

Math 5707 Spring 2017 (Darij Grinberg): midterm 2**due: Mon, 5 Apr 2017, in class** or by email (dgrinber@umn.edu) before class

See the website for relevant material.

Results proven in the notes, or in the handwritten notes, or in class, or in previous homework sets can be used without proof; but they should be referenced clearly (e.g., not “by a theorem done in class” but “by the theorem that states that a strongly connected digraph has a Eulerian circuit if and only if each vertex has indegree equal to its outdegree”). If you reference results from the lecture notes, please **mention the date and time** of the version of the notes you are using (as the numbering changes during updates).

As always, proofs need to be provided, and they have to be clear and rigorous. Obvious details can be omitted, but they actually have to be obvious.

This is a midterm, so you are **not allowed to collaborate or contact others** (apart from me) for help with the problems. (Feel free to ask me for clarifications, but I will not give hints towards solving the problems.) Reading up (in books or on the internet) is allowed, but asking for help is not. If you get your solution from a book (or paper, or website), do cite the source¹, and do explain the solution in your own words.

0.1. Exercise 1: assigning to each vertex an edge avoiding it

Exercise 1. Let $G = (V, E)$ be a simple graph such that $|E| \geq |V|$. Show that there exists an injective map $f : V \rightarrow E$ such that each $v \in V$ satisfies $v \notin f(v)$.

(In other words, show that we can assign to each vertex v of G an edge that does not contain v , in such a way that edges assigned to distinct vertices are distinct.)

0.2. Exercise 2: assigning to each vertex an edge containing it

Exercise 2. Let $G = (V, E)$ be a **connected** simple graph such that $|E| \geq |V|$. Show that there exists an injective map $f : V \rightarrow E$ such that each $v \in V$ satisfies $v \in f(v)$.

(In other words, show that we can assign to each vertex v of G an edge that contains v , in such a way that edges assigned to distinct vertices are distinct.)

0.3. Exercise 3: a “transitivity” property for arc-disjoint paths

Exercise 3. Let $D = (V, A)$ be a digraph. Let $k \in \mathbb{N}$. Let u, v and w be three vertices of D . Assume that there exist k arc-disjoint paths from u to v . Assume furthermore that there exist k arc-disjoint paths from v to w .

Prove that there exist k arc-disjoint paths from u to w .

[**Note:** If $u = w$, then the trivial path (u) counts as being arc-disjoint from itself (so in this case, there exist arbitrarily many arc-disjoint paths from u to w).]

¹You won't be penalized for this.

0.4. Exercise 4: the chromatic polynomial

Exercise 4. Let $G = (V, E)$ be a simple graph. Define a polynomial χ_G in a single indeterminate x (with integer coefficients) by

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\text{conn}(V, F)}.$$

(Here, as usual, $\text{conn } H$ denotes the number of connected components of any graph H .) This polynomial χ_G is called the *chromatic polynomial* of G .

Fix $k \in \mathbb{N}$. Recall that a k -coloring of G means a map $f : V \rightarrow \{1, 2, \dots, k\}$. (The image $f(v)$ of a vertex $v \in V$ under this map is called the *color* of v under this k -coloring f .) A k -coloring f of G is said to be *proper* if each edge $\{u, v\}$ of G satisfies $f(u) \neq f(v)$. (In other words, a k -coloring f of G is proper if and only if no two adjacent vertices share the same color.)

Prove that the number of proper k -colorings of G is $\chi_G(k)$.

[**Hint:** Show that $k^{\text{conn}(V, F)}$ also counts certain k -colorings (I like to call them “ F -improper colorings” – what could that mean?). Then, analyze how often (and with what signs) a given k -coloring of G appears in the sum $\sum_{F \subseteq E} (-1)^{|F|} k^{\text{conn}(V, F)}$.]

Note that most graph-theoretical literature defines the chromatic polynomial differently than I do in Exercise 4. Use the literature at your own peril! Most authors define χ_G as the polynomial whose value at each $k \in \mathbb{N}$ is the number of proper k -colorings. This may be more intuitive, but it leaves a question unanswered: Why is there such a polynomial in the first place? Exercise 4 answers this question.

0.5. Exercise 5: some concrete chromatic polynomials

Exercise 5. In Exercise 4, we have defined the chromatic polynomial χ_G of a simple graph G . In this exercise, we shall compute it on some examples.

(a) For each $n \in \mathbb{N}$, prove that the complete graph K_n has chromatic polynomial $\chi_{K_n} = x(x-1) \cdots (x-n+1)$.

(b) Let T be a tree (regarded as a simple graph). Let $n = |V(T)|$. Prove that $\chi_T = x(x-1)^{n-1}$.

(c) Find the chromatic polynomial χ_{P_3} of the path graph P_3 .

[You are allowed to use both the definition of χ_G and the claim of Exercise 4 even if you have not solved that exercise. You are also allowed to use the following fact without proof: If a polynomial p with integer coefficients satisfies $p(k) = 0$ for all $k \in \mathbb{N}$, then $p = 0$.]

0.6. Exercise 6: the distances between four points on a tree

Exercise 6. Let G be a tree. Let x, y, z and w be four vertices of G .

Show that the two larger ones among the three numbers $d(x, y) + d(z, w)$, $d(x, z) + d(y, w)$ and $d(x, w) + d(y, z)$ are equal.

0.7. Exercise 7: on triple intersections

Exercise 7. Let $G = (V, E, \phi)$ be a multigraph.

For any subset U of V , we let $G[U]$ denote the sub-multigraph $(U, E_U, \phi|_{E_U})$ of G , where E_U is the subset $\{e \in E \mid \phi(e) \subseteq U\}$ of E . (Thus, $G[U]$ is the sub-multigraph obtained from G by removing all vertices that don't belong to U , and subsequently removing all edges that don't have both their endpoints in U .) This sub-multigraph $G[U]$ is called the *induced sub-multigraph of G on the subset U* .

Let A, B and C be three subsets of V such that the sub-multigraphs $G[A]$, $G[B]$ and $G[C]$ are connected.

A cycle of G will be called *eclectic* if it contains at least one edge of $G[A]$, at least one edge of $G[B]$ and at least one edge of $G[C]$ (although these three edges are not required to be distinct).

(a) If the sets $B \cap C$, $C \cap A$ and $A \cap B$ are nonempty, but $A \cap B \cap C$ is empty, then prove that G has an eclectic cycle.

(b) If the subgraphs $G[B \cap C]$, $G[C \cap A]$ and $G[A \cap B]$ are connected, but the subgraph $G[A \cap B \cap C]$ is not connected, then prove that G has an eclectic cycle.

[Note: Keep in mind that the multigraph with 0 vertices does not count as connected.]