i)$ (since each $\{u, v\} \in B$ satisfies $f(u) = f(v)$). In other words, $f(w_0) = f(w_1) = \cdots = f(w_k)$. Hence, $f(w_0) = f(w_k)$. Since $w_0 = p$ and $w_k = q$, this rewrites as $f(p) = f(q)$. Qed.

[12] *Proof.* Assume that every two vertices p and q that are connected in (V, B) satisfy $f(p) = f(q)$. We must then prove that each $\{u, v\} \in B$ satisfies $f(u) = f(v)$.

Fix any $\{u, v\} \in B$. Then, the vertices u and v are adjacent in the graph (V, B), and thus are connected in (V, B). Hence, $f(u) = f(v)$ (since every two vertices p and q that are connected in (V, B) satisfy $f(p) = f(q)$). Qed.

satisfies $B \subseteq E_f$ if and only if it has the property that whenever C is a connected component of the graph (V, B), all the vertices in C have the same color (in f). Therefore, all k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $B \subseteq E_f$ can be obtained by the following procedure:

- **For each** connected component C of the graph (V, B), pick a color c_C (i.e., an element c_C of $\{1, 2, \ldots, k\}$) and then color each vertex in C with this color c_C (i.e., set $f(v) = c_C$ for each $v \in C$).

[13] This procedure involves choices (because for each connected component C of (V, B), we get to pick a color), and there is a total of $k^{\operatorname{conn}(V,B)}$ possible choices that can be made (because we get to choose a color from $\{1, 2, \ldots, k\}$ for each of the $\operatorname{conn}(V, B)$ connected components of (V, B)). Each of these choices gives rise to a different k-coloring $f : V \to \{1, 2, \ldots, k\}$. Therefore, the number of all k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $B \subseteq E_f$ is $k^{\operatorname{conn}(V,B)}$ (because all of these k-colorings can be obtained by this procedure). This proves Lemma 0.8. $\square$

Corollary 0.9. Let (V, E) be a simple graph. Let F be a subset of E. Then,

$$k^{\operatorname{conn}(V,F)} = \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1.$$

Proof of Corollary 0.9 (sketched). We have

$$\begin{aligned}
\sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1 &= \left(\text{the number of all } f : V \to \{1, 2, \ldots, k\} \text{ satisfying } F \subseteq E_f\right) \cdot 1 \\
&= \left(\text{the number of all } f : V \to \{1, 2, \ldots, k\} \text{ satisfying } F \subseteq E_f\right) \\
&= k^{\operatorname{conn}(V,F)}
\end{aligned}$$

(because Lemma 0.8 (applied to $B = F$) shows that the number of all k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $F \subseteq E_f$ is $k^{\operatorname{conn}(V,F)}$). This proves Corollary 0.9. $\square$

Solution to Exercise 4 (sketched). Substituting k for x in the equality

$$\chi_G = \sum_{F\subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)},$$

[13]Let us restate this more rigorously: All k-colorings $f : V \to \{1, 2, \ldots, k\}$ satisfying $B \subseteq E_f$ can be obtained by the following procedure:

- **For each** connected component C of the graph (V, B),
  - pick any number $c_C \in \{1, 2, \ldots, k\}$;
  - set $f(v) = c_C$ for each $v \in C$.

we obtain

$$\begin{aligned}
\chi_G(k) &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{k^{\operatorname{conn}(V,F)}}_{\substack{=\sum\limits_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1 \\ \text{(by Corollary 0.9)}}} = \sum_{F\subseteq E} (-1)^{|F|} \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}} 1 \\
&= \underbrace{\sum_{F\subseteq E} \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ F\subseteq E_f}}}_{=\sum\limits_{f:V\to\{1,2,\ldots,k\}} \sum\limits_{\substack{F\subseteq E;\\ F\subseteq E_f}}} \underbrace{(-1)^{|F|} 1}_{=(-1)^{|F|}} = \sum_{f:V\to\{1,2,\ldots,k\}} \underbrace{\sum_{\substack{F\subseteq E;\\ F\subseteq E_f}}}_{\substack{=\sum\limits_{F\subseteq E_f} \\ \text{(since } E_f\subseteq E)}} (-1)^{|F|} \\
&= \sum_{f:V\to\{1,2,\ldots,k\}} \sum_{F\subseteq E_f} (-1)^{|F|} = \sum_{f:V\to\{1,2,\ldots,k\}} \underbrace{\sum_{A\subseteq E_f} (-1)^{|A|}}_{\substack{=[E_f=\varnothing] \\ \text{(by Lemma 0.5,} \\ \text{applied to } P=E_f)}} \\
&\qquad \left(\begin{array}{c}\text{here, we have renamed the summation index } F \\ \text{in the second sum as } A\end{array}\right) \\
&= \sum_{f:V\to\{1,2,\ldots,k\}} [E_f=\varnothing] \\
&= \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ E_f=\varnothing}} \underbrace{[E_f=\varnothing]}_{\substack{=1 \\ \text{(since } E_f=\varnothing \text{ is true)}}} + \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ \text{not } E_f=\varnothing}} \underbrace{[E_f=\varnothing]}_{\substack{=0 \\ \text{(since } E_f=\varnothing \text{ is false)}}} \\
&\qquad \left(\begin{array}{c}\text{since each } f:V\to\{1,2,\ldots,k\} \text{ satisfies either } E_f=\varnothing \\ \text{or } E_f\neq\varnothing \text{ (but not both)}\end{array}\right) \\
&= \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ E_f=\varnothing}} 1 + \underbrace{\sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ \text{not } E_f=\varnothing}} 0}_{=0} = \sum_{\substack{f:V\to\{1,2,\ldots,k\};\\ E_f=\varnothing}} 1 \\
&= \left(\text{the number of all } f:V\to\{1,2,\ldots,k\} \text{ such that } E_f=\varnothing\right)\cdot 1 \\
&= \left(\text{the number of all } f:V\to\{1,2,\ldots,k\} \text{ such that } \underbrace{E_f=\varnothing}_{\substack{\iff \text{(the } k\text{-coloring } f \text{ is proper)} \\ \text{(by Proposition 0.7)}}}\right) \\
&= (\text{the number of all } f:V\to\{1,2,\ldots,k\} \text{ such that the } k\text{-coloring } f \text{ is proper}) \\
&= (\text{the number of all proper } k\text{-colorings}).
\end{aligned}$$

In other words, the number of proper k-colorings of G is $\chi_G(k)$. This solves Exercise 4. □

0.6. Exercise 5: some concrete chromatic polynomials

Exercise 5. In Exercise 4, we have defined the chromatic polynomial χ_G of a simple graph G. In this exercise, we shall compute it on some examples.

(a) For each $n \in \mathbb{N}$, prove that the complete graph K_n has chromatic polynomial $\chi_{K_n} = x(x-1)\cdots(x-n+1)$.

(b) Let T be a tree (regarded as a simple graph). Let $n = |\operatorname{V}(T)|$. Prove that $\chi_T = x(x-1)^{n-1}$.

(c) Find the chromatic polynomial χ_{P_3} of the path graph P_3.

Before we start solving Exercise 5, let us make a general remark about it. In order to prove a formula for the chromatic polynomial χ_G of a graph G, at least two approaches are available: One is to use the definition of χ_G; another is to use the claim of Exercise 4. In order to use the second approach, one needs to know that a polynomial p is uniquely determined by its values at infinitely many points. In other words, one needs to know the following fact:

Lemma 0.10. Let p and q be two polynomials in one variable x with rational coefficients. If $p(k) = q(k)$ holds for infinitely many rational numbers k, then we have $p = q$.

Lemma 0.10 is easy to derive from the following well-known fact:

Lemma 0.11. Let p be a polynomial in one variable x with rational coefficients. If $p(k) = 0$ holds for infinitely many rational numbers k, then we have $p = 0$.

Proof of Lemma 0.11. Assume that $p(k) = 0$ holds for infinitely many rational numbers k. In other words, the polynomial p has infinitely many rational roots.

It is known[14] that any nonzero polynomial (in one variable x) over a field[15] has finitely many roots. Hence, if the polynomial p were nonzero, then p would have finitely many roots, which would contradict the fact that p has infinitely many roots. Hence, p cannot be nonzero. In other words, we have $p = 0$. This proves Lemma 0.11. □

Proof of Lemma 0.10. Assume that $p(k) = q(k)$ holds for infinitely many rational numbers k. Thus, for infinitely many rational numbers k, we have $(p-q)(k) = \underbrace{p(k)}_{=q(k)} - q(k) = q(k) - q(k) = 0$. Hence, Lemma 0.11 (applied to $p-q$ instead of p) shows that $p - q = 0$. In other words, $p = q$. This proves Lemma 0.10. □

Now, we can attack Exercise 5 **(a)**:

[14] see, e.g., `https://proofwiki.org/wiki/Polynomial_over_Field_has_Finitely_Many_Roots`

[15] This includes polynomials with rational coefficients, but also polynomials with real or complex coefficients.

Lemma 0.12. Let $n \in \mathbb{N}$. Then, the complete graph K_n has chromatic polynomial $\chi_{K_n} = x(x-1)\cdots(x-n+1)$.

Proof of Lemma 0.12. Fix an integer $k \geq n$. Recall the concept of k-colorings defined in Exercise 4 (applied to $G = K_n$).

Exercise 4 (applied to $G = K_n$) shows that the number of proper k-colorings of K_n is $\chi_{K_n}(k)$. In other words:

$$(\text{the number of proper } k\text{-colorings of } K_n) = \chi_{K_n}(k). \tag{5}$$

Now, let us compute this number in a different way. Namely, recall that the complete graph K_n has n vertices $1, 2, \ldots, n$, and that any two distinct vertices of K_n are adjacent. Thus, a k-coloring of K_n is proper if and only if no two distinct vertices of K_n have the same color. Hence, we obtain the following procedure for constructing a proper k-coloring of K_n:

- First, choose a color for the vertex 1. This color must belong to the set $\{1, 2, \ldots, k\}$.
- Next, choose a color for the vertex 2. This color must belong to the set $\{1, 2, \ldots, k\}$, and must be distinct from the color chosen for the vertex 1 (since no two distinct vertices of K_n may have the same color).
- Next, choose a color for the vertex 3. This color must belong to the set $\{1, 2, \ldots, k\}$, and must be distinct from the two colors chosen for the vertices 1 and 2 (since no two distinct vertices of K_n may have the same color).
- Next, choose a color for the vertex 4. This color must belong to the set $\{1, 2, \ldots, k\}$, and must be distinct from the three colors chosen for the vertices 1, 2 and 3 (since no two distinct vertices of K_n may have the same color).
- And so on. Keep going until all n vertices $1, 2, \ldots, n$ have been assigned colors.

This procedure clearly produces each proper k-coloring of K_n. Furthermore, this procedure involves choices, and there is a total of $k(k-1)(k-2)\cdots(k-n+1)$ possible choices that can be made[16]. Each of these choices produces a different proper k-coloring of K_n. Thus, the number of proper k-colorings of K_n is exactly

[16] *Proof.* In our procedure, we first choose a color for the vertex 1, then choose a color for the vertex 2, then choose a color for the vertex 3, and so on. In other words, for each $i \in \{1, 2, \ldots, n\}$, we choose a color for the vertex i after having chosen colors for the vertices $1, 2, \ldots, i-1$. The color we choose for a given vertex i must belong to the k-element set $\{1, 2, \ldots, k\}$, but must be distinct from the $i-1$ colors already chosen for the vertices $1, 2, \ldots, i-1$; thus, the number of ways in which we can choose this color is $k-(i-1)$ (because the $i-1$ colors already chosen for the vertices $1, 2, \ldots, i-1$ are distinct (because we have chosen the color for each vertex to be distinct from all the colors chosen before)).

Thus, altogether, for each $i \in \{1, 2, \ldots, n\}$, we have to choose a color for the vertex i, and there

$k(k-1)(k-2)\cdots(k-n+1)$ (because all proper k-colorings of K_n can be obtained by this procedure). In other words,

$$(\text{the number of proper } k\text{-colorings of } K_n) = k(k-1)(k-2)\cdots(k-n+1).$$

Comparing this with (5), we obtain

$$\chi_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1). \tag{6}$$

Now, forget that we fixed k. We thus have shown that (6) holds for every integer $k \geq n$. Thus, (6) holds for infinitely many rational numbers k. Hence, Lemma 0.10 (applied to $p = \chi_{K_n}$ and $q = x(x-1)(x-2)\cdots(x-n+1)$) shows that $\chi_{K_n} = x(x-1)(x-2)\cdots(x-n+1)$. This proves Lemma 0.12. □

Next, we recall a classical fact:

Proposition 0.13. Let T be a tree such that $|\operatorname{V}(T)| \geq 2$. Let v be a leaf of T. Let T' denote the multigraph obtained from T by removing this leaf v and the unique edge that contains v.

Then, the multigraph T' is a tree again.

Next, let us deal with Exercise 5 **(b)**.

Lemma 0.14. Let T be a tree (regarded as a simple graph). Let $n = |\operatorname{V}(T)|$. Then, $\chi_T = x(x-1)^{n-1}$.

Proof of Lemma 0.14. We shall prove Lemma 0.14 by induction on n.

The *induction base* (the case when $n = 1$) is simple and is left to the reader.

Induction step: Fix a positive integer $N > 1$. Assume that Lemma 0.14 has been proven in the case when $n = N - 1$. We must now prove Lemma 0.14 in the case when $n = N$.

We have assumed that Lemma 0.14 has been proven in the case when $n = N - 1$. Thus, the following fact holds:

Fact 1: Let T be a tree (regarded as a simple graph) such that $N - 1 = |\operatorname{V}(T)|$. Then, $\chi_T = x(x-1)^{(N-1)-1}$.

Now, let T be a tree (regarded as a simple graph) such that $N = |\operatorname{V}(T)|$. We shall show that $\chi_T = x(x-1)^{N-1}$.

Indeed, $|\operatorname{V}(T)| = N > 1$, so that $|\operatorname{V}(T)| \geq 2$. Hence, the tree T has at least 2 vertices. Thus, the tree T has a leaf (since each tree that has at least 2 vertices must have a leaf). Fix such a leaf, and denote it by v. Let u be the unique neighbor of

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are $k-(i-1)$ ways to choose this color. Therefore, the total number of possible choices is

$$(k-(1-1))(k-(2-1))\cdots(k-(n-1)) = k(k-1)(k-2)\cdots(k-n+1).$$

Qed.

v. Hence, the vertices u and v are adjacent, and u is the only vertex of T that is adjacent to v.

Consider the multigraph T' defined as in Proposition 0.13. Then, Proposition 0.13 shows that T' is a tree again. Regard T' as a simple graph.

The multigraph T' is obtained from T by removing a single vertex and a single edge. Hence, $|\mathrm{V}(T')| = \underbrace{|\mathrm{V}(T)|}_{=N} - 1 = N - 1$, so that $N - 1 = |\mathrm{V}(T')|$. Hence, Fact 1 (applied to T' instead of T) shows that $\chi_{T'} = x(x-1)^{(N-1)-1}$.

Fix an integer $k \geq 1$. Recall the concept of k-colorings defined in Exercise 4 (applied to $G = T$, and to $G = T'$).

Exercise 4 (applied to $G = T'$) shows that the number of proper k-colorings of T' is $\chi_{T'}(k)$.

Exercise 4 (applied to $G = T$) shows that the number of proper k-colorings of T is $\chi_T(k)$. In other words:

$$(\text{the number of proper } k\text{-colorings of } T) = \chi_T(k). \tag{7}$$

Now, let us compute this number in a different way. Namely, observe that a k-coloring of T differs from a k-coloring of T' only in that the former assigns a color to the vertex v whereas the latter does not. Furthermore, a k-coloring of T is proper if and only if no two adjacent vertices of T have the same color. This condition implies that the color assigned to v must be distinct from the color assigned to u (since v and u are adjacent). No other restrictions apply to the color assigned to v (since u is the only vertex of T that is adjacent to v).

Hence, we obtain the following procedure for constructing a proper k-coloring of T:

- First, choose colors for the vertices of T distinct from v. These colors must belong to the set $\{1, 2, \ldots, k\}$, and must have the property that no two adjacent vertices have the same color. In other words, these colors must form a proper k-coloring of the tree T' (because two vertices of T distinct from v are adjacent in T if and only if they are adjacent in T').
- Next, choose a color for the vertex v. This color must belong to the set $\{1, 2, \ldots, k\}$, and must be distinct from the color chosen for the vertex u.

This procedure clearly produces each proper k-coloring of T. Furthermore, this procedure involves two choices, and there is a total of $\chi_{T'}(k) \cdot (k-1)$ possible choices that can be made[17]. Each of these choices produces a different proper k-coloring of T. Thus, the number of proper k-colorings of T is exactly $\chi_{T'}(k) \cdot (k-1)$

[17] *Proof.* In our procedure, we first choose colors for the vertices distinct from v, and then choose a color for the vertex v.

The colors that we choose for the vertices distinct from v must form a proper k-coloring of the tree T'. Hence, the number of ways in which we can choose these colors is the number of proper k-colorings of T'; but as we know, the latter number is $\chi_{T'}(k)$. Hence, the number of ways in which we can choose the colors for the vertices distinct from v is $\chi_{T'}(k)$.

(because all proper k-colorings of T can be obtained by this procedure). In other words,

$$(\text{the number of proper } k\text{-colorings of } T) = \chi_{T'}(k) \cdot (k-1).$$

Comparing this with (7), we obtain

$$\chi_T(k) = \chi_{T'}(k) \cdot (k-1) \tag{8}$$

Now, forget that we fixed k. We thus have shown that (8) holds for every integer $k \geq 1$. Thus, (8) holds for infinitely many rational numbers k. Hence, Lemma 0.10 (applied to $p = \chi_T$ and $q = \chi_{T'} \cdot (x-1)$) shows that $\chi_T = \chi_{T'} \cdot (x-1)$. Hence,

$$\chi_T = \underbrace{\chi_{T'}}_{=x(x-1)^{(N-1)-1}} \cdot (x-1) = x \underbrace{(x-1)^{(N-1)-1} \cdot (x-1)}_{=(x-1)^{N-1}} = x(x-1)^{N-1}.$$

Now, forget that we fixed T. We thus have shown that if T is a tree (regarded as a simple graph) satisfying $N = |\operatorname{V}(T)|$, then $\chi_T = x(x-1)^{N-1}$. In other words, Lemma 0.14 holds for $n = N$. This completes the induction step. Thus, Lemma 0.14 is proven by induction. □

Now, we have essentially solved Exercise 5:

Solution to Exercise 5. **(a)** Exercise 5 **(a)** follows from Lemma 0.12.

(b) Exercise 5 **(b)** follows from Lemma 0.14.

(c) The graph P_3 is a tree with $|\operatorname{V}(P_3)| = 3$. Thus, Lemma 0.14 (applied to $T = P_3$ and $n = 3$) shows that $\chi_T = x(x-1)^{3-1} = x(x-1)^2 = x^3 - 2x^2 + x$. □

0.7. Exercise 6: the distances between four points on a tree

Exercise 6. Let G be a tree. Let x, y, z and w be four vertices of G.

Show that the two largest ones among the three numbers $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ are equal.

Before solving this exercise, let us state some facts about distances in multigraphs:

Lemma 0.15. Let u and v be two vertices of a connected multigraph $G = (V, E, \phi)$. Then, $d(u,v) \leq |V| - 1$.

The color that we choose for the vertex v must belong to the k-element set $\{1, 2, \ldots, k\}$, and must be distinct from the color chosen for the vertex u. Thus, the number of ways in which we can choose this color is $k-1$.

Hence, the first of our two choices can be done in $\chi_{T'}(k)$ different ways, whereas the second choice can be done in $k-1$ different ways. Therefore, the total number of possible choices is $\chi_{T'}(k) \cdot (k-1)$. Qed.

Lemma 0.16. Let u and v be two vertices of a multigraph G. Let $k \in \mathbb{N}$. If there exists a walk from u to v in G having length k, then $d(u,v) \leq k$.

Lemma 0.17. Let $G = (V, E, \phi)$ be a multigraph.
(a) Each $u \in V$ satisfies $d(u,u) = 0$.
(b) Each $u \in V$ and $v \in V$ satisfy $d(u,v) = d(v,u)$.
(c) Each $u \in V$, $v \in V$ and $w \in V$ satisfy $d(u,v) + d(v,w) \geq d(u,w)$. (This inequality has to be interpreted appropriately when one of the distances is infinite: For example, we understand ∞ to be greater than any integer, and we understand $\infty + m$ to be ∞ whenever $m \in \mathbb{Z}$.)
(d) If $u \in V$ and $v \in V$ satisfy $d(u,v) = 0$, then $u = v$.

Lemma 0.15, Lemma 0.16 and Lemma 0.17 are analogues of three lemmas encountered in the solutions to midterm #1 (namely, Lemma 0.1, Lemma 0.2 and Lemma 0.3 in the latter solutions). More precisely, the former three lemmas differ from the latter three lemmas only in that the simple graph has been replaced by a multigraph. Proofs of the former three lemmas can be obtained from proofs of the latter three lemmas by making straightforward minor modifications[18]. We leave the details of these modifications to the reader.

Let us further state a basic property of trees:

Lemma 0.18. Let u and v be two vertices of a tree G. Let $k \in \mathbb{N}$. If there exists a path from u to v in G having length k, then $d(u,v) = k$.

Proof of Lemma 0.18. It is known that the multigraph G is a tree if and only if for every two vertices x and y of G, there is a unique path from x to y in G [19]. Hence, for every two vertices x and y of G, there is a unique path from x to y in G (since the multigraph G is a tree). Applying this to $x = u$ and $y = v$, we conclude that there is a unique path from u to v in G. Hence, any two paths from u to v must be equal.

We have assumed that there exists a path from u to v in G having length k. Fix such a path, and denote it by $\mathbf{p}$. Thus, (the length of $\mathbf{p}$) $= k$.

We know that there exists a path from u to v. Hence, $d(u,v)$ is the minimum length of a path from u to v (by the definition of $d(u,v)$). Thus, there exists a path from u to v having length $d(u,v)$. Fix such a path, and denote it by $\mathbf{q}$. Hence, (the length of $\mathbf{q}$) $= d(u,v)$.

Now, both $\mathbf{p}$ and $\mathbf{q}$ are paths from u to v. Hence, $\mathbf{p}$ and $\mathbf{q}$ are equal (since any two paths from u to v must be equal). In other words, $\mathbf{p} = \mathbf{q}$. Hence, (the length of $\mathbf{p}$) $=$ (the length of $\mathbf{q}$) $= d(u,v)$. Comparing this with (the length of $\mathbf{p}$) $= k$, we obtain $d(u,v) = k$. This proves Lemma 0.18. □

Next, let us show a further useful lemma:

[18] The most important modification is to include the edges in the paths.
[19] This is the equivalence $\mathcal{T}_1 \iff \mathcal{T}_2$ in Theorem 13 in Lecture 9.

Lemma 0.19. Let G be a connected multigraph. Let x, y and z be three vertices of G.

Let $\left(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g\right)$ be a path from x to y.

Let i be an element of $\{0, 1, \ldots, g\}$ minimizing the distance $d(z, p_i)$.

Let $h = d(z, p_i)$.

Then:

(a) There exists a path from x to z having length $i + h$.

(b) There exists a path from z to y having length $g - i + h$.

(c) If $j \in \{0, 1, \ldots, g\}$ is such that $i \leq j$, then there exists a path from z to p_j having length $j - i + h$.

Proof of Lemma 0.19 (sketched). We have $h = d(z, p_i)$. In other words, h is the minimum length of a path from z to p_i (since $d(z, p_i)$ is defined as the minimum length of a path from z to p_i). Thus, there exists a path from z to p_i having length h. Fix such a path, and denote it by $(a_0, f_1, a_1, f_2, a_2, \ldots, f_h, a_h)$. Thus, $a_0 = z$ and $a_h = p_i$.

Recall that $\left(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g\right)$ is a path from x to y. Thus, $p_0 = x$ and $p_g = y$.

The element $i \in \{0, 1, \ldots, g\}$ minimizes the distance $d(z, p_i)$. Hence,

$$d\left(z, p_j\right) \geq d(z, p_i) \qquad \text{for each } j \in \{0, 1, \ldots, g\}. \tag{9}$$

The $g + 1$ vertices $p_0, p_1, \ldots, p_g$ are distinct (since $\left(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g\right)$ is a path).

The $h + 1$ vertices $a_0, a_1, \ldots, a_h$ are distinct (since $(a_0, f_1, a_1, f_2, a_2, \ldots, f_h, a_h)$ is a path). Thus, in particular, the h vertices $a_0, a_1, \ldots, a_{h-1}$ are distinct. In other words, the vertices $a_{h-1}, a_{h-2}, \ldots, a_0$ are distinct.

We have

$$\{a_0, a_1, \ldots, a_{h-1}\} \cap \left\{p_0, p_1, \ldots, p_g\right\} = \varnothing \tag{10}$$

[20].

Recall that $(a_0, f_1, a_1, f_2, a_2, \ldots, f_h, a_h)$ is a path, and thus is a walk. Hence, $(a_h, f_h, a_{h-1}, f_{h-1}, a_{h-2}, \ldots, f_1, a_0)$ is a walk as well (being the reversal of the walk

[20] *Proof of* (10)*:* Let $v \in \{a_0, a_1, \ldots, a_{h-1}\} \cap \left\{p_0, p_1, \ldots, p_g\right\}$. We shall derive a contradiction.

We have $v \in \{a_0, a_1, \ldots, a_{h-1}\} \cap \left\{p_0, p_1, \ldots, p_g\right\} \subseteq \{a_0, a_1, \ldots, a_{h-1}\}$. Hence, $v = a_k$ for some $k \in \{0, 1, \ldots, h-1\}$. Consider this k.

We have $v \in \{a_0, a_1, \ldots, a_{h-1}\} \cap \left\{p_0, p_1, \ldots, p_g\right\} \subseteq \left\{p_0, p_1, \ldots, p_g\right\}$. Hence, $v = p_j$ for some $j \in \{0, 1, \ldots, g\}$. Consider this j.

Recall that $(a_0, f_1, a_1, f_2, a_2, \ldots, f_h, a_h)$ is a path, and thus is a walk. Hence, $(a_0, f_1, a_1, f_2, a_2, \ldots, f_k, a_k)$ is a walk as well. This walk $(a_0, f_1, a_1, f_2, a_2, \ldots, f_k, a_k)$ is a walk from z to v (since $a_0 = z$ and $a_k = v$) and has length p. Hence, there is a walk from z to v in G having length k (namely, the walk $(a_0, f_1, a_1, f_2, a_2, \ldots, f_k, a_k)$). Consequently, Lemma 0.16 (applied to $u = z$) shows that $d(z, v) \leq k \leq h - 1$ (since $k \in \{0, 1, \ldots, h-1\}$).

But (9) yields $d(z, p_i) \leq d\left(z, p_j\right) = d(z, v)$ (since $p_j = v$). Thus, $d(z, v) \geq d(z, p_i) = h > h - 1$. This contradicts $d(z, v) \leq h - 1$.

Now, forget that we fixed v. Thus, we have obtained a contradiction for each $v \in \{a_0, a_1, \ldots, a_{h-1}\} \cap \left\{p_0, p_1, \ldots, p_g\right\}$. Hence, there exists no $v \in \{a_0, a_1, \ldots, a_{h-1}\} \cap \left\{p_0, p_1, \ldots, p_g\right\}$. Thus, $\{a_0, a_1, \ldots, a_{h-1}\} \cap \left\{p_0, p_1, \ldots, p_g\right\} = \varnothing$.

$(a_0, f_1, a_1, f_2, a_2, \ldots, f_h, a_h))$. This walk is a walk from p_i to z (since $a_h = p_i$ and $a_0 = z$).

(a) Recall that $(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g)$ is a path, and thus is a walk. Hence, $(p_0, e_1, p_1, e_2, p_2, \ldots, e_i, p_i)$ is a walk as well. This walk is a walk from x to p_i (since $p_0 = x$ and $p_i = p_i$).

We have

$$\{p_0, p_1, \ldots, p_i\} \cap \{a_{h-1}, a_{h-2}, \ldots, a_0\} = \varnothing \tag{11}$$

[21]. The vertices $p_0, p_1, \ldots, p_i, a_{h-1}, a_{h-2}, \ldots, a_0$ are distinct[22].

Now, we know that $(p_0, e_1, p_1, e_2, p_2, \ldots, e_i, p_i)$ is a walk from x to p_i, whereas $(a_h, f_h, a_{h-1}, f_{h-1}, a_{h-2}, \ldots, f_1, a_0)$ is a walk from p_i to z. Since the ending point of the former walk is the starting point of the latter walk[23], we can combine these two walks. We thus obtain a new walk $(p_0, e_1, p_1, e_2, p_2, \ldots, e_i, p_i, f_h, a_{h-1}, f_{h-1}, a_{h-2}, \ldots, f_1, a_0)$, which is a walk from x to z and has length $i + h$. Furthermore, this new walk is actually a path (since the vertices $p_0, p_1, \ldots, p_i, a_{h-1}, a_{h-2}, \ldots, a_0$ are distinct), and therefore is a path from x to z having length $i + h$. Hence, there exists a path from x to z having length $i + h$ (namely, the path that we have just constructed). This proves Lemma 0.19 **(a)**.

(c) Let $j \in \{0, 1, \ldots, g\}$ be such that $i \leq j$. Thus, $j \in \{i, i+1, \ldots, g\}$ (since $j \in \{0, 1, \ldots, g\}$ and $j \geq i$).

Recall that $(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g)$ is a path, and thus is a walk. Hence, $(p_i, e_{i+1}, p_{i+1}, e_{i+2}, p_{i+2}, \ldots, e_g, p_g)$ is a walk as well. Thus, $(p_i, e_{i+1}, p_{i+1}, e_{i+2}, p_{i+2}, \ldots, e_j, p_j)$ is a walk as well (since $j \in \{i, i+1, \ldots, g\}$). This walk is a walk from p_i to p_j (since $p_i = p_i$ and $p_j = p_j$).

We have

$$\{a_0, a_1, \ldots, a_{h-1}\} \cap \{p_i, p_{i+1}, \ldots, p_j\} = \varnothing \tag{12}$$

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[21] *Proof of* (11)*:* We have

$$\underbrace{\{p_0, p_1, \ldots, p_i\}}_{\subseteq \{p_0, p_1, \ldots, p_g\}} \cap \underbrace{\{a_{h-1}, a_{h-2}, \ldots, a_0\}}_{= \{a_0, a_1, \ldots, a_{h-1}\}} \subseteq \{p_0, p_1, \ldots, p_g\} \cap \{a_0, a_1, \ldots, a_{h-1}\}$$
$$= \{a_0, a_1, \ldots, a_{h-1}\} \cap \{p_0, p_1, \ldots, p_g\} = \varnothing$$

(by (10)). Thus, $\{p_0, p_1, \ldots, p_i\} \cap \{a_{h-1}, a_{h-2}, \ldots, a_0\} = \varnothing$.

[22] *Proof.* This follows from the following three observations:

- The vertices $p_0, p_1, \ldots, p_i$ are distinct (since the vertices $p_0, p_1, \ldots, p_g$ are distinct).
- The vertices $a_{h-1}, a_{h-2}, \ldots, a_0$ are distinct.
- The vertices $p_0, p_1, \ldots, p_i$ are distinct from the vertices $a_{h-1}, a_{h-2}, \ldots, a_0$ (because of (11)).

[23] Indeed, the ending point is the former walk is p_i, while the starting point of the latter walk is p_i as well.

[24]. The vertices $a_0, a_1, \ldots, a_{h-1}, p_i, p_{i+1}, \ldots, p_j$ are distinct[25].

Now, the path $(a_0, f_1, a_1, f_2, a_2, \ldots, f_h, a_h)$ is a walk from z to p_i (since it is a path from z to p_i), whereas $(p_i, e_{i+1}, p_{i+1}, e_{i+2}, p_{i+2}, \ldots, e_j, p_j)$ is a walk from p_i to p_j. Since the ending point of the former walk is the starting point of the latter walk[26], we can combine these two walks. We thus obtain a new walk $(a_0, f_1, a_1, f_2, a_2, \ldots, f_{h-1}, a_{h-1}, f_h, p_i, e_{i+1}, p_{i+1}, e_{i+2}, p_{i+2}, \ldots, e_j, p_j)$, which is a walk from z to p_j and has length $h + (j - i)$. Furthermore, this new walk is actually a path (since the vertices $a_0, a_1, \ldots, a_{h-1}, p_i, p_{i+1}, \ldots, p_j$ are distinct), and therefore is a path from z to p_j having length $h + (j - i)$. Hence, there exists a path from z to p_j having length $h + (j - i)$ (namely, the path that we have just constructed). In other words, there exists a path from z to p_j having length $j - i + h$ (since $h + (j - i) = j - i + h$). This proves Lemma 0.19 **(c)**.

(b) We have $g \in \{0, 1, \ldots, g\}$ (since $g \in \mathbb{N}$) and $i \leq g$ (since $i \in \{0, 1, \ldots, g\}$). Thus, Lemma 0.19 **(c)** (applied to $j = g$) shows that there exists a path from z to p_g having length $g - i + h$. Since $p_g = y$, this rewrites as follows: There exists a path from z to y having length $g - i + h$. This proves Lemma 0.19 **(b)**. □

Corollary 0.20. Let G be a tree. Let x, y and z be three vertices of G.

Let $(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g)$ be a path from x to y.

Let i be an element of $\{0, 1, \ldots, g\}$ minimizing the distance $d(z, p_i)$.

Let $h = d(z, p_i)$.

Then:

(a) We have $d(x, z) = i + h$.

(b) We have $d(y, z) = g - i + h$.

(c) If $j \in \{0, 1, \ldots, g\}$ is such that $i \leq j$, then $d(z, p_j) = j - i + h$.

Proof of Corollary 0.20. **(a)** Lemma 0.19 **(a)** shows that there exists a path from x to z having length $i + h$. Hence, Lemma 0.18 (applied to $u = x$, $v = z$ and $k = i + h$) shows that $d(x, z) = i + h$. This proves Corollary 0.20 **(a)**.

(b) Lemma 0.19 **(b)** shows that there exists a path from z to y having length $g - i + h$. Hence, Lemma 0.18 (applied to $u = z$, $v = y$ and $k = g - i + h$) shows

[24]*Proof of* (12)*:* We have

$$\{a_0, a_1, \ldots, a_{h-1}\} \cap \underbrace{\{p_i, p_{i+1}, \ldots, p_j\}}_{\subseteq \{p_0, p_1, \ldots, p_g\}} \subseteq \{a_0, a_1, \ldots, a_{h-1}\} \cap \{p_0, p_1, \ldots, p_g\} = \varnothing$$

(by (10)). Thus, $\{a_0, a_1, \ldots, a_{h-1}\} \cap \{p_i, p_{i+1}, \ldots, p_j\} = \varnothing$.

[25]*Proof.* This follows from the following three observations:

- The vertices $a_0, a_1, \ldots, a_{h-1}$ are distinct.
- The vertices $p_i, p_{i+1}, \ldots, p_j$ are distinct (since the vertices $p_0, p_1, \ldots, p_g$ are distinct).
- The vertices $a_0, a_1, \ldots, a_{h-1}$ are distinct from the vertices $p_i, p_{i+1}, \ldots, p_j$ (because of (12)).

[26]Indeed, the ending point is the former walk is p_i, while the starting point of the latter walk is p_i as well.

that $d(z,y) = g - i + h$.

Write the tree G in the form $G = (V, E, \phi)$. Then, y and z are elements of V (since y and z are vertices of G). Thus, Lemma 0.17 **(b)** (applied to $u = y$ and $v = z$) yields $d(y,z) = d(z,y) = g - i + h$. This proves Corollary 0.20 **(b)**.

(c) Let $j \in \{0, 1, \ldots, g\}$ be such that $i \leq j$. Lemma 0.19 **(c)** shows that there exists a path from z to p_j having length $j - i + h$. Hence, Lemma 0.18 (applied to $u = z$, $v = p_j$ and $k = j - i + h$) shows that $d(z, p_j) = j - i + h$. This proves Corollary 0.20 **(c)**. $\square$

Proposition 0.21. Let G be a tree. Let x, y, z and w be four vertices of G.

Let $(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g)$ be a path from x to y.

Let i be an element of $\{0, 1, \ldots, g\}$ minimizing the distance $d(z, p_i)$.

Let j be an element of $\{0, 1, \ldots, g\}$ minimizing the distance $d(w, p_j)$.

(a) If $i \leq j$, then $d(x,w) + d(y,z) \geq d(x,y) + d(z,w)$.

(b) If $i \geq j$, then $d(x,z) + d(y,w) \geq d(x,y) + d(z,w)$.

Proof of Proposition 0.21. **(a)** Assume that $i \leq j$.

Let $h = d(z, p_i)$. Let $k = d(w, p_j)$.

The path $(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g)$ is a path from x to y, and has length g. Hence, there exists a path from u to v in G having length g (namely, the path $(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g)$). Thus, Lemma 0.18 (applied to x, y and g instead of u, v and k) yields $d(x,y) = g$.

Corollary 0.20 **(a)** (applied to w, j and k instead of z, i and h) yields $d(x,w) = j + k$.

Corollary 0.20 **(b)** yields $d(y,z) = g - i + h$.

Corollary 0.20 **(c)** yields $d(z, p_j) = j - i + h$.

Also, $d(z,w) \leq d(z, p_j) + d(w, p_j)$ [27]. Thus, $d(z,w) \leq \underbrace{d(z, p_j)}_{= j - i + h} + \underbrace{d(w, p_j)}_{= k} = j - i + h + k$.

Now,

$$\underbrace{d(x,y)}_{= g} + \underbrace{d(z,w)}_{= j - i + h - k} \leq g + j - i + h - k = \underbrace{j + k}_{= d(x,w)} + \underbrace{g - i + h}_{= d(y,z)} = d(x,w) + d(y,z).$$

This proves Proposition 0.21 **(a)**.

(b) Assume that $i \geq j$. Thus, $j \leq i$. Hence, Proposition 0.21 **(a)** (applied to w, z, j and i instead of z, w, i and j) yields $d(x,z) + d(y,w) \geq d(x,y) + d(w,z)$. But

[27] *Proof.* Write the tree G in the form $G = (V, E, \phi)$. Then, z, w and p_j are elements of V (since z, w and p_j are vertices of G). Thus, Lemma 0.17 **(b)** (applied to $u = w$ and $v = p_j$) yields $d(w, p_j) = d(p_j, w)$. But Lemma 0.17 **(c)** (applied to $u = z$ and $v = p_j$) yields $d(z, p_j) + d(p_j, w) \geq d(z,w)$. Hence, $d(z,w) \leq d(z, p_j) + \underbrace{d(p_j, w)}_{= d(w, p_j)} = d(z, p_j) + d(w, p_j)$.

$d(w,z) = d(z,w)$ [28]. Hence,

$$d(x,z) + d(y,w) \geq d(x,y) + \underbrace{d(w,z)}_{=d(z,w)} = d(x,y) + d(z,w).$$

This proves Proposition 0.21 **(b)**. □

Corollary 0.22. Let G be a tree. Let x, y, z and w be four vertices of G.

(a) We have

$$d(x,y) + d(z,w) \leq \max\{d(x,z) + d(y,w), d(x,w) + d(y,z)\}. \tag{13}$$

(b) We have

$$d(x,z) + d(y,w) \leq \max\{d(x,y) + d(z,w), d(x,w) + d(y,z)\}. \tag{14}$$

(c) We have

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\}. \tag{15}$$

Proof of Corollary 0.22. We have $d(w,z) = d(z,w)$ [29]. Similarly, $d(y,w) = d(w,y)$ and $d(y,z) = d(z,y)$.

(a) Assume the contrary. Thus,

$$d(x,y) + d(z,w) > \max\{d(x,z) + d(y,w), d(x,w) + d(y,z)\}.$$

Hence,

$$d(x,y) + d(z,w) > \max\{d(x,z) + d(y,w), d(x,w) + d(y,z)\} \geq d(x,z) + d(y,w) \tag{16}$$

and

$$d(x,y) + d(z,w) > \max\{d(x,z) + d(y,w), d(x,w) + d(y,z)\} \geq d(x,w) + d(y,z). \tag{17}$$

The multigraph G is a tree, and thus is connected. Hence, there exists a walk from x to y. Thus, there exists a path from x to y. Fix such a path, and denote it by $(p_0, e_1, p_1, e_2, p_2, \ldots, e_g, p_g)$.

Fix an element i of $\{0, 1, \ldots, g\}$ minimizing the distance $d(z, p_i)$. Fix an element j of $\{0, 1, \ldots, g\}$ minimizing the distance $d(w, p_j)$.

We are now in one of the following two cases:

- *Case 1:* We have $i \leq j$.

[28] *Proof.* Write the tree G in the form $G = (V, E, \phi)$. Then, w and z are elements of V (since w and z are vertices of G). Thus, Lemma 0.17 **(b)** (applied to $u = w$ and $v = z$) yields $d(w,z) = d(z,w)$.

[29] *Proof.* Write the tree G in the form $G = (V, E, \phi)$. Then, w and z are elements of V (since w and z are vertices of G). Thus, Lemma 0.17 **(b)** (applied to $u = w$ and $v = z$) yields $d(w,z) = d(z,w)$.

- *Case 2:* We have $i \geq j$.

We shall derive a contradiction in each of these two cases.

Indeed, let us first consider Case 1. In this case, we have $i \leq j$. Hence, Proposition 0.21 **(a)** shows that $d(x,w) + d(y,z) \geq d(x,y) + d(z,w)$. This contradicts (17). Hence, we have found a contradiction in Case 1.

Let us now consider Case 2. In this case, we have $i \geq j$. Hence, Proposition 0.21 **(b)** shows that $d(x,z) + d(y,w) \geq d(x,y) + d(z,w)$. This contradicts (16). Hence, we have found a contradiction in Case 2.

We thus have found a contradiction in each of the two Cases 1 and 2. Thus, we always get a contradiction. This shows that our assumption was false. Hence, the proof of Corollary 0.22 **(a)** is complete.

(b) Corollary 0.22 **(a)** (applied to z and y instead of y and z) yields

$$\begin{aligned} d(x,z) + d(y,w) &\leq \max \left\{ d(x,y) + d(z,w), d(x,w) + \underbrace{d(z,y)}_{=d(y,z)} \right\} \\ &= \max \{ d(x,y) + d(z,w), d(x,w) + d(y,z) \}. \end{aligned}$$

This proves Corollary 0.22 **(b)**.

(c) Corollary 0.22 **(a)** (applied to w and y instead of y and w) yields

$$\begin{aligned} d(x,w) + d(z,y) &\leq \max \left\{ d(x,z) + \underbrace{d(w,y)}_{=d(y,w)}, d(x,y) + \underbrace{d(w,z)}_{=d(z,w)} \right\} \\ &= \max \{ d(x,z) + d(y,w), d(x,y) + d(z,w) \} \\ &= \max \{ d(x,y) + d(z,w), d(x,z) + d(y,w) \}. \end{aligned}$$

This proves Corollary 0.22 **(c)**. $\square$

Solution to Exercise 6 (sketched). Let a, b and c be the three numbers $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$, sorted in increasing order (so that $a \leq b \leq c$). Then, the two largest ones among the three numbers $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ are b and c.

The three equalities (13), (14) and (15) (combined) show that each of the three numbers $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ is less than or equal to the maximum of the two others. Since the three numbers a, b and c are precisely the three numbers $d(x,y) + d(z,w)$, $d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ (except possibly in a different order), we can rewrite this as follows: Each of the three numbers a, b and c is less than or equal to the maximum of the two others. In other words, we have $a \leq \max\{b,c\}$ and $b \leq \max\{c,a\}$ and $c \leq \max\{a,b\}$. But $a \leq b \leq c$ (since the three numbers a, b and c are sorted in increasing order). Now, $c \leq \max\{a,b\} = b$ (since $a \leq b$). Combined with $b \leq c$, this yields $b = c$. In other words, b and c are equal. In other words, two largest ones among the three numbers $d(x,y)+d(z,w)$, $d(x,z)+d(y,w)$ and $d(x,w)+d(y,z)$ are equal (since the two largest ones among the three numbers $d(x,y)+d(z,w)$, $d(x,z)+d(y,w)$ and $d(x,w)+d(y,z)$ are b and c). This solves the exercise. $\square$

0.8. Exercise 7: on triple intersections

Definition 0.23. Let $G=(V,E,\phi)$ be a multigraph.

For any subset U of V, we let $G[U]$ denote the sub-multigraph $(U, E_U, \phi\mid_{E_U})$ of G, where E_U is the subset $\{e\in E \mid \phi(e)\subseteq U\}$ of E. (Thus, $G[U]$ is the sub-multigraph obtained from G by removing all vertices that don't belong to U, and subsequently removing all edges that don't have both their endpoints in U.) This sub-multigraph $G[U]$ is called the *induced sub-multigraph of G on the subset U*.

Exercise 7. Let $G=(V,E,\phi)$ be a multigraph.

Let A, B and C be three subsets of V such that the sub-multigraphs $G[A]$, $G[B]$ and $G[C]$ are connected.

A cycle of G will be called *eclectic* if it contains at least one edge of $G[A]$, at least one edge of $G[B]$ and at least one edge of $G[C]$ (although these three edges are not required to be distinct).

(a) If the sets $B\cap C$, $C\cap A$ and $A\cap B$ are nonempty, but $A\cap B\cap C$ is empty, then prove that G has an eclectic cycle.

(b) If the sub-multigraphs $G[B\cap C]$, $G[C\cap A]$ and $G[A\cap B]$ are connected, but the sub-multigraph $G[A\cap B\cap C]$ is not connected, then prove that G has an eclectic cycle.

[**Note:** Keep in mind that the multigraph with 0 vertices does not count as connected.]

Solution to Exercise 7 (sketched). **(a)** Assume that the sets $B\cap C$, $C\cap A$ and $A\cap B$ are nonempty, but $A\cap B\cap C$ is empty. There exists at least one triple $(u,v,w)\in(B\cap C)\times(C\cap A)\times(A\cap B)$ (since the sets $B\cap C$, $C\cap A$ and $A\cap B$ are nonempty), and for each such triple, the integers $d_{G[A]}(v,w)$, $d_{G[B]}(w,u)$ and $d_{G[C]}(u,v)$ are well-defined (i.e., not equal to ∞) [30]. Hence, we can pick a triple $(u,v,w)\in(B\cap C)\times(C\cap A)\times(A\cap B)$ minimizing the sum $d_{G[A]}(v,w)+d_{G[B]}(w,u)+d_{G[C]}(u,v)$. Pick such a triple.

The vertices u, v and w are distinct (because if any two of them were equal, then these two equal vertices would lie in the set $A\cap B\cap C$, which however was assumed to be empty). Pick any path p from v to w in $G[A]$ having length $d_{G[A]}(v,w)$. Pick any path q from w to u in $G[B]$ having length $d_{G[B]}(w,u)$. Pick any path r from u to v in $G[C]$ having length $d_{G[C]}(u,v)$. Each of the paths p, q and r has length ≥ 1 (since the vertices u, v and w are distinct).

[30] This is because the multigraphs $G[A]$, $G[B]$ and $G[C]$ are connected.

The paths p and q have no vertices in common apart from the vertex w (at which the path p ends and the path q starts)[31]. Similarly, the paths q and r have no vertices in common apart from the vertex u (at which the path q ends and the path r starts). Similarly, the paths r and p have no vertices in common apart from the vertex v (at which the path r ends and the path p starts). Thus, we can combine the three paths p, q and r to form a cycle. This cycle contains an edge of $G[A]$ (since the path p is nonempty, and thus contributes an edge), an edge of $G[B]$ (similarly) and an edge of $G[C]$ (similarly). Hence, this cycle is eclectic. Thus, G has an eclectic cycle. This solves Exercise 7 **(a)**.

(b) Assume that the sub-multigraphs $G[B \cap C]$, $G[C \cap A]$ and $G[A \cap B]$ are connected, but the sub-multigraph $G[A \cap B \cap C]$ is not connected. We must prove that G has an eclectic cycle.

The sets $B \cap C$, $C \cap A$ and $A \cap B$ are nonempty (since the multigraphs $G[B \cap C]$, $G[C \cap A]$ and $G[A \cap B]$ are connected). If the set $A \cap B \cap C$ is empty, then Exercise 7 **(a)** proves that G has an eclectic cycle. Hence, for the rest of this proof, we WLOG assume that the set $A \cap B \cap C$ is nonempty.

The graph $G[A \cap B \cap C]$ is not connected, but it has at least one vertex (since the set $A \cap B \cap C$ is nonempty). Thus, there exists at least one pair $(u, v) \in (A \cap B \cap C)^2$ such that there exists no path from u to v in $G[A \cap B \cap C]$. Fix such a pair (u, v) minimizing the sum $d_{G[C \cap A]}(u, v) + d_{G[A \cap B]}(v, u)$. (This sum is an integer, since both sub-multigraphs $G[C \cap A]$ and $G[A \cap B]$ are connected.)

The vertices u and v are distinct (since there exists no path from u to v in $G[A \cap B \cap C]$). Pick any path p from u to v in $G[C \cap A]$ having length $d_{G[C \cap A]}(u, v)$. Pick any path q from v to u in $G[A \cap B]$ having length $d_{G[A \cap B]}(v, u)$. Each of the paths p and q has length ≥ 1 (since the vertices u and v are distinct).

The paths p and q have no vertices in common apart from the vertices u and v [32]. Thus, we can combine these two paths p and q to form a cycle. This cycle

[31] *Proof.* Assume the contrary. Thus, the paths p and q have a vertex $w' \neq w$ in common. Consider this w'. Each vertex on the path p belongs to A (since p is a path in $G[A]$). Thus, $w' \in A$. Similarly, $w' \in B$. Hence, $w' \in A \cap B$. Furthermore, $d_{G[A]}(v, w') < d_{G[A]}(v, w)$ (because w' lies on the path p, which has length $d_{G[A]}(v, w)$, but is distinct from w) and $d_{G[B]}(w', u) < d_{G[B]}(w, u)$ (similarly). Hence, $d_{G[A]}(v, w') + d_{G[B]}(w', u) + d_{G[C]}(u, v) < d_{G[A]}(v, w) + d_{G[B]}(w, u) + d_{G[C]}(u, v)$. This contradicts the fact that our triple (u, v, w) was chosen to minimize the sum $d_{G[A]}(v, w) + d_{G[B]}(w, u) + d_{G[C]}(u, v)$.

[32] *Proof.* Assume the contrary. Thus, the paths p and q have a vertex $w' \notin \{u, v\}$ in common. Consider this w'. Each vertex on the path p belongs to $C \cap A$ (since p is a path in $G[C \cap A]$). Thus, $w' \in C \cap A$. Similarly, $w' \in A \cap B$. Hence, $w' \in (C \cap A) \cap (A \cap B) = A \cap B \cap C$.

Furthermore, $d_{G[C \cap A]}(u, w') < d_{G[C \cap A]}(u, v)$ (because w' lies on the path p, which has length $d_{G[C \cap A]}(u, v)$, but is distinct from v) and $d_{G[A \cap B]}(w', u) < d_{G[A \cap B]}(v, u)$ (similarly). Hence, $d_{G[C \cap A]}(u, w') + d_{G[A \cap B]}(w', u) < d_{G[C \cap A]}(u, v) + d_{G[A \cap B]}(v, u)$. This contradicts the fact that our pair (u, v) was chosen to minimize the sum $d_{G[C \cap A]}(u, v) + d_{G[A \cap B]}(v, u)$, unless there exists a path from u to w' in $G[A \cap B \cap C]$. Hence, we conclude that there exists a path from u to w' in $G[A \cap B \cap C]$.

Moreover, $d_{G[C \cap A]}(w', v) < d_{G[C \cap A]}(u, v)$ (because w' lies on the path p, which has length $d_{G[C \cap A]}(u, v)$, but is distinct from u) and $d_{G[A \cap B]}(v, w') < d_{G[A \cap B]}(v, u)$ (similarly). Hence, contains an edge of $G[C \cap A]$ (since the path p is nonempty, and thus contributes an edge) and an edge of $G[A \cap B]$ (similarly). Therefore, this cycle contains an edge of $G[A]$ (since any edge of $G[C \cap A]$ is an edge of $G[A]$), an edge of $G[B]$ (since any edge of $G[A \cap B]$ is an edge of $G[B]$), and an edge of $G[C]$ (since any edge of $G[C \cap A]$ is an edge of $G[C]$). Hence, this cycle is eclectic. Thus, G has an eclectic cycle. This solves Exercise 7 **(b)**. □

Let us make some comments about the origin of Exercise 7. Namely, I came up with it when generalizing the following classical fact:

Theorem 0.24. Let $G = (V, E, \phi)$ be a tree.

Let A, B and C be three subsets of V such that the sub-multigraphs $G[A]$, $G[B]$ and $G[C]$ are connected (and thus are trees as well). Assume further that the sets $B \cap C$, $C \cap A$ and $A \cap B$ are nonempty. Then, the sub-multigraph $G[A \cap B \cap C]$ of G is a tree.

Proof of Theorem 0.24 (sketched). The multigraph G has no cycles (since it is a tree).

Let us first show that the sub-multigraph $G[B \cap C]$ is connected.

Indeed, assume the contrary. Thus, there exist two vertices u and v of $G[B \cap C]$ such that there exists no path from u to v in $G[B \cap C]$ (since $B \cap C$ is nonempty). Fix two such vertices u and v.

The multigraph G is a tree. Hence, there exists a unique path from u to v in G. Denote this path by p.

But $u \in B \cap C \subseteq B$ and $v \in B \cap C \subseteq B$. Thus, u and v are two vertices of the multigraph $G[B]$. Since this multigraph $G[B]$ is connected, we thus conclude that there exists a path from u to v in $G[B]$. This path must clearly be a path from u to v in G (because $G[B]$ is a sub-multigraph of G), and therefore must be the path p (since p is the unique path from u to v in G). Therefore, the path p is a path from u to v in $G[B]$. In particular, the path p is a path in $G[B]$. Thus, all vertices of p belong to B.

Similarly, all vertices of p belong to C.

Now, we have shown that all vertices of p belong to B, and that all vertices of p belong to C. Hence, all vertices of p belong to $B \cap C$ (since they belong to B and to C at the same time). Consequently, p is a path from u to v in $G[B \cap C]$. This contradicts the fact that there exists no path from u to v in $G[B \cap C]$. This contradiction proves that our assumption was false.

Hence, we have proven that the sub-multigraph $G[B \cap C]$ is connected.

$d_{G[C \cap A]}(w', v) + d_{G[A \cap B]}(v, w') < d_{G[C \cap A]}(u, v) + d_{G[A \cap B]}(v, u)$. This contradicts the fact that our pair (u, v) was chosen to minimize the sum $d_{G[C \cap A]}(u, v) + d_{G[A \cap B]}(v, u)$, unless there exists a path from w' to v in $G[A \cap B \cap C]$. Hence, we conclude that there exists a path from w' to v in $G[A \cap B \cap C]$.

We now know that there exists a path from u to w' in $G[A \cap B \cap C]$, and that there exists a path from w' to v in $G[A \cap B \cap C]$. Concatenating these paths, we obtain a walk from u to v in $G[A \cap B \cap C]$. Hence, there exists a path from u to v in $G[A \cap B \cap C]$ as well. This contradicts the fact that there exists no path from u to v in $G[A \cap B \cap C]$. This contradiction shows that our assumption was wrong, qed.

Similarly, the sub-multigraphs $G[C \cap A]$ and $G[A \cap B]$ are connected.

Now, we claim that the sub-multigraph $G[A \cap B \cap C]$ is connected.

Indeed, assume the contrary. Thus, the sub-multigraph $G[A \cap B \cap C]$ is not connected. Hence, Exercise 7 **(b)** shows that G has an eclectic cycle. Thus, G has a cycle. This contradicts the fact that G has no cycles. This contradiction proves that our assumption was wrong.

Hence, we have proven that the sub-multigraph $G[A \cap B \cap C]$ is connected. Furthermore, this sub-multigraph $G[A \cap B \cap C]$ has no cycles (since it is a sub-multigraph of the tree G, which has no cycles). Hence, this sub-multigraph $G[A \cap B \cap C]$ is a forest. Thus, $G[A \cap B \cap C]$ is a connected forest, i.e., a tree. This proves Theorem 0.24. □

A generalization of Theorem 0.24 is known as "Helly's theorem for trees" (see, e.g., [Horn71, Theorem 4.1]):

Theorem 0.25. Let $G = (V, E, \phi)$ be a tree.

Let $A_1, A_2, \ldots, A_k$ be k subsets of V such that for each $i \in \{1, 2, \ldots, k\}$, the sub-multigraph $G[A_i]$ is connected. Assume further that for each $1 \leq i < j \leq k$, the set $A_i \cap A_j$ is nonempty. Then, the sub-multigraph $G[A_1 \cap A_2 \cap \cdots \cap A_k]$ of G is a tree.

It is not hard to derive Theorem 0.25 from Theorem 0.24 by induction over k. But I am wondering:

Question 0.26. Is there a generalization of Exercise 7 that extends it to k subsets of V, similarly to how Theorem 0.25 extends Theorem 0.24 ?

Let me observe one more curiosity. Namely, Exercise 7 has an analogue for multidigraphs. To state this analogue, let us define induced sub-multidigraphs[33]:

Definition 0.27. Let $G = (V, E, \phi)$ be a multidigraph.

For any subset U of V, we let $G[U]$ denote the sub-multidigraph $(U, E_U, \phi\mid_{E_U})$ of G, where E_U is the subset $\{e \in E \mid \phi(e) \in U \times U\}$ of E. (Thus, $G[U]$ is the sub-multidigraph obtained from G by removing all vertices that don't belong to U, and subsequently removing all arcs that don't have both their source and their target in U.) This sub-multidigraph $G[U]$ is called the *induced sub-multidigraph of G on the subset U*.

Now, the analogue of Exercise 7 states the following:

Proposition 0.28. Let $G = (V, E, \phi)$ be a multidigraph.

Let A, B and C be three subsets of V such that the sub-multidigraphs $G[A]$, $G[B]$ and $G[C]$ are strongly connected.

[33] We shall use the notation $G = (V, E, \phi)$ instead of the more common notation $D = (V, A, \phi)$ for our multidigraph in order to make the analogy to Exercise 7 more obvious.

A cycle of G will be called *eclectic* if it contains at least one arc of $G[A]$, at least one arc of $G[B]$ and at least one arc of $G[C]$ (although these three arcs are not required to be distinct).

(a) If the sets $B \cap C$, $C \cap A$ and $A \cap B$ are nonempty, but $A \cap B \cap C$ is empty, then prove that G has an eclectic cycle.

(b) If the sub-multidigraphs $G[B \cap C]$, $G[C \cap A]$ and $G[A \cap B]$ are strongly connected, but the sub-multidigraph $G[A \cap B \cap C]$ is not strongly connected, then prove that G has an eclectic cycle.

[**Note:** Keep in mind that the multidigraph with 0 vertices does not count as strongly connected.]

Proof of Proposition 0.28 (sketched). The proof of Proposition 0.28 is completely analogous to the solution to Exercise 7. (Of course, the obvious changes need to be made – e.g., replacing "multigraph" by "multidigraph", and replacing "connected" by "strongly connected".) □

References

[Grinbe16] Darij Grinberg, *A note on non-broken-circuit sets and the chromatic polynomial*, preprint, version 1.2, April 9, 2017.
`http://www.cip.ifi.lmu.de/~grinberg/algebra/chromatic.pdf`

[Grinbe17] Darij Grinberg, *Notes on graph theory*, unfinished draft, version 24 February 2017.
`http://www.cip.ifi.lmu.de/~grinberg/t/17s/nogra.pdf`

[Horn71] W. A. Horn, *Three Results for Trees, Using Mathematical Induction*, Journal of Research of the National Bureau of Standards - B. Mathematical Sciences, Volume 76B, Nos. 1 and 2, January-June 1972.
`http://nistdigitalarchives.contentdm.oclc.org/cdm/ref/collection/p13011coll6/id/59825`

[LeLeMe16] Eric Lehman, F. Thomson Leighton, Albert R. Meyer, *Mathematics for Computer Science*, revised 4th April 2017,
`https://courses.csail.mit.edu/6.042/spring17/mcs.pdf` .

[Whitne32] Hassler Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc., Volume 38, Number 8 (1932), pp. 572–579.
`https://projecteuclid.org/euclid.bams/1183496087`