Math 5707 Spring 2017 (Darij Grinberg): midterm 1
due: Mon, 27 Feb 2017, in class or by email (`dgrinber@umn.edu`) before class

See the website for relevant material.

Results proven in the notes, or in the handwritten notes, or in class, or in previous homework sets can be used without proof; but they should be referenced clearly (e.g., not "by a theorem done in class" but "by the theorem that states that a strongly connected digraph has a Eulerian circuit if and only if each vertex has indegree equal to its outdegree"). If you reference results from the lecture notes, please **mention the date and time** of the version of the notes you are using (as the numbering changes during updates).

As always, proofs need to be provided, and they have to be clear and rigorous. Obvious details can be omitted, but they actually have to be obvious.

This is a midterm, so you are **not allowed to collaborate or contact others** (apart from me) for help with the problems. (Feel free to ask me for clarifications, but I will not give hints towards solving the problems.) Reading up (in books or on the internet) is allowed, but asking for help is not. If you get your solution from a book (or paper, or website), do cite the source[1], and do explain the solution in your own words.

Exercise 1. Let $D = (V, A)$ be a digraph. A *from-dominating set* of D shall mean a subset S of V such that for each vertex $v \in V \setminus S$, there exists at least one arc $uv \in A$ with $u \in S$.

Assume that D has a Hamiltonian path. Prove that D has a from-dominating set of size $\leq \dfrac{|V| + 1}{2}$.

Exercise 2. Let $G = (V, E)$ be a simple graph. The *line graph* $L(G)$ is defined as the simple graph (E, F), where

$$F = \{\{e_1, e_2\} \in \mathcal{P}_2(E) \mid e_1 \cap e_2 \neq \varnothing\}.$$

(In other words, $L(G)$ is the graph whose **vertices** are the **edges** of G, and in which two vertices e_1 and e_2 are adjacent if and only if the edges e_1 and e_2 of G share a common vertex.)

Assume that $|V| > 1$.

(a) If G has a Hamiltonian path, then prove that $L(G)$ has a Hamiltonian path.

(b) If G has a Eulerian walk, then prove that $L(G)$ has a Hamiltonian path.

Exercise 3. Let $D = (V, A)$ be a digraph with $|V| > 0$. Assume that each vertex $v \in V$ satisfies $\deg^- v > 0$. Prove that D has at least one cycle.

(Keep in mind that a length-1 circuit (v, v) counts as a cycle when A contains the loop (v, v).)

[1] You won't be penalized for this.

Exercise 4. Let G be a multigraph with at least one edge. Assume that each vertex of G has even degree. Prove that G has a cycle.

Exercise 5. Let $k \in \mathbb{N}$. Let $p_1, p_2, \ldots, p_k$ be k nonnegative real numbers such that $p_1 + p_2 + \cdots + p_k \geq 1$.

Let $G = (V, E)$ be a simple graph. A *k-coloring* of G shall mean a map $f : V \to \{1, 2, \ldots, k\}$.

Prove that there exists a k-coloring f of G with the following property: For each vertex $v \in V$, at most $p_{f(v)} \deg v$ neighbors of v have the same color as v. Here, the *color* of a vertex $w \in V$ (under the k-coloring f) means the value $f(w)$.

Exercise 6. Let G be a connected multigraph. Let m be the number of vertices of G that have odd degree. Prove that we can add $m/2$ new edges to G in such a way that the resulting multigraph will have an Eulerian circuit. (It is allowed to add an edge even if there is already an edge between the same two vertices.)

If u and v are two vertices of a simple graph G, then $d(u, v)$ denotes the *distance* between u and v. This is defined to be the minimum length of a path from u to v if such a path exists; otherwise it is defined to be the symbol ∞.

Exercise 7. Let a, b and c be three vertices of a connected simple graph $G = (V, E)$. Prove that $d(b, c) + d(c, a) + d(a, b) \leq 2|V| - 2$.

Exercise 8. Let $G = (V, E)$ be a simple graph such that $|E| > |V|$. Prove that G has a cycle of length $\leq \dfrac{2n+2}{3}$, where $n = |V|$.