Math 5707 Spring 2017 (Darij Grinberg): homework set 5
Solution sketches.
extended and generalized version, with an introduction to sandpile theory
(June 9, 2023).

This tale grew in the telling.

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Contents

0.1. Reminders

See the lecture notes and also the handwritten notes for relevant material. See also the solutions to homework set 2 [Grinbe17a] for various conventions and notations that are in use here.

0.2. Sandpiles: the basic results

Let me recall the definitions of the basic concepts on chip-firing done in class. Various sources on this material are [BjoLov92] (and, less directly, [BjLoSh91]), [HLMPPW13], [Musike09, Lectures 29–31] and [CorPet16]. (None of these is as readable as I would like to have it, but the whole subject is about 30 years old, with

most activity very recent... Also, be aware of incompatible notations, as well as of the fact that some of the sources only consider undirected graphs.) The particular case of the "integer lattice" graph has attracted particular attention due to the mysterious pictures it generates; see `http://www.math.cmu.edu/~wes/sand.html#next-page` for some of these pictures, as well as `http://www.math.cornell.edu/~levine/apollonian-slides.pdf` for a talk with various illustrations.

Let me give a survey of the basics of the theory. (Due to a reliance on constructive proofs, this survey is unfortunately longer than I expected it to be.)

0.2.1. Configurations, $\mathbb{Z}$-configurations and toppling

We refer to [Grinbe17a] for the definition of a multidigraph, as well as for the definitions of walks, paths, circuits and cycles in a multidigraph.

If v is a vertex of a multidigraph (V, A, ϕ), then the *outdegree* $\deg^+ v$ of v is defined to be the number of all arcs $a \in A$ whose source is v. Similarly, if v is a vertex of a multidigraph (V, A, ϕ), then the *indegree* $\deg^- v$ of v is defined to be the number of all arcs $a \in A$ whose target is v.

An arc a of a multidigraph is said to be a *loop* if the source of a is the target of a.

Fix a multidigraph $D = (V, A, \phi)$.

Definition 0.1. A *configuration* (on D) means a map $f : V \to \mathbb{N}$. (Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

A configuration is also called a *chip configuration* or *sandpile*.

We like to think of a configuration as a way to place a finite number of game chips on the vertices of D: Namely, the configuration f corresponds to placing $f(v)$ chips on the vertex v for each $v \in V$. The chips are understood to be undistinguishable, so the only thing that matters is how many of them are placed on each given vertex. Sometimes, we speak of grains of sand instead of chips.

Definition 0.2. A $\mathbb{Z}$*-configuration* (on D) means a map $f : V \to \mathbb{Z}$. We shall regard each configuration as a $\mathbb{Z}$-configuration (since $\mathbb{N} \subseteq \mathbb{Z}$).

Definition 0.3. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration.

(a) A vertex $v \in V$ is said to be *active* in f if and only if $f(v) \geq \deg^+ v$. (Recall that $\deg^+ v$ is the outdegree of v.)

(b) The $\mathbb{Z}$-configuration f is said to be *stable* if no vertex $v \in V$ is active in f.

Thus, a $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ is stable if and only if each vertex $v \in V$ satisfies $f(v) < \deg^+ v$.

Notice that there are only finitely many stable configurations (because if f is a stable configuration, then, for each $v \in V$, the stability of f implies $f(v) \leq \deg^+ v$, whereas the fact that f is a configuration implies $f(v) \geq 0$; but these two inequalities combined leave only finitely many possible values for $f(v)$).

Definition 0.4. The set $\mathbb{Z}^V$ of all $\mathbb{Z}$-configurations can be equipped with operations of addition and subtraction, defined as follows:

- For any two $\mathbb{Z}$-configurations $f : V \to \mathbb{Z}$ and $g : V \to \mathbb{Z}$, we define a $\mathbb{Z}$-configuration $f + g : V \to \mathbb{Z}$ by setting
$$(f+g)(v) = f(v) + g(v) \qquad \text{for each } v \in V.$$
- For any two $\mathbb{Z}$-configurations $f : V \to \mathbb{Z}$ and $g : V \to \mathbb{Z}$, we define a $\mathbb{Z}$-configuration $f - g : V \to \mathbb{Z}$ by setting
$$(f-g)(v) = f(v) - g(v) \qquad \text{for each } v \in V.$$

These operations of addition and subtraction satisfy the standard rules (e.g., we always have $(f+g)+h = f+(g+h)$ and $(f-g)-h = f-(g+h)$). Hence, we can write terms like $f+g+h$ or $f-g-h$ without having to explicitly place parentheses.

Also, we can define a "zero configuration" $0 : V \to \mathbb{Z}$, which is the configuration that sends each $v \in V$ to the number 0. (Hopefully, the dual use of the symbol 0 for both the number 0 and this zero configuration is not too confusing.)

Also, for each $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ and each integer N, we define a $\mathbb{Z}$-configuration $Nf : V \to \mathbb{Z}$ by
$$(Nf)(v) = Nf(v) \qquad \text{for each } v \in V.$$

Definition 0.5. Let $f : V \to \mathbb{Z}$ be any $\mathbb{Z}$-configuration. Then, $\sum f$ shall denote the integer $\sum\limits_{v \in V} f(v)$.

This integer $\sum f$ is called the *degree* of f.

If f is a configuration, then $\sum f$ is the total number of chips in f.

It is easy to see that any two $\mathbb{Z}$-configurations f and g satisfy $\sum (f+g) = \sum f + \sum g$ and $\sum (f-g) = \sum f - \sum g$. Also, any $N \in \mathbb{Z}$ and any $\mathbb{Z}$-configuration f satisfy $\sum (Nf) = N \sum f$.

Proposition 0.6. Let f be a configuration. Let $h = \sum f$ and $w \in V$. Then, $f(w) \leq h$.

Proof of Proposition 0.6 (sketched). We have $h = \sum f = \sum\limits_{v \in V} f(v)$; this is a sum of nonnegative integers (since f is a configuration). But a sum of nonnegative integers is always $\geq$ to each of its addends. Hence, $\sum\limits_{v \in V} f(v) \geq f(w)$. Thus, $h = \sum\limits_{v \in V} f(v) \geq f(w)$. This proves Proposition 0.6. $\square$

Definition 0.7. We shall use the *Iverson bracket notation*: If $\mathcal{A}$ is any logical statement, then we define an integer $[\mathcal{A}] \in \{0,1\}$ by

$$[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}.$$

For example, $[1+1=2] = 1$ (since $1+1=2$ is true), whereas $[1+1=1] = 0$ (since $1+1=1$ is false).

Definition 0.8. If v and w are any two vertices of D, then $a_{v,w}$ shall denote the number of all arcs of D having source v and target w. This is a nonnegative integer.

(Note that $a_{v,w}$ might be > 1, since D is a multidigraph. Note also that $a_{v,v}$ might be nonzero, since loops are allowed.)

Definition 0.9. Let $v \in V$ be a vertex. Then, a $\mathbb{Z}$-configuration Δv is defined by setting

$$(\Delta v)(w) = [w = v] \deg^+ v - a_{v,w} \qquad \text{for all } w \in V.$$

Let us unpack the definition of Δv we just gave: It says that

$$(\Delta v)(w) = -a_{v,w} \qquad \text{for each vertex } w \in V \text{ distinct from } v$$

(because if w is distinct from v, then $[w = v] = 0$); and it says that

$$(\Delta v)(v) = \deg^+ v - a_{v,v}.$$

Definition 0.10. Let $v \in V$ be a vertex. Then, *firing* v is the operation on $\mathbb{Z}$-configurations (i.e., formally speaking, the mapping from $\mathbb{Z}^V$ to $\mathbb{Z}^V$) that sends each $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ to $f - \Delta v$.

We sometimes say "toppling v" instead of "firing v".

If $f : V \to \mathbb{N}$ is a configuration, then the $\mathbb{Z}$-configuration $f - \Delta v$ obtained by firing v can be described as follows: The vertex v "donates" $\deg^+ v$ of its chips to its neighbors, by sending one chip along each of its outgoing arcs (i.e., for each arc having source v, the vertex v sends one chip along this arc to the target of this arc). Thus, the number of chips on v (weakly) decreases, while the number of chips on each other vertex (weakly) increases. Of course, the resulting $\mathbb{Z}$-configuration $f - \Delta v$ is not necessarily a configuration. In fact, it is a configuration if the vertex v is active in f, and also in some other cases:

Proposition 0.11. Let $f : V \to \mathbb{N}$ be a configuration. Let $v \in V$.

(a) If the vertex v is active in f, then $f - \Delta v$ is a configuration.

(b) If $f - \Delta v$ is a configuration, and if D has no loops with source and target v, then the vertex v is active in f.

Note that the converse of Proposition 0.11 **(a)** does not hold when D has loops with source and target v.

Proof of Proposition 0.11 (sketched). **(a)** Assume that the vertex v is active in f. Thus, $f(v) \geq \deg^+ v$. Hence,

$$(f - \Delta v)(v) = \underbrace{(f(v))}_{\geq \deg^+ v} - \underbrace{((\Delta v)(v))}_{= \deg^+ v - a_{v,v}} \geq \deg^+ v - \left(\deg^+ v - a_{v,v}\right) = a_{v,v} \geq 0.$$

Also, each vertex $w \in V$ distinct from v satisfies

$$(f - \Delta v)(w) = f(w) - \underbrace{((\Delta v)(w))}_{= -a_{v,w} \leq 0} \geq f(w) \geq 0$$

(since f is a configuration). Combining these two inequalities, we conclude that $(f - \Delta v)(w) \geq 0$ for each $w \in V$. In other words, $f - \Delta v$ is a configuration. This proves Proposition 0.11 **(a)**.

(b) Assume that $f - \Delta v$ is a configuration. Also, assume that D has no loops with source and target v. Thus, $a_{v,v} = 0$. Now, $(f - \Delta v)(v) \geq 0$ (since $f - \Delta v$ is a configuration), so that

$$\begin{aligned} 0 \leq (f - \Delta v)(v) &= f(v) - \underbrace{((\Delta v)(v))}_{= \deg^+ v - a_{v,v}} = f(v) - \left(\deg^+ v - a_{v,v}\right) \\ &= f(v) - \deg^+ v + \underbrace{a_{v,v}}_{=0} = f(v) - \deg^+ v. \end{aligned}$$

In other words, $f(v) \geq \deg^+ v$. In other words, the vertex v is active in f. This proves Proposition 0.11 **(b)**. □

Proposition 0.12. We have $\sum (\Delta v) = 0$ for each vertex $v \in V$.

Proof of Proposition 0.12 (sketched). Let $v \in V$ be any vertex. Then,

$$\begin{aligned} \sum (\Delta v) &= \sum_{w \in V} \underbrace{(\Delta v)(w)}_{= [w=v] \deg^+ v - a_{v,w}} = \sum_{w \in V} \left([w = v] \deg^+ v - a_{v,w}\right) \\ &= \underbrace{\sum_{w \in V} [w = v]}_{=1} \deg^+ v - \underbrace{\sum_{w \in V} a_{v,w}}_{\substack{= (\text{the number of all arcs with source } v) \\ = \deg^+ v}} \\ &= \deg^+ v - \deg^+ v = 0. \end{aligned}$$

This proves Proposition 0.12. □

Thus,

$$\sum (f - \Delta v) = \sum f - \underbrace{\sum (\Delta v)}_{\substack{=0 \\ \text{(by Proposition 0.12)}}} = \sum f$$

for each $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ and each vertex v. In other words, firing a vertex v does not change the degree of a $\mathbb{Z}$-configuration.

0.2.2. Legal and stabilizing sequences

The word "sequence" shall always mean "finite sequence" (which is the same as "finite list" or "tuple").

We shall now study the effects of repeatedly firing vertices of D (that is, firing several vertices one after the other). If $v_1, v_2, \ldots, v_k$ are finitely many elements of V, and if $f : V \to \mathbb{Z}$ is any $\mathbb{Z}$-configuration, then firing the vertices $v_1, v_2, \ldots, v_k$ (one after the other, in this order) results in the $\mathbb{Z}$-configuration $f - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_k$. This shows, in particular, that the order in which we fire the vertices does not matter for the final result (i.e., firing them in any other order would yield the same resulting $\mathbb{Z}$-configuration); however, the intermediate configurations of course do depend on the order.

Definition 0.13. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration.

Let $(v_1, v_2, \ldots, v_k)$ be a sequence of vertices of D.

(a) The sequence $(v_1, v_2, \ldots, v_k)$ is said to be *legal* for f if for each $i \in \{1, 2, \ldots, k\}$, the vertex v_i is active in the $\mathbb{Z}$-configuration $f - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_{i-1}$.

(b) The sequence $(v_1, v_2, \ldots, v_k)$ is said to be *stabilizing* for f if the $\mathbb{Z}$-configuration $f - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_k$ is stable.

What is the rationale behind the notions of "legal" and "stabilizing"? A sequence of vertices provides a way to modify a $\mathbb{Z}$-configuration by first firing the first vertex in the sequence, then firing the second, and so on. The sequence is said to be *legal* (for f) if only active vertices are being fired in this process (i.e., each vertex that gets fired is active at the time of its firing); thus, in particular, if f was a configuration, then it remains a configuration throughout this process (i.e., at no point does a vertex have a negative number of chips)[1]. The sequence is said to be *stabilizing* (for f) if the $\mathbb{Z}$-configuration resulting from it at the very end is stable.

We notice some obvious consequences of the definitions:

Proposition 0.14. Let f be a $\mathbb{Z}$-configuration.

(a) If f is a configuration, and if a sequence $(v_1, v_2, \ldots, v_k)$ is legal for f, then all of the $\mathbb{Z}$-configurations $f - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_i$ for $i \in \{0, 1, \ldots, k\}$ are actually configurations.

(b) If a sequence $(v_1, v_2, \ldots, v_k)$ is legal for f, then each prefix of this sequence (i.e., each sequence of the form $(v_1, v_2, \ldots, v_i)$ for some $i \in \{0, 1, \ldots, k\}$) is legal for f as well.

(c) If a sequence $(v_1, v_2, \ldots, v_k)$ is stabilizing for f, then each permutation of this sequence (i.e., each sequence of the form $\left(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}\right)$ for a permutation σ of $\{1, 2, \ldots, k\}$) is stabilizing for f as well.

[1] This follows from Proposition 0.11 **(a)** (applied repeatedly).

If our multidigraph D has no loops, then the converse is also true: If the configuration remains a configuration throughout the process, then the sequence is legal. (This follows from Proposition 0.11 **(b)**.) But if D has loops, this is not always the case.

(d) If $(v_1, v_2, \ldots, v_k)$ is a legal sequence for f, then $(v_1, v_2, \ldots, v_k)$ is stabilizing if and only if there exist no $v \in V$ such that the sequence $(v_1, v_2, \ldots, v_k, v)$ is legal.

An important property of chip-firing is the following result (sometimes called the "least action principle"):

Theorem 0.15. Let $f : V \to \mathbb{Z}$ be any $\mathbb{Z}$-configuration. Let ℓ and s be two sequences of vertices of D such that ℓ is legal for f while s is stabilizing for f. Then, ℓ is a subpermutation of s.

Here, we are using the following notation:

Definition 0.16. Let $(p_1, p_2, \ldots, p_u)$ and $(q_1, q_2, \ldots, q_v)$ be two finite sequences. Then, we say that $(p_1, p_2, \ldots, p_u)$ is a *subpermutation* of $(q_1, q_2, \ldots, q_v)$ if and only if, for each object t, the following holds: The number of $i \in \{1, 2, \ldots, u\}$ satisfying $p_i = t$ is less or equal to the number of $j \in \{1, 2, \ldots, v\}$ satisfying $q_j = t$.

Equivalently, the sequence $(p_1, p_2, \ldots, p_u)$ is a subpermutation of the sequence $(q_1, q_2, \ldots, q_v)$ if and only if you can obtain the former from the latter by removing some entries and permuting the remaining entries. ("Some" allows for the possibility of "zero".)

Our proof of Theorem 0.15 relies on the following simple facts about subpermutations (whose proofs are left to the reader):

Lemma 0.17. **(a)** Any permutation of a finite sequence s is a subpermutation of s.

(b) If three finite sequences a, b and c have the property that a is a subpermutation of b, and that b is a subpermutation of c, then a is a subpermutation of c.

(c) If $(a_1, a_2, \ldots, a_N)$ and $(b_1, b_2, \ldots, b_M)$ are two finite sequences such that $N > 0$ and $M > 0$ and $a_1 = b_1$, and if the sequence $(a_2, a_3, \ldots, a_N)$ is a subpermutation of $(b_2, b_3, \ldots, b_M)$, then the sequence $(a_1, a_2, \ldots, a_N)$ is a subpermutation of $(b_1, b_2, \ldots, b_M)$.

(d) If two finite sequences a and b are such that a is a subpermutation of b and b is a subpermutation of a, then a is a permutation of b.

(e) Any subpermutation of a finite sequence s is at most as long as s.

Proof of Theorem 0.15 (sketched). We shall prove Theorem 0.15 by induction on the length of ℓ.

Induction base: Theorem 0.15 is obvious when the length of ℓ is 0.

Induction step: Fix a positive integer N. Assume (as the induction hypothesis) that Theorem 0.15 is true when the sequence ℓ has length $N - 1$. We must prove that Theorem 0.15 is true when the sequence ℓ has length N.

So let f, ℓ and s be as in Theorem 0.15, and assume that ℓ has length N. We must then prove that ℓ is a subpermutation of s.

Write the sequence s in the form $s = (s_1, s_2, \ldots, s_M)$.

Write the sequence ℓ in the form $\ell = (\ell_1, \ell_2, \ldots, \ell_N)$ (this is possible, since ℓ has length N). The entry ℓ_1 exists (since N is positive). Clearly, the vertex ℓ_1 is active in f (because ℓ_1 is the first entry of a sequence that is legal for f). In other words, $f(\ell_1) \geq \deg^+(\ell_1)$.

Now, we claim that $\ell_1 \in \{s_1, s_2, \ldots, s_M\}$. Indeed, assume the contrary. Thus, $\ell_1 \notin \{s_1, s_2, \ldots, s_M\}$. In other words, $\ell_1 \neq s_i$ for each $i \in \{1, 2, \ldots, M\}$. Hence, for each $i \in \{1, 2, \ldots, M\}$, the definition of Δs_i yields

$$(\Delta s_i)(\ell_1) = \underbrace{[\ell_1 = s_i]}_{\substack{=0 \\ (\text{since } \ell_1 \neq s_i)}} \deg^+(s_i) - \underbrace{a_{s_i, \ell_1}}_{\geq 0} \leq 0. \tag{1}$$

But the sequence $(s_1, s_2, \ldots, s_M) = s$ is stabilizing; in other words, the $\mathbb{Z}$-configuration $f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_M$ is stable. Thus, in particular,

$$(f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_M)(\ell_1) < \deg^+(\ell_1) \leq f(\ell_1)$$

(since $f(\ell_1) \geq \deg^+(\ell_1)$). Hence,

$$f(\ell_1) > (f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_M)(\ell_1) = f(\ell_1) - \sum_{i=1}^{M} \underbrace{(\Delta s_i)(\ell_1)}_{\substack{\leq 0 \\ (\text{by } (1))}} \geq f(\ell_1),$$

which is absurd. This contradiction completes our proof of $\ell_1 \in \{s_1, s_2, \ldots, s_M\}$.

In other words, ℓ_1 is an entry of the sequence s. Thus, there exists a permutation t of the sequence s such that ℓ_1 is the **first** entry of t. Consider such a t. Write t in the form $t = (t_1, t_2, \ldots, t_M)$. (This is possible, since t has length M, being a permutation of the length-M sequence s.)

We know that ℓ_1 is the first entry of the sequence $t = (t_1, t_2, \ldots, t_M)$. In other words, $\ell_1 = t_1$. In particular, $M > 0$.

Recall that $(t_1, t_2, \ldots, t_M) = t$ is a permutation of the sequence $(s_1, s_2, \ldots, s_M) = s$. Thus, $f - \Delta t_1 - \Delta t_2 - \cdots - \Delta t_M = f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_M$.

We have

$$\begin{aligned}
&\left(f - \Delta \underbrace{\ell_1}_{=t_1}\right) - \Delta t_2 - \Delta t_3 - \cdots - \Delta t_M \\
&= (f - \Delta t_1) - \Delta t_2 - \Delta t_3 - \cdots - \Delta t_M \\
&= f - \Delta t_1 - \Delta t_2 - \cdots - \Delta t_M = f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_M.
\end{aligned}$$

Hence, the $\mathbb{Z}$-configuration $(f - \Delta \ell_1) - \Delta t_2 - \Delta t_3 - \cdots - \Delta t_M$ is stable (since the $\mathbb{Z}$-configuration $f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_M$ is stable). In other words, the sequence $(t_2, t_3, \ldots, t_M)$ is stabilizing for the $\mathbb{Z}$-configuration $f - \Delta \ell_1$.

On the other hand, the sequence $(\ell_1, \ell_2, \ldots, \ell_N) = \ell$ is legal for f. Hence, for each $i \in \{2, 3, \ldots, N\}$, the vertex ℓ_i is active in the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1}$. In other words, for each $i \in \{2, 3, \ldots, N\}$, the vertex ℓ_i is active in the $\mathbb{Z}$-configuration $(f - \Delta\ell_1) - \Delta\ell_2 - \Delta\ell_3 - \cdots - \Delta\ell_{i-1}$ (since $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1} = (f - \Delta\ell_1) - \Delta\ell_2 - \Delta\ell_3 - \cdots - \Delta\ell_{i-1}$). Renaming i as $i+1$ in this fact, we obtain the following: For each $i \in \{1, 2, \ldots, N-1\}$, the vertex ℓ_{i+1} is active in the $\mathbb{Z}$-configuration $(f - \Delta\ell_1) - \Delta\ell_2 - \Delta\ell_3 - \cdots - \Delta\ell_i$. In other words, the sequence $(\ell_2, \ell_3, \ldots, \ell_N)$ is legal for the $\mathbb{Z}$-configuration $f - \Delta\ell_1$. Moreover, this sequence has length $N - 1 < N$.

Hence, by the induction hypothesis, we can apply Theorem 0.15 to $f - \Delta\ell_1$, $(\ell_2, \ell_3, \ldots, \ell_N)$ and $(t_2, t_3, \ldots, t_M)$ instead of f, ℓ and t. We thus conclude that the sequence $(\ell_2, \ell_3, \ldots, \ell_N)$ is a subpermutation of $(t_2, t_3, \ldots, t_M)$. Hence, Lemma 0.17 **(c)** (applied to $a_i = \ell_i$ and $b_j = t_j$) shows that the sequence $(\ell_1, \ell_2, \ldots, \ell_N)$ is a subpermutation of $(t_1, t_2, \ldots, t_M)$ (since $N > 0$ and $M > 0$ and $\ell_1 = t_1$). In other words, the sequence ℓ is a subpermutation of t (since $\ell = (\ell_1, \ell_2, \ldots, \ell_N)$ and $t = (t_1, t_2, \ldots, t_M)$).

But the sequence t is a permutation of s, and thus a subpermutation of s (by Lemma 0.17 **(a)**). Hence, Lemma 0.17 **(b)** (applied to $a = \ell$, $b = t$ and $c = s$) shows that the sequence ℓ is a subpermutation of s. This completes the induction step. Thus, Theorem 0.15 is proven. □

Corollary 0.18. Let $f : V \to \mathbb{N}$ be any configuration. Let ℓ and ℓ' be two sequences of vertices of D that are both legal and stabilizing for f. Then:

(a) The sequence ℓ' is a permutation of ℓ.

In particular:

(b) The sequences ℓ and ℓ' have the same length.

(c) For each $t \in V$, the number of times t appears in ℓ' equals the number of times t appears in ℓ.

(d) The configuration obtained from f by firing all vertices in ℓ (one after the other) equals the configuration obtained from f by firing all vertices in ℓ' (one after the other).

Proof of Corollary 0.18 (sketched). Theorem 0.15 (applied to ℓ' instead of s) shows that ℓ is a subpermutation of ℓ'. But Theorem 0.15 (applied to ℓ' and ℓ instead of ℓ and s) shows that ℓ' is a subpermutation of ℓ. Hence, Lemma 0.17 **(d)** (applied to $a = \ell'$ and $b = \ell$) shows that ℓ' is a permutation of ℓ. This proves part **(a)** of Corollary 0.18. Parts **(b)** and **(c)** follow immediately from part **(a)**. Part **(d)** also follows from part **(a)**, because we know that the order in which we fire a sequence of vertices does not matter for the final result (so any permutation of a sequence yields the same configuration as the sequence itself). □

0.2.3. Concatenation of legal sequences

For the next lemma, we need a simple piece of notation:

Definition 0.19. **(a)** If $u = (u_1, u_2, \ldots, u_p)$ and $v = (v_1, v_2, \ldots, v_q)$ are two finite sequences, then $u * v$ denote the sequence $(u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q)$. This sequence $u * v$ is called the *concatenation* of the sequences u and v.

(b) If u is a finite sequence, and if $k \in \mathbb{N}$, then u^{*k} denotes the sequence $\underbrace{u * u * \cdots * u}_{k \text{ times}}$. (This is well-defined, since the operation of concatenation is associative. Notice that u^{*0} is the empty sequence $()$.)

Lemma 0.20. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration. Let u be a legal sequence for f. Let g be the $\mathbb{Z}$-configuration obtained from f by firing all vertices in u (that is, $g = f - \Delta u_1 - \Delta u_2 - \cdots - \Delta u_p$, where u is written in the form $u = (u_1, u_2, \ldots, u_p)$). Let v be a legal sequence for g. Then, $u * v$ is a legal sequence for f.

Proof of Lemma 0.20 (sketched). Write the sequence u in the form $u = (u_1, u_2, \ldots, u_p)$. Write the sequence v in the form $v = (v_1, v_2, \ldots, v_q)$. Then, the definition of $u * v$ yields $u * v = (u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q)$.

We want to prove that the sequence $u * v$ is legal for f. In other words, we want to prove that the sequence $(u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q)$ is legal for f (since $u * v = (u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q)$). In other words, we want to prove that if we start with the $\mathbb{Z}$-configuration f, and fire the vertices $u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q$ one by one (from first to last), then each vertex we fire is active at the time of firing.

This process can be subdivided into two phases: The phase where we fire the first p vertices $u_1, u_2, \ldots, u_p$ will be called *phase* 1. It is followed by a phase where we fire the next q vertices, which are $v_1, v_2, \ldots, v_q$; this latter phase will be called *phase* 2.

We must prove that each vertex we fire in our process is active at the time of firing. We call this the *activity claim*. This claim is obvious for the first p vertices being fired (because these are the vertices $u_1, u_2, \ldots, u_p$, and we already know that the sequence $(u_1, u_2, \ldots, u_p) = u$ is legal for f); in other words, the activity claim is obvious for the vertices fired in phase 1. It thus remains to prove the activity claim for the vertices fired in phase 2.

In phase 1, the $\mathbb{Z}$-configuration f is transformed by firing all vertices in u (because the vertices $u_1, u_2, \ldots, u_p$ are precisely the vertices in u). Thus, the $\mathbb{Z}$-configuration obtained at the end of phase 1 is g (because g is **defined** as the $\mathbb{Z}$-configuration obtained from f by firing all vertices in u). In phase 2, this $\mathbb{Z}$-configuration g is transformed further by firing the vertices $v_1, v_2, \ldots, v_q$. Since the sequence $(v_1, v_2, \ldots, v_q)$ is legal for g, we thus conclude that these vertices are active at the time of firing (in phase 2). Hence, the activity claim for the vertices fired in phase 2 is proven. As we have explained, this completes our proof of Lemma 0.20. $\square$

Lemma 0.21. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration. Let $u = (u_1, u_2, \ldots, u_p)$ be a legal sequence for f such that $\Delta u_1 + \Delta u_2 + \cdots + \Delta u_p = 0$. Let $k \in \mathbb{N}$. Then, the sequence u^{*k} is legal for f.

Proof of Lemma 0.21 (sketched). We want to prove that

$$\text{the sequence } u^{*k} \text{ is legal for } f. \tag{2}$$

We shall prove this by induction over k:

Induction base: The sequence $u^{*0} = ()$ (the empty sequence) is clearly legal for f. In other words, (2) holds for $k = 0$. This completes the induction base.

Induction step: Let $K \in \mathbb{N}$. Assume that (2) holds for $k = K$. We must prove that (2) holds for $k = K + 1$.

We have assumed that (2) holds for $k = K$. In other words, the sequence u^{*K} is legal for f.

Let g be the $\mathbb{Z}$-configuration obtained from f by firing all vertices in u. Thus,

$$g = f - \Delta u_1 - \Delta u_2 - \cdots - \Delta u_p = f - \underbrace{\left(\Delta u_1 + \Delta u_2 + \cdots + \Delta u_p\right)}_{=0} = f.$$

Hence, the sequence u^{*K} is legal for g (since the sequence u^{*K} is legal for f). Thus, Lemma 0.20 (applied to $v = u^{*K}$) shows that the sequence $u * u^{*K}$ is legal for f. In other words, the sequence $u^{*(K+1)}$ is legal for f (since $u^{*(K+1)} = u * u^{*K}$). In other words, (2) holds for $k = K + 1$. This completes the induction step.

Thus, (2) is proven. This proves Lemma 0.21. □

Lemma 0.22. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration. Let $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ be a legal sequence for f. Let i and j be two elements of $\{0, 1, \ldots, k\}$ such that $i < j$ and

$$f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_j. \tag{3}$$

Then:

(a) For every $r \in \mathbb{N}$, the sequence $(\ell_1, \ell_2, \ldots, \ell_i) * \left(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j\right)^{*r}$ is legal for f. (This is the sequence which begins with $\ell_1, \ell_2, \ldots, \ell_i$ and then goes on repeating the $j - i$ elements $\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j$ for a total of r times.)

(b) There exist legal sequences (for f) of arbitrary length.

Proof of Lemma 0.22 (sketched). If we subtract $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ from both sides of (3), we obtain

$$0 = -\Delta\ell_{i+1} - \Delta\ell_{i+2} - \cdots - \Delta\ell_j.$$

Thus,

$$\Delta\ell_{i+1} + \Delta\ell_{i+2} + \cdots + \Delta\ell_j = 0. \tag{4}$$

(a) Let $r \in \mathbb{N}$. The sequence $(\ell_1, \ell_2, \ldots, \ell_k)$ is legal for f. Thus, its prefix $(\ell_1, \ell_2, \ldots, \ell_i)$ is also legal for f.

Let g be the $\mathbb{Z}$-configuration obtained from f by firing all vertices in $(\ell_1, \ell_2, \ldots, \ell_i)$. Thus, $g = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$. Then, the sequence $\left(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j\right)$ is le-

gal for g [2]. Hence, Lemma 0.21 (applied to g, $j-i$, $(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)$, ℓ_{i+h} and r instead of f, p, u, u_h and k) shows that the sequence $(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)^{*r}$ is legal for g (because of (4)). Hence, Lemma 0.20 (applied to $(\ell_1, \ell_2, \ldots, \ell_i)$ and $(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)^{*r}$ instead of u and v) yields that $(\ell_1, \ell_2, \ldots, \ell_i) * (\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)^{*r}$ is a legal sequence for f. This proves Lemma 0.22 **(a)**.

(b) We must show that for each $r \in \mathbb{N}$, there exists a legal sequence for f of length r.

So let us fix $r \in \mathbb{N}$. From $i < j$, we obtain $j - i \geq 1$. Lemma 0.22 **(a)** shows that the sequence $(\ell_1, \ell_2, \ldots, \ell_i) * (\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)^{*r}$ is legal for f. This sequence has length $i + r\underbrace{(j-i)}_{\geq 1} \geq \underbrace{i}_{\geq 0} + r \geq r$. Hence, the first r entries of this sequence form a sequence of length r. This sequence is legal for f (being a prefix of the legal sequence $(\ell_1, \ell_2, \ldots, \ell_i) * (\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)^{*r}$). Hence, we have found a legal sequence for f of length r. This proves Lemma 0.22 **(b)**. □

0.2.4. Bounds on legal sequences

This section is of technical nature; it will later be helpful in constructive proofs and algorithms. You should probably skip it at first read.

Let us state a simple lemma for future use:

Lemma 0.23. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration. Let $v_1, v_2, \ldots, v_N$ be any vertices of D. Then,

$$\sum (f - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_N) = \sum f.$$

Proof of Lemma 0.23. We have

$$\begin{aligned} &\sum (f - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_N) \\ &= \sum f - \sum (\Delta v_1) - \sum (\Delta v_2) - \cdots - \sum (\Delta v_N) = \sum f \end{aligned}$$

(since Proposition 0.12 yields $\sum (\Delta v_j) = 0$ for each $j \in \{1, 2, \ldots, N\}$). This proves Lemma 0.23. □

[2] *Proof.* We need to show that for each $p \in \{1, 2, \ldots, j-i\}$, the vertex ℓ_{i+p} is active in the $\mathbb{Z}$-configuration $g - \Delta\ell_{i+1} - \Delta\ell_{i+2} - \cdots - \Delta\ell_{i+p-1}$.

So let $p \in \{1, 2, \ldots, j-i\}$. Then, the vertex ℓ_{i+p} is active in the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i+p-1}$ (since the sequence $(\ell_1, \ell_2, \ldots, \ell_k)$ is legal for f). In view of

$$\begin{aligned} f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i+p-1} &= \underbrace{(f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i)}_{=g} - \Delta\ell_{i+1} - \Delta\ell_{i+2} - \cdots - \Delta\ell_{i+p-1} \\ &= g - \Delta\ell_{i+1} - \Delta\ell_{i+2} - \cdots - \Delta\ell_{i+p-1}, \end{aligned}$$

this rewrites as follows: The vertex ℓ_{i+p} is active in the $\mathbb{Z}$-configuration $g - \Delta\ell_{i+1} - \Delta\ell_{i+2} - \cdots - \Delta\ell_{i+p-1}$. But this is precisely what we needed to show.

Next we state some inequality-type facts about legal sequences:

Lemma 0.24. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let $(\ell_1, \ell_2, \ldots, \ell_N)$ be a legal sequence for f. Then, $\deg^+ (\ell_i) \leq h$ for each $i \in \{1, 2, \ldots, N\}$.

Proof of Lemma 0.24 (sketched). Fix $i \in \{1, 2, \ldots, N\}$.

The sequence $(\ell_1, \ell_2, \ldots, \ell_N)$ is legal for f. Thus, the vertex ℓ_i is active in the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1}$ (by the definition of a legal sequence). In other words, $(f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1})(\ell_i) \geq \deg^+ (\ell_i)$.

But the sequence $(\ell_1, \ell_2, \ldots, \ell_N)$ is legal for f. Thus, $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1}$ is a configuration (since f is a configuration). Moreover, Lemma 0.23 (applied to $i-1$ and ℓ_p instead of N and v_p) yields $\sum (f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1}) = \sum f = h$. Hence, Proposition 0.6 (applied to $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1}$ and ℓ_i instead of f and w) yields $(f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1})(\ell_i) \leq h$. Hence,

$$h \geq (f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1})(\ell_i) \geq \deg^+ (\ell_i).$$

This proves Lemma 0.24. □

Lemma 0.25. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let ℓ be a legal sequence for f.

Let a be an arc of D. Let u be the source of a, and let v be the target of a.

(a) If u appears more than h times in the sequence ℓ, then v must appear at least once in the sequence ℓ.

(b) Fix $k \in \mathbb{N}$. If u appears more than kh times in the sequence ℓ, then v must appear at least k times in the sequence ℓ.

Proof of Lemma 0.25 (sketched). **(b)** Assume that u appears more than kh times in the sequence ℓ. We must prove that v must appear at least k times in the sequence ℓ.

Assume the contrary. Thus, v appears less than k times in the sequence ℓ. In other words, v appears at most $k-1$ times in the sequence ℓ.

Write the sequence ℓ in the form $\ell = (\ell_1, \ell_2, \ldots, \ell_N)$. Then, $(\ell_1, \ell_2, \ldots, \ell_N) = \ell$ is a legal sequence for f. Hence, $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_N$ is a configuration (since f is a configuration). Denote this configuration by f'. Thus, $f' = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_N$.

At least one arc of D has source u and target v (namely, a). Thus, $a_{u,v} \geq 1$. Thus,

$$a_{w,v} \geq [w = u] \qquad \text{for each vertex } w \in V \tag{5}$$

(because if $w \neq u$, then this is obvious, but for $w = u$ it follows from $a_{u,v} \geq 1$).

From $f' = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_N$, we obtain

$$\begin{aligned} \sum f' &= \sum (f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_N) \\ &= \sum f \qquad \text{(by Lemma 0.23, applied to } v_p = \ell_p) \\ &= h. \end{aligned}$$

Hence, Proposition 0.6 (applied to f' and v instead of f and w) yields $f'(v) \leq h$.

Also, $f(v) \geq 0$ (since f is a configuration). Furthermore, every $i \in \{1, 2, \ldots, N\}$ satisfies

$$\begin{aligned}(\Delta\ell_i)(v) &= [v = \ell_i] \underbrace{\deg^+(\ell_i)}_{\substack{\leq h \\ \text{(by Lemma 0.24)}}} - \underbrace{a_{\ell_i, v}}_{\substack{\geq [\ell_i = u] \\ \text{(by (5))}}} \qquad \text{(by the definition of } \Delta\ell_i) \\ &\leq [v = \ell_i] h - [\ell_i = u]. \end{aligned} \tag{6}$$

But $f' = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_N$. Hence,

$$\begin{aligned} f'(v) &= (f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_N)(v) = \underbrace{f(v)}_{\geq 0} - \sum_{i=1}^{N} (\Delta\ell_i)(v) \\ &\geq -\sum_{i=1}^{N} \underbrace{(\Delta\ell_i)(v)}_{\substack{\leq [v=\ell_i]h - [\ell_i = u] \\ \text{(by (6))}}} \geq -\sum_{i=1}^{N} ([v = \ell_i] h - [\ell_i = u]) \\ &= - \underbrace{\sum_{i=1}^{N} [v = \ell_i]}_{\substack{=(\text{the number of all } i \in \{1,2,\ldots,N\} \text{ such that } v = \ell_i) \\ \leq k-1 \\ \text{(since } v \text{ appears at most } k-1 \text{ times in the sequence } \ell)}} h \\ &\quad + \underbrace{\sum_{i=1}^{N} [\ell_i = u]}_{\substack{=(\text{the number of all } i \in \{1,2,\ldots,N\} \text{ such that } \ell_i = u) \\ > kh \\ \text{(since } u \text{ appears more than } kh \text{ times in the sequence } \ell)}} \\ &> -(k-1)h + kh = h. \end{aligned}$$

This contradicts $f'(v) \leq h$. This contradiction proves that our assumption was wrong. Thus, Lemma 0.25 **(b)** is proven.

(a) Lemma 0.25 **(a)** follows by applying Lemma 0.25 **(b)** to $k = 1$. □

Lemma 0.26. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let ℓ be a legal sequence for f.

Let u and v be two vertices of D such that there exists a path of length d from u to v.

If u appears at least $h^0 + h^1 + \cdots + h^d$ times in the sequence ℓ, then v must appear at least once in the sequence ℓ.

Proof of Lemma 0.26. We shall prove Lemma 0.26 by induction over d:

Induction base: If $d = 0$, then $u = v$. Thus, if $d = 0$, then Lemma 0.26 holds for obvious reasons. This completes the induction base.

Induction step: Fix a positive integer e. Assume (as the induction hypothesis) that Lemma 0.26 holds for $d = e - 1$. We must show that Lemma 0.26 holds for $d = e$.

Let f, h, ℓ, u, v and d be as in Lemma 0.26, and assume that $d = e$. Assume that u appears at least $h^0 + h^1 + \cdots + h^d$ times in the sequence ℓ. We must show that v must appear at least once in the sequence ℓ.

There exists a path of length d from u to v. Fix such a path, and let w be its second vertex. (This is well-defined, since this path has length $d = e > 0$.) Then, the path can be split into a single arc from u to w, and a path of length $d - 1$ from w to v. Let us denote this arc from u to w by a. Thus, a is an arc with source u and target w.

We have assumed that u appears at least $h^0 + h^1 + \cdots + h^d$ times in the sequence ℓ. Since

$$h^0 + h^1 + \cdots + h^d = \underbrace{h^0}_{=1>0} + \underbrace{h^1 + h^2 + \cdots + h^d}_{=\left(h^0+h^1+\cdots+h^{d-1}\right)h} > \left(h^0 + h^1 + \cdots + h^{d-1}\right) h,$$

we thus conclude that u appears more than $\left(h^0 + h^1 + \cdots + h^{d-1}\right) h$ times in the sequence ℓ. Thus, Lemma 0.25 **(b)** (applied to $h^0 + h^1 + \cdots + h^{d-1}$ and w instead of k and v) shows that w must appear at least $h^0 + h^1 + \cdots + h^{d-1}$ times in the sequence ℓ.

But $d - 1 = e - 1$ (since $d = e$). Hence, by the induction hypothesis, we can apply Lemma 0.26 to w and $d - 1$ instead of u and d (since there exists a path of length $d - 1$ from w to v). We thus conclude that v must appear at least once in the sequence ℓ. This completes the induction step. Thus, Lemma 0.26 is proven. □

Proposition 0.27. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let ℓ be a legal sequence for f. Let $n = |V|$.

Let q be a vertex of D such that for each vertex $u \in V$, there exists a path from u to q.

If the length of ℓ is $> (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right)$, then q must appear at least once in the sequence ℓ.

Proof of Proposition 0.27. Assume that the length of ℓ is $> (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right)$. We must prove that q must appear at least once in the sequence ℓ.

Assume the contrary. Thus, q never appears in the sequence ℓ. Hence, all entries of ℓ are elements of $V \setminus \{q\}$.

Now, let $u \in V \setminus \{q\}$ be arbitrary. Then,

$$u \text{ appears at most } h^1 + h^2 + \cdots + h^{n-1} \text{ times in the sequence } \ell. \tag{7}$$

[*Proof of (7):* Assume the contrary.[3] Thus, u appears more than $h^1 + h^2 + \cdots + h^{n-1}$ times in the sequence ℓ. In other words, u appears at least $\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$ times in the sequence ℓ.

[3] Yes, this is a proof by contradiction inside a proof by contradiction.

But $u \in V \setminus \{q\} \subseteq V$. Thus, there exists a path from u to q (by one of the assumptions of Proposition 0.27). Consider such a path $\mathbf{p}$. Let d be the length of this path $\mathbf{p}$. The vertices of this path $\mathbf{p}$ are distinct (since it is a path), and belong to the n-element set V. Thus, this path $\mathbf{p}$ has at most n vertices; in other words, it has length $\leq n-1$. In other words, $d \leq n-1$ (since we have denoted the length of the path $\mathbf{p}$ by d). Hence, $n-1 \geq d$.

But recall that u appears at least $\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$ times in the sequence ℓ. Thus, u appears at least

$$\begin{aligned}\left(h^1 + h^2 + \cdots + h^{n-1}\right) + \underbrace{1}_{=h^0} &= \left(h^1 + h^2 + \cdots + h^{n-1}\right) + h^0 = h^0 + h^1 + \cdots + h^{n-1} \\ &\geq h^0 + h^1 + \cdots + h^d \qquad (\text{since } n-1 \geq d)\end{aligned}$$

times in the sequence ℓ. Hence, Lemma 0.26 (applied to $v = q$) yields that q must appear at least once in the sequence ℓ (since there exists a path of length d from u to q (namely, the path $\mathbf{p}$)). This contradicts the fact that q never appears in the sequence ℓ. This proves (7).]

Now, forget that we fixed u. We thus have proven (7) for each $u \in V \setminus \{q\}$.

Notice that $|V \setminus \{q\}| = \underbrace{|V|}_{=n} - 1 = n-1$. In other words, there are $n-1$ elements of $V \setminus \{q\}$.

The sequence ℓ consists of elements of $V \setminus \{q\}$ (since all entries of ℓ are elements of $V \setminus \{q\}$), and contains each of these elements at most $h^1 + h^2 + \cdots + h^{n-1}$ times (by (7)). Hence, it has at most $(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right)$ elements in total (because there are $n-1$ elements of $V \setminus \{q\}$). In other words, the length of ℓ is $\leq (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right)$. This contradicts the fact that the length of ℓ is $> (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right)$. This shows that our assumption was wrong; this proves Proposition 0.27. □

Proposition 0.28. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ be a legal sequence for f. Let $i \in \{0, 1, \ldots, k\}$. Let $g = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ be the configuration obtained from f by firing the vertices $\ell_1, \ell_2, \ldots, \ell_i$.

Then, $g \in \{0, 1, \ldots, h\}^V$. (In other words, $g(v) \in \{0, 1, \ldots, h\}$ for each $v \in V$.)

Proof of Proposition 0.28. Fix $v \in V$.

The sequence ℓ is legal for f. Thus, the sequence $(\ell_1, \ell_2, \ldots, \ell_i)$ (being a prefix of ℓ) is also legal for f. Hence, $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ is a configuration (since f is a configuration). In other words, g is a configuration (since $g = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$). Thus, $g(v) \geq 0$.

On the other hand, from $g = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$, we obtain

$$\begin{aligned}\sum g &= \sum (f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i) \\ &= \sum f \qquad (\text{by Lemma 0.23, applied to } N = i \text{ and } v_p = \ell_p) \\ &= h.\end{aligned}$$

Hence, Proposition 0.6 (applied to g and v instead of f and w) shows that $g(v) \leq h$. Combined with $g(v) \geq 0$, this yields $g(v) \in \{0, 1, \ldots, h\}$.

Now, forget that we fixed v. We thus have shown that $g(v) \in \{0, 1, \ldots, h\}$ for each $v \in V$. In other words, $g \in \{0, 1, \ldots, h\}^V$. This proves Proposition 0.28. □

Proposition 0.29. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let $n = |V|$. Let ℓ be a legal sequence for f. If the sequence ℓ has length $\geq (h+1)^n$, then there exist legal sequences (for f) of arbitrary length.

Proof of Proposition 0.29. Assume that the sequence ℓ has length $\geq (h+1)^n$. We must prove that there exist legal sequences (for f) of arbitrary length.

Write the sequence ℓ in the form $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$. Thus, $k \geq (h+1)^n$ (since ℓ has length $\geq (h+1)^n$).

For each $i \in \{0, 1, \ldots, k\}$, we have $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i \in \{0, 1, \ldots, h\}^V$ (by Proposition 0.28, applied to $g = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$). In other words, all the $k+1$ configurations $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ (with $i \in \{0, 1, \ldots, k\}$) belong to the set $\{0, 1, \ldots, h\}^V$. But the set $\{0, 1, \ldots, h\}^V$ has exactly $(h+1)^n$ elements[4]. Since $k+1 > k \geq (h+1)^n$, we thus conclude (by the pigeonhole principle) that two of the $k+1$ configurations $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ (with $i \in \{0, 1, \ldots, k\}$) must be equal (because all these $k+1$ configurations belong to $\{0, 1, \ldots, h\}^V$). In other words, there exist two elements i and j of $\{0, 1, \ldots, k\}$ satisfying $i < j$ and

$$f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_j.$$

Consider these i and j. Then, Lemma 0.22 **(b)** yields that there exist legal sequences (for f) of arbitrary length. This proves Proposition 0.29. □

0.2.5. Finitary and infinitary configurations, and stabilization

Definition 0.30. Let $f : V \to \mathbb{N}$ be a configuration.

We say that f is *finitary* if there exists a sequence of vertices that is stabilizing for f. Otherwise, we say that f is *infinitary*.

Theorem 0.31. Let $f : V \to \mathbb{N}$ be a configuration. Then, exactly one of the following two statements holds:

- *Statement 1:* The configuration f is finitary.

  There exists a sequence s of vertices that is both legal and stabilizing for f.

  All such sequences are permutations of s.

[4] since

$$\left|\{0, 1, \ldots, h\}^V\right| = |\{0, 1, \ldots, h\}|^{|V|} = (h+1)^n \qquad \text{(because } |\{0, 1, \ldots, h\}| = h+1 \text{ and } |V| = n\text{)}$$

All legal sequences (for f) are subpermutations of s, and in particular are at most as long as s.

- *Statement 2:* The configuration f is infinitary.

  There exists no stabilizing sequence for f.

  There exist legal sequences for f of arbitrary length. More precisely, each legal sequence for f can be extended to a longer legal sequence.

Proof of Theorem 0.31. Set $h = \sum f$ and $n = |V|$.

Consider the following simple algorithm:

- Start with the configuration f.
- Search for an active vertex. If you find one, fire it. Keep doing so until you **either** can no longer find an active vertex **or** you have fired $(h+1)^n$ many vertices[5].
- Say you are in *Case 1* if you have stopped because you could no longer find an active vertex. Say you are in *Case 2* if you have stopped because you have fired $(h+1)^n$ many vertices.

Now, let us consider Case 1 first. In this case, you have stopped because you could no longer find an active vertex. Thus, you have arrived at a stable configuration. Let s be the sequence of vertices you have fired during the algorithm (from first to last). Then, the sequence s is legal (because you have only fired active vertices) and stabilizing (since you have arrived at a stable configuration). Thus, there exists a sequence of vertices that is both legal and stabilizing for f (namely, s). In particular, there exists a sequence of vertices that is stabilizing for f. In other words, f is finitary. Thus, f is not infinitary. Hence, Statement 2 cannot hold.

Furthermore, Corollary 0.18 **(a)** shows that every sequence of vertices that is both legal and stabilizing is a permutation of s. In other words, all such sequences are permutations of s.

Finally, Theorem 0.15 shows that all legal sequences (for f) are subpermutations of s (since s is stabilizing), and therefore in particular are at most as long as s (since a subpermutation of s is always at most as long as s).

Hence, Statement 1 holds. Thus, exactly one of the two Statements 1 and 2 holds (namely, Statement 1, but not Statement 2). Theorem 0.31 is thus proven in Case 1.

Let us now consider Case 2. In this case, you have stopped because you have fired $(h+1)^n$ many vertices. Let ℓ be the sequence of vertices you have fired during the algorithm (from first to last). Then, the sequence ℓ is legal (because you have only fired active vertices) and has length $(h+1)^n$ (since you have fired $(h+1)^n$ many vertices). Hence, Proposition 0.29 shows that there exist legal sequences (for f) of

[5] Of course, many of these vertices will be equal.

arbitrary length. Thus, there exists no stabilizing sequence for f [6]. Hence, the configuration f is infinitary. In other words, f is not finitary. Therefore, Statement 1 does not hold.

Each legal sequence for f can be extended to a longer legal sequence[7]. Thus, Statement 2 holds. Thus, exactly one of the two Statements 1 and 2 holds (namely, Statement 2, but not Statement 1). Theorem 0.31 is thus proven in Case 2.

We have now proven Theorem 0.31 in both Cases 1 and 2. Hence, Theorem 0.31 is always proven. □

How do we actually tell whether a given configuration f is finitary or infinitary? We simply follow the algorithm given in the proof of Proposition 0.29: Keep firing active vertices over and over until either no more active vertices remain, or you have fired $(h+1)^n$ many times (where $h = \sum f$ and $n = |V|$). In the former case, f is finitary (and the sequence of vertices you have fired is a legal stabilizing sequence for f). In the latter case, f is infinitary (by Proposition 0.29, since the sequence of vertices you have fired is a legal sequence of length $\geq (h+1)^n$). Needless to say, this algorithm is extremely slow and inefficient in practice, but to some extent this slowness is unavoidable: A finitary configuration may require a huge number of firings before it stabilizes. Nevertheless, there are faster algorithms in many particular cases; this is a subject of ongoing research.

There are also criteria which, in certain cases, guarantee that a configuration is finitary or infinitary. For example, the following is not hard to show:

Proposition 0.32. Let $f : V \to \mathbb{N}$ be a configuration with $\sum f > |A| - |V|$. Then, f is infinitary.

Proof of Proposition 0.32. Let s be a sequence of vertices that is stabilizing for f. We shall derive a contradiction.

Let g be the configuration obtained from f by firing the vertices in s. Thus, g is stable (since s is stabilizing for f). In other words, there are no active vertices in g.

[6] *Proof.* Let s be a stabilizing sequence for f. Then, there exists a legal sequence ℓ for f that is longer than s (since there exist legal sequences (for f) of arbitrary length). Consider such an ℓ. But Theorem 0.15 shows that ℓ is a subpermutation of s. Hence, ℓ is at most as long as s (since a subpermutation of s is always at most as long as s). This contradicts the fact that ℓ is longer than s.

Now, forget that we fixed s. We thus have found a contradiction for each stabilizing sequence s for f. Thus, there exists no stabilizing sequence for f.

[7] *Proof.* Let ℓ be a legal sequence for f. We must prove that ℓ can be extended to a longer legal sequence.

The sequence ℓ is not stabilizing (since there exists no stabilizing sequence for f). Thus, if we start with f and fire all vertices of ℓ, we end up with a configuration f' that is not stable. Hence, the configuration f' has at least one active vertex; consider such a vertex, and denote it by v. Append the vertex v to the end of the sequence ℓ, and denote the resulting sequence by ℓ'. Then, ℓ' is a legal sequence for f (since v is an active vertex in f'). Thus, ℓ can be extended to a longer legal sequence (namely, to ℓ'). Qed.

Write the sequence s in the form $s = (s_1, s_2, \ldots, s_N)$. Then, $g = f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_N$ (since g is obtained from f by firing the vertices in s). Hence, $\sum g = \sum (f - \Delta s_1 - \Delta s_2 - \cdots - \Delta s_N) = \sum f$ (by Lemma 0.23, applied to $v_i = s_i$).

A well-known fact (shown, e.g., during [Grinbe17a, solution to Exercise 4]) says that $\sum\limits_{v \in V} \deg^+ v = |A|$. Thus,

$$\sum_{v \in V} \left(\deg^+ v - 1\right) = \underbrace{\sum_{v \in V} \deg^+ v}_{=|A|} - \underbrace{\sum_{v \in V} 1}_{=|V|} = |A| - |V| .$$

But the definition of $\sum g$ yields

$$\sum_{v \in V} g(v) = \sum g = \sum f > |A| - |V| = \sum_{v \in V} \left(\deg^+ v - 1\right) . \tag{8}$$

Hence, there exists at least one $v \in V$ satisfying $g(v) > \deg^+ v - 1$ (because otherwise, each $v \in V$ would satisfy $g(v) \leq \deg^+ v - 1$, so that we would get $\sum\limits_{v \in V} \underbrace{g(v)}_{\leq \deg^+ v - 1} \leq \sum\limits_{v \in V} \left(\deg^+ v - 1\right)$, which would contradict (8)). In other words, there exists at least one $v \in V$ satisfying $g(v) \geq \deg^+ v$ (since $g(v)$ and $\deg^+ v$ are integers). In other words, there exists at least one active vertex $v \in V$ in g. This contradicts the fact that there are no active vertices in g.

Now, forget that we fixed s. We thus have obtained a contradiction whenever s is a sequence of vertices that is stabilizing for f. Hence, there exists no sequence of vertices that is stabilizing for f. In other words, f is infinitary. This proves Proposition 0.32. □

Conversely, the results of Exercise 3 can be viewed as bounds on $\sum f$ that guarantee that f must be finitary.

Definition 0.33. Let $f : V \to \mathbb{N}$ be a finitary configuration. Then, Statement 1 in Theorem 0.31 must hold. Therefore, there exists a sequence s of vertices that is both legal and stabilizing for f. The *stabilization* of f means the configuration obtained from f by firing all vertices in s (one after the other). (This does not depend on the choice of s, because of Corollary 0.18 **(d)**.)

The stabilization of f is denoted by f°.

0.2.6. q-stabilization

Something similar holds if we forbid firing a specific vertex:

Definition 0.34. Let $q \in V$. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration.

The $\mathbb{Z}$-configuration f is said to be *q-stable* if no vertex $v \in V$ except (possibly) q is active in f.

So the vertex q may and may not be active in a q-stable configuration; but no other vertex is allowed to be active.

Definition 0.35. Let $q \in V$.

Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration.

Let $(v_1, v_2, \ldots, v_k)$ be a sequence of vertices of D.

(a) The sequence $(v_1, v_2, \ldots, v_k)$ is said to be *q-legal* for f if it is legal and does not contain the vertex q.

(b) The sequence $(v_1, v_2, \ldots, v_k)$ is said to be *q-stabilizing* for f if the $\mathbb{Z}$-configuration $f - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_k$ is q-stable (i.e., has no active vertices except (possibly) q).

We can now define "q-finitary" and "q-infinitary" and obtain an analogue of Theorem 0.31. But the most commonly considered case is that when q is a "global sink" (a vertex with no outgoing arcs, and which is reachable from any vertex), and in this case **every** configuration is q-finitary. Let us state this as its own result:

Theorem 0.36. Let $f : V \to \mathbb{N}$ be a configuration. Let $q \in V$. Assume that for each vertex $u \in V$, there exists a path from u to q. Then, there exists a sequence s of vertices that is both q-legal and q-stabilizing for f. All such sequences are permutations of s. All q-legal sequences (for f) are subpermutations of s, and in particular are at most as long as s.

Before we can prove this, let us state the analogue of Theorem 0.15 for q-legal and q-stabilizing sequences:

Theorem 0.37. Let $f : V \to \mathbb{Z}$ be any $\mathbb{Z}$-configuration. Let $q \in V$. Let ℓ and s be two sequences of vertices of D such that ℓ is q-legal for f while s is q-stabilizing for f. Then, ℓ is a subpermutation of s.

Proof of Theorem 0.37. This proof is completely analogous to the proof of Theorem 0.15 above. (Of course, you should use that none of the ℓ_i can be equal to q.) $\square$

We also get an analogue of Corollary 0.18:

Corollary 0.38. Let $f : V \to \mathbb{N}$ be any configuration. Let $q \in V$. Let ℓ and ℓ' be two sequences of vertices of D that are both q-legal and q-stabilizing for f. Then:

(a) The sequence ℓ' is a permutation of ℓ.

In particular:

(b) The sequences ℓ and ℓ' have the same length.

(c) For each $t \in V$, the number of times t appears in ℓ' equals the number of times t appears in ℓ.

(d) The configuration obtained from f by firing all vertices in ℓ (one after the other) equals the configuration obtained from f by firing all vertices in ℓ' (one after the other).

Proof of Corollary 0.38. This proof is completely analogous to the proof of Corollary 0.18 above. (Of course, Theorem 0.37 needs to be used now instead of Theorem 0.15.) □

Proof of Theorem 0.36. This is somewhat similar to how we proved Theorem 0.31, but there will be a twist.

Set $h = \sum f$ and $n = |V|$.

Consider the following simple algorithm:

- Start with the configuration f.
- Search for an active vertex distinct from q. If you find one, fire it. Keep doing so until you **either** can no longer find an active vertex distinct from q **or** you have fired $(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$ many vertices[8].
- Say you are in *Case 1* if you have stopped because you could no longer find an active vertex distinct from q. Say you are in *Case 2* if you have stopped because you have fired $(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$ many vertices.

Now, let us consider Case 1 first. In this case, you have stopped because you could no longer find an active vertex distinct from q. Thus, you have arrived at a q-stable configuration. Let s be the sequence of vertices you have fired during the algorithm (from first to last). Then, the sequence s is q-legal (because you have only fired active vertices distinct from q) and q-stabilizing (since you have arrived at a q-stable configuration). Thus, there exists a sequence of vertices that is both q-legal and q-stabilizing for f (namely, s). Hence, Corollary 0.38 **(a)** shows that every sequence of vertices that is both q-legal and q-stabilizing is a permutation of s. In other words, all such sequences are permutations of s.

Finally, Theorem 0.37 shows that all q-legal sequences (for f) are subpermutations of s (since s is q-stabilizing), and therefore in particular are at most as long as s (since a subpermutation of s is always at most as long as s). Theorem 0.36 is thus proven in Case 1.

Let us now consider Case 2. In this case, you have stopped because you have fired $(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$ many vertices. Let ℓ be the sequence of vertices you have fired during the algorithm (from first to last). Then, the sequence ℓ is legal (because you have only fired active vertices) and has length $(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$ (since you have fired $(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$ many vertices). Also, the sequence ℓ does not contain the vertex q (since you have only fired vertices distinct from q).

The length of the sequence ℓ is
$(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1 > (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right)$. Hence, Proposition 0.27 shows that q must appear at least once in the sequence ℓ. This contradicts the fact that the sequence ℓ does not contain the vertex q. This contradiction shows that Case 2 cannot happen. Thus, the only possible case is Case 1.

[8] Of course, many of these vertices will be equal.

But we have proven Theorem 0.36 in Case 1. Thus, Theorem 0.36 always holds. □

Definition 0.39. Let $f : V \to \mathbb{N}$ be a configuration. Let $q \in V$. Assume that for each vertex $u \in V$, there exists a path from u to q. Then, Theorem 0.36 shows that there exists a sequence s of vertices that is both q-legal and q-stabilizing for f. The *q-stabilization* of f means the configuration obtained from f by firing all vertices in s (one after the other). (This does not depend on the choice of s, because of Corollary 0.38 **(d)**.)

0.3. Exercise 1: better bounds for legal sequences

The following exercise improves on the bound given in Proposition 0.29 and also on the one given in Proposition 0.27[9]. I don't know whether the improved bounds can be further improved.

Exercise 1. Fix a multidigraph $D = (V, A, \phi)$. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let $n = |V|$. Assume that $n > 0$.

Let $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ be a legal sequence for f having length $k \geq \binom{n+h-1}{n-1}$.

Prove the following:

(a) There exist legal sequences (for f) of arbitrary length.

(b) Let q be a vertex of D such that for each vertex $u \in V$, there exists a path from u to q. Then, q must appear at least once in the sequence ℓ.

[**Hint:** For **(a)**, apply the same pigeonhole-principle argument as for Proposition 0.29.]

In the solution of Exercise 1, the following classical fact will turn out useful:

Proposition 0.40. Let $n \in \mathbb{N}$ and $h \in \mathbb{N}$ be such that $n > 0$. The number of n-tuples $(a_1, a_2, \ldots, a_n)$ of nonnegative integers satisfying $a_1 + a_2 + \cdots + a_n = h$ is $\binom{n+h-1}{n-1}$.

Proof of Proposition 0.40. See, for example, `https://math.stackexchange.com/questions/36250/number-of-monomials-of-certain-degree` for a proof of this fact (in the language of monomials). Or see [Stanle11, §1.2] (search for "weak composition" and read the first paragraph that comes up). Or see [Galvin17, Proposition 13.3] (in the case $h > 0$; but the case $h = 0$ is trivial). □

[9] To see that Exercise 1 **(b)** improves on the bound given in Proposition 0.27, we need to check that $(n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1 \geq \binom{n+h-1}{n-1}$. This is easy for $n \leq 1$ (in fact, the case $n = 0$ is impossible due to the existence of a $q \in V$, and the case $n = 1$ is an equality case). In the remaining case $n \geq 2$, the stronger inequality $1 + h^1 + h^2 + \cdots + h^{n-1} \geq \binom{n+h-1}{n-1}$ can be proven by a simple induction over n.

Hints to Exercise 1. For each $i \in \{0, 1, \ldots, k\}$, the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ is a configuration (since the sequence $(\ell_1, \ell_2, \ldots, \ell_k)$ is legal for f). Moreover, each of the $k+1$ configurations $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ (with $i \in \{0, 1, \ldots, k\}$) has exactly h chips[10].

There exist precisely $\binom{n+h-1}{n-1}$ configurations having exactly h chips[11]. Since $k+1 > k \geq \binom{n+h-1}{n-1}$, we thus conclude (by the pigeonhole principle) that two of the $k+1$ configurations $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ (with $i \in \{0, 1, \ldots, k\}$) must be equal (because all these $k+1$ configurations are configurations having exactly h chips). In other words, there exist two elements i and j of $\{0, 1, \ldots, k\}$ satisfying $i < j$ and

$$f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i = f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_j. \tag{9}$$

Consider these i and j. Hence, Lemma 0.22 **(a)** shows that there exist legal sequences (for f) of arbitrary length. This solves Exercise 1 **(a)**.

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[10] *Proof.* Let $i \in \{0, 1, \ldots, k\}$. Then,

$$\sum \underbrace{(f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i)}_{=f-\sum_{j=1}^{i}\Delta\ell_j} = \sum\left(f - \sum_{j=1}^{i}\Delta\ell_j\right) = \underbrace{\sum f}_{=h} - \sum_{j=1}^{i} \underbrace{\sum(\Delta\ell_j)}_{\substack{=0 \\ \text{(since } \sum(\Delta v)=0 \\ \text{for each vertex } v)}}$$

$$= h - \underbrace{\sum_{j=1}^{i} 0}_{=0} = h.$$

In other words, the configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ has exactly h chips.

[11] *Proof.* Recall that $\mathbb{N}$ denotes the set of nonnegative integers.

Let $v_1, v_2, \ldots, v_n$ be the vertices of D (listed with no repetitions). Then, there is a bijection

$$\begin{aligned} \{\text{configurations on } D\} &\to \mathbb{N}^n, \\ f &\mapsto (f(v_1), f(v_2), \ldots, f(v_n)). \end{aligned}$$

This bijection restricts to a bijection

$$\begin{aligned} \{\text{configurations on } D \text{ having exactly } h \text{ chips}\} &\to \{(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n \mid a_1 + a_2 + \cdots + a_n = h\}, \\ f &\mapsto (f(v_1), f(v_2), \ldots, f(v_n)) \end{aligned}$$

(since the number of chips in a configuration f on D is $\sum f = f(v_1) + f(v_2) + \cdots + f(v_n)$). Hence,

$$\begin{aligned} &|\{\text{configurations on } D \text{ having exactly } h \text{ chips}\}| \\ &= |\{(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n \mid a_1 + a_2 + \cdots + a_n = h\}| \\ &= \binom{n+h-1}{n-1} \qquad \text{(by Proposition 0.40)}. \end{aligned}$$

In other words, there exist precisely $\binom{n+h-1}{n-1}$ configurations having exactly h chips.

(b) Define an $r \in \mathbb{N}$ by $r = (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1$. Let ℓ' be the sequence $(\ell_1, \ell_2, \ldots, \ell_i) * \left(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j\right)^{*r}$. Lemma 0.22 **(b)** thus shows that this sequence ℓ' is legal for f. But the length of this sequence ℓ' is

$$\begin{aligned}\underbrace{i}_{\geq 0} + r \underbrace{(j-i)}_{\substack{\geq 1 \\ (\text{since } i<j)}} &\geq 0 + r \cdot 1 = r = (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right) + 1 \\ &> (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right).\end{aligned}$$

Hence, Proposition 0.27 (applied to ℓ' instead of ℓ) yields that q must appear at least once in the sequence ℓ'. Therefore, q must also appear at least once in the sequence ℓ (since each entry of ℓ' is an entry of ℓ). This solves Exercise 1 **(b)**. $\square$

0.4. Exercise 2: examples of chip-firing

We identify each simple digraph (V, A) with the multidigraph (V, A, ι), where ι is the map $V \to A \times A$ that sends each arc $a \in A$ to itself.

Exercise 2. **(a)** Let D be the following simple digraph:

$$u \longrightarrow v \longrightarrow q$$

(i.e., the digraph D with three vertices u, v, q and two arcs uv, vq).

Let k be a positive integer. Consider the configuration g_k on D which has k chips at u and 0 chips at each other vertex.

Find the q-stabilization of g_k.

(b) Let D be the following simple digraph:

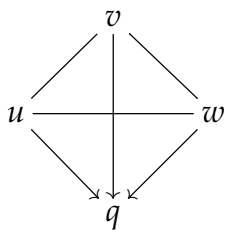
$$u \longrightarrow v \longrightarrow q$$

where a curve without an arrow stands for one arc in each direction. (Thus, formally speaking, the digraph D has three vertices u, v, q and three arcs uv, vu, vq.)

Let k be a positive integer. Consider the configuration g_k on D which has k chips at u and 0 chips at each other vertex.

Find the q-stabilization of g_k.

(c) Let D be the following simple digraph:



where a curve without an arrow stands for one arc in each direction. (Thus, formally speaking, the digraph D has four vertices u, v, w, q and nine arcs $uv, vu, vw, wv, wu, uw, uq, vq, wq$.)

Let $k \geq 2$ be an integer. Consider the configuration f_k on D which has k chips at each vertex (i.e., which has $f_k(v') = k$ for each $v' \in \{u, v, w, q\}$).

Find the q-stabilization of f_k.

Before we approach this exercise, let us introduce a simple notation:

Definition 0.41. If k and k' are two configurations on D, then $k \overset{*}{\to} k'$ shall mean that there exists a legal sequence ℓ for k such that firing all vertices in ℓ (one after the other) transforms k into k'. Thus, we have defined a binary relation $\overset{*}{\to}$ on the set of all configurations of D.

Proposition 0.42. **(a)** The binary relation $\overset{*}{\to}$ is reflexive (i.e., each configuration k satisfies $k \overset{*}{\to} k$) and transitive (i.e., every three configurations k_1, k_2 and k_3 satisfying $k_1 \overset{*}{\to} k_2$ and $k_2 \overset{*}{\to} k_3$ satisfy $k_1 \overset{*}{\to} k_3$).

(b) If c, k and k' are three configurations satisfying $k \overset{*}{\to} k'$, then $c + k \overset{*}{\to} c + k'$.

Proof of Proposition 0.42 (sketched). **(a)** Each configuration k satisfies $k \overset{*}{\to} k$ (because the empty sequence $()$ is legal for k, and clearly firing all vertices in the empty sequence $()$ transforms k into k). Thus, the relation $\overset{*}{\to}$ is reflexive. It now remains to prove that the relation $\overset{*}{\to}$ is transitive.

Let k_1, k_2 and k_3 be three configurations satisfying $k_1 \overset{*}{\to} k_2$ and $k_2 \overset{*}{\to} k_3$. We shall show that $k_1 \overset{*}{\to} k_3$.

From $k_1 \overset{*}{\to} k_2$, we conclude that there exists a legal sequence ℓ_1 for k_1 such that firing all vertices in ℓ_1 transforms k_1 into k_2. Consider this ℓ_1.

From $k_2 \overset{*}{\to} k_3$, we conclude that there exists a legal sequence ℓ_2 for k_2 such that firing all vertices in ℓ_2 transforms k_2 into k_3. Consider this ℓ_2.

Write the sequences ℓ_1 and ℓ_2 as $\ell_1 = (v_1, v_2, \ldots, v_p)$ and $\ell_2 = (w_1, w_2, \ldots, w_q)$. Let ℓ be the sequence $\ell_1 * \ell_2 = (v_1, v_2, \ldots, v_p, w_1, w_2, \ldots, w_q)$. (We are using the notation from Definition 0.19 here.) Then, firing all vertices in ℓ transforms k_1 into k_3 (because the first p firings transform k_1 into k_2, and from there on the remaining q firings take us to k_3). Moreover, this sequence $\ell = \ell_1 * \ell_2$ is legal for k_1 (by Lemma 0.20, applied to k_1, ℓ_1, k_2 and ℓ_2 instead of f, u, g and v).

We thus have found a legal sequence ℓ for k_1 such that firing all vertices in ℓ (one after the other) transforms k_1 into k_3. Hence, such a legal sequence ℓ exists. In other words, $k_1 \overset{*}{\to} k_3$ (by the definition of the relation $\overset{*}{\to}$).

We thus have shown that every three configurations k_1, k_2 and k_3 satisfying $k_1 \overset{*}{\to} k_2$ and $k_2 \overset{*}{\to} k_3$ satisfy $k_1 \overset{*}{\to} k_3$. In other words, the relation $\overset{*}{\to}$ is transitive. This proves Proposition 0.42 **(a)**.

(b) Let c, k and k' be three configurations satisfying $k \overset{*}{\to} k'$.

From $k \overset{*}{\to} k'$, we conclude that there exists a legal sequence ℓ for k such that firing all vertices in ℓ transforms k into k'. Consider this ℓ.

The sequence ℓ is legal for k, and thus also for $c + k$ (since $(c + k)(v) = \underbrace{c(v)}_{\geq 0} + k(v) \geq k(v)$ for each vertex v of D). Moreover, firing all vertices in ℓ transforms $c + k$ into $c + k'$ (since firing all vertices in ℓ transforms k into k'). Hence, we have found a legal sequence ℓ for $c + k$ such that firing all vertices in ℓ (one after the other) transforms $c + k$ into $c + k'$. Hence, such a legal sequence ℓ exists. In other words, $c + k \overset{*}{\to} c + k'$ (by the definition of the relation $\overset{*}{\to}$). This proves Proposition 0.42 **(b)**. $\square$

Similar facts hold for q-stabilization:

Definition 0.43. Let q be a vertex of D. If k and k' are two configurations on D, then $k \underset{q}{\overset{*}{\to}} k'$ shall mean that there exists a q-legal sequence ℓ for k such that firing all vertices in ℓ (one after the other) transforms k into k'. Thus, we have defined a binary relation $\underset{q}{\overset{*}{\to}}$ on the set of all configurations of D.

Proposition 0.44. Let q be a vertex of D.

(a) The binary relation $\overset{*}{\underset{q}{\to}}$ is reflexive (i.e., each configuration k satisfies $k \overset{*}{\underset{q}{\to}} k$) and transitive (i.e., every three configurations k_1, k_2 and k_3 satisfying $k_1 \overset{*}{\underset{q}{\to}} k_2$ and $k_2 \overset{*}{\underset{q}{\to}} k_3$ satisfy $k_1 \overset{*}{\underset{q}{\to}} k_3$).

(b) If c, k and k' are three configurations satisfying $k \overset{*}{\underset{q}{\to}} k'$, then $c + k \overset{*}{\underset{q}{\to}} c + k'$.

Proof of Proposition 0.44 (sketched). Analogous to the proof of Proposition 0.42. □

Hints to Exercise 2. **(a)** We shall write each configuration f on D as the triple $(f(u), f(v), f(q))$. Thus, $g_k = (k, 0, 0)$.

For each $h \in \{0, 1, \ldots, k\}$, define a configuration z_h by $z_h = (k - h, 0, h)$.

Now, for each $h \in \{0, 1, \ldots, k-1\}$, we have

$$z_h \overset{*}{\underset{q}{\to}} z_{h+1}$$

[12]. Hence,

$$z_0 \overset{*}{\underset{q}{\to}} z_1 \overset{*}{\underset{q}{\to}} z_2 \overset{*}{\underset{q}{\to}} z_3 \overset{*}{\underset{q}{\to}} \cdots \overset{*}{\underset{q}{\to}} z_{k-1} \overset{*}{\underset{q}{\to}} z_k.$$

Since the relation $\overset{*}{\underset{q}{\to}}$ is transitive, we thus have $z_0 \overset{*}{\underset{q}{\to}} z_k$. In other words, $g_k \overset{*}{\underset{q}{\to}} (0, 0, k)$ (since $z_0 = (k - 0, 0, 0) = (k, 0, 0) = g_k$ and $z_k = (k - k, 0, k) = (0, 0, k)$). Since the configuration $(0, 0, k)$ is q-stable, this shows that $(0, 0, k)$ is the q-stabilization of g_k.

(b) We shall write each configuration f on D as the triple $(f(u), f(v), f(q))$. Thus, $g_k = (k, 0, 0)$.

For each $h \in \{0, 1, \ldots, k\}$, define a configuration z_h by $z_h = (k - h, 0, h)$.

Now, for each $h \in \{0, 1, \ldots, k-2\}$, we have

$$z_h \overset{*}{\underset{q}{\to}} z_{h+1}$$

[13]. Hence,

$$z_0 \overset{*}{\underset{q}{\to}} z_1 \overset{*}{\underset{q}{\to}} z_2 \overset{*}{\underset{q}{\to}} z_3 \overset{*}{\underset{q}{\to}} \cdots \overset{*}{\underset{q}{\to}} z_{k-2} \overset{*}{\underset{q}{\to}} z_{k-1}.$$

[12] *Proof.* Let $h \in \{0, 1, \ldots, k-1\}$. The sequence (u, v) is q-legal for z_h, and firing all vertices in this sequence transforms z_h into z_{h+1} (because this firing process looks as follows:

$$z_h = (k - h, 0, h) \overset{\text{fire } u}{\longrightarrow} (k - h - 1, 1, h) \overset{\text{fire } v}{\longrightarrow} (k - h - 1, 0, h + 1) = (k - (h + 1), 0, h + 1) = z_{h+1}$$

). Hence, $z_h \overset{*}{\underset{q}{\to}} z_{h+1}$.

[13] *Proof.* Let $h \in \{0, 1, \ldots, k-2\}$. The sequence (u, u, v) is q-legal for z_h, and firing all vertices in this

[14] Since the relation $\underset{q}{\overset{*}{\to}}$ is transitive and reflexive, we thus have $z_0 \underset{q}{\overset{*}{\to}} z_{k-1}$. In other words, $g_k \underset{q}{\overset{*}{\to}} (1,0,k-1)$ (since $z_0 = (k-0,0,0) = (k,0,0) = g_k$ and $z_{k-1} = (k-(k-1),0,k-1) = (1,0,k-1)$).

Combining this with $(1,0,k-1) \underset{q}{\overset{*}{\to}} (0,1,k-1)$ (because firing the legal sequence (u) transforms the configuration $(1,0,k-1)$ into $(0,1,k-1)$), we obtain $g_k \underset{q}{\overset{*}{\to}} (0,1,k-1)$. Since the configuration $(0,1,k-1)$ is q-stable, this shows that $(0,1,k-1)$ is the q-stabilization of g_k.

(c) We shall write each configuration f on D as the 4-tuple $(f(u), f(v), f(w), f(q))$. Thus, $f_k = (k,k,k,k)$.

For each $h \in \{0,1,\ldots,k\}$, define a configuration z_h by $z_h = (k-h, k-h, k-h, k+3h)$.

Now, for each $h \in \{0,1,\ldots,k-3\}$, we have

$$z_h \underset{q}{\overset{*}{\to}} z_{h+1}$$

[15]. Hence,

$$z_0 \underset{q}{\overset{*}{\to}} z_1 \underset{q}{\overset{*}{\to}} z_2 \underset{q}{\overset{*}{\to}} z_3 \underset{q}{\overset{*}{\to}} \cdots \underset{q}{\overset{*}{\to}} z_{k-3} \underset{q}{\overset{*}{\to}} z_{k-2}.$$

[16] Since the relation $\underset{q}{\overset{*}{\to}}$ is transitive and reflexive, we thus have $z_0 \underset{q}{\overset{*}{\to}} z_{k-2}$. In other words, $f_k \underset{q}{\overset{*}{\to}} (2,2,2,4k-6)$ (since $z_0 = (k-0, k-0, k-0, k+3\cdot 0) = (k,k,k,k) = f_k$ and $z_{k-2} = (k-(k-2), k-(k-2), k-(k-2), k+3(k-2)) = (2,2,2,4k-6)$). Since the configuration $(2,2,2,4k-6)$ is q-stable, this shows that $(2,2,2,4k-6)$ is the q-stabilization of f_k. $\square$

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sequence transforms z_h into z_{h+1} (because this firing process looks as follows:

$$\begin{aligned} z_h &= (k-h,0,h) \overset{\text{fire } u}{\longrightarrow} (k-h-1,1,h) \overset{\text{fire } u}{\longrightarrow} (k-h-2,2,h) \\ &\overset{\text{fire } v}{\longrightarrow} (k-h-1,0,h+1) = (k-(h+1),0,h+1) = z_{h+1} \end{aligned}$$

). Hence, $z_h \underset{q}{\overset{*}{\to}} z_{h+1}$.

[14] This chain of relations can consist of a single configuration (and 0 relation signs) when $k=1$. There is nothing wrong about this!

[15] *Proof.* Let $h \in \{0,1,\ldots,k-3\}$. The sequence (u,v,w) is q-legal for z_h, and firing all vertices in this sequence transforms z_h into z_{h+1} (because this firing process looks as follows:

$$\begin{aligned} z_h &= (k-h, k-h, k-h, k+3h) \\ &\overset{\text{fire } u}{\longrightarrow} (k-h-3, k-h+1, k-h+1, k+3h+1) \overset{\text{fire } v}{\longrightarrow} (k-h-2, k-h-2, k-h+2, k+3h+2) \\ &\overset{\text{fire } w}{\longrightarrow} (k-h-1, k-h-1, k-h-1, k+3h+3) \\ &= (k-(h+1), k-(h+1), k-(h+1), k+3(h+1)) = z_{h+1} \end{aligned}$$

). Hence, $z_h \underset{q}{\overset{*}{\to}} z_{h+1}$.

[16] This chain of relations can consist of a single configuration (and 0 relation signs) when $k=2$. There is nothing wrong about this!

0.5. Exercise 3: a lower bound on the degree of an infinitary configuration

0.5.1. The exercise

Exercise 3. Assume that the multidigraph D is strongly connected. Let $f : V \to \mathbb{N}$ be an infinitary configuration.

(a) Prove that D cannot have more than $\sum f$ vertex-disjoint cycles. (A set of cycles is said to be *vertex-disjoint* if no two distinct cycles in the set have a vertex in common.)

(b) Prove that D cannot have more than $\sum f$ arc-disjoint cycles. (A set of cycles is said to be *arc-disjoint* if no two distinct cycles in the set have an arc in common.)

Exercise 3 **(b)** is [BjoLov92, Theorem 2.2], but the proof given there is vague and unrigorous.

0.5.2. Solution to part (a)

We shall first solve Exercise 3 **(a)**. First, let us introduce a simple notation:

Definition 0.45. Let $g : V \to \mathbb{N}$ be a configuration. Let c be a cycle of D. We say that the cycle c is *non-void* in g if and only if there exists at least one vertex v on c satisfying $g(v) \geq 1$. (In other words, the cycle c is non-void in g if and only if at least one vertex of c has at least one chip in g.)

In the following lemma, we shall use the notation from Definition 0.41.

Lemma 0.46. Let $g : V \to \mathbb{N}$ and $g' : V \to \mathbb{N}$ be two configurations such that $g \overset{*}{\to} g'$. Let c be a cycle of D such that c is non-void in g. Then, c is non-void in g'.

Proof of Lemma 0.46 (sketched). We must merely show that whenever we fire a vertex w in g, the cycle c remains non-void. But this is easy:

We know that there exists at least one vertex v on c satisfying $g(v) \geq 1$ (since c is non-void in g). Consider this v. Now:

- If $w = v$, then the vertex that follows v on the cycle c gains at least one chip when we fire w; thus, the cycle c remains non-void in this case.
- If $w \neq v$, then the vertex v does not lose any chips when we fire w; thus, the cycle c remains non-void in this case as well.

This concludes the proof of Lemma 0.46. □

Hints to Exercise 3. **(a)** Let $h = \sum f$. Thus, we must prove that D cannot have more than h vertex-disjoint cycles.

The configuration f is infinitary. Thus, there exist arbitrarily long legal sequences for f. In particular, there exists a legal sequence $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ for f of length $k > (n-1)\left(h^1 + h^2 + \cdots + h^{n-1}\right)$. Consider this ℓ.

Each vertex of D appears at least once in the sequence ℓ [17].

Let f' be the configuration obtained from f by firing the vertices in ℓ (one after the other). Each cycle of D is non-void in f' [18].

Recall that we must prove that D cannot have more than h vertex-disjoint cycles.

Assume the contrary. Thus, D has more than h vertex-disjoint cycles. In other words, there exist $h+1$ vertex-disjoint cycles $c_1, c_2, \ldots, c_{h+1}$. Consider these cycles $c_1, c_2, \ldots, c_{h+1}$. All these cycles are non-void in f' (since each cycle of D is non-void in f'), and of course are vertex-disjoint. Thus, the configuration f' has $h+1$ vertex-disjoint non-void cycles. Thus, it has a chip on each of these cycles. Consequently, it has at least $h+1$ chips. In other words, $\sum f' \geq h+1$.

But f' is obtained from f by firing vertices. Hence, $\sum f' = \sum f$ (since firing vertices does not change the total number of chips). Thus, $\sum f = \sum f' \geq h+1$, which contradicts $\sum f = h < h+1$. This contradiction proves that our assumption was wrong, qed. $\square$

0.5.3. Solution to part (b)

Our next goal is to solve Exercise 3 **(b)**. Our solution is going to be a rigorous version of [BjoLov92, proof of Theorem 2.2].[19] We first need to prepare by introducing notations and showing lemmas.

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[17] *Proof.* Let q be a vertex of D. We must show that q appears at least once in the sequence ℓ.

For each vertex $u \in V$, there exists a path from u to q (since D is strongly connected). Hence, Proposition 0.27 shows that q appears at least once in the sequence ℓ.

[18] *Proof.* Let c be a cycle of D. We must then show that c is non-void in f'.

For each $i \in \{0, 1, \ldots, k\}$, we define f_i to be the configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ obtained from f by firing the vertices $\ell_1, \ell_2, \ldots, \ell_i$ (in this order). (This is indeed a configuration, since the sequence ℓ is legal.) Thus, $f_0 \overset{*}{\to} f_1 \overset{*}{\to} \cdots \overset{*}{\to} f_k$. Notice that $f_0 = f$ and $f_k = f'$ (by the definition of f').

Pick a vertex v on the cycle c. Then, v appears at least once in the sequence ℓ (since each vertex of D appears at least once in the sequence ℓ). In other words, there exists some $i \in \{1, 2, \ldots, k\}$ satisfying $\ell_i = v$. Consider this i.

The vertex v lies on the cycle c, and thus there exists at least one arc with source v (namely, the arc of the cycle c emanating from v). In other words, $\deg^+ v \geq 1$.

The sequence $(\ell_1, \ell_2, \ldots, \ell_k)$ is legal for f. Thus, the vertex ℓ_i is active in the configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{i-1} = f_{i-1}$ (since this is how f_{i-1} is defined). In other words, $f_{i-1}(\ell_i) \geq \deg^+(\ell_i)$. In other words, $f_{i-1}(v) \geq \deg^+(v)$ (since $\ell_i = v$). Hence, $f_{i-1}(v) \geq \deg^+(v) \geq 1$. Thus, the cycle c is non-void in f_{i-1}. But $f_{i-1} \overset{*}{\to} f_k$ (since $f_0 \overset{*}{\to} f_1 \overset{*}{\to} \cdots \overset{*}{\to} f_k$, and since the relation $\overset{*}{\to}$ is transitive). Hence, the cycle c is non-void in f_k (by Lemma 0.46, applied to $g = f_{i-1}$ and $g' = f_k$). In other words, the cycle c is non-void in f' (since $f_k = f'$). Qed.

[19] This is one of those cases where making a proof rigorous is difficult and makes the proof much longer.

We begin by introducing some notations:

Definition 0.47. Let $f : V \to \mathbb{Z}$ and $g : V \to \mathbb{Z}$ be two $\mathbb{Z}$-configurations on a multidigraph $D = (V, A, \phi)$. Then, we say that $f \geq g$ if and only if each $v \in V$ satisfies $f(v) \geq g(v)$.

Thus, a $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ is a configuration if and only if $f \geq 0$ (where 0 means the "zero configuration" $0 : V \to \mathbb{Z}$).

Definition 0.48. Let $v \in V$ be a vertex of a multidigraph $D = (V, A, \phi)$. Then, a configuration $\delta_v : V \to \mathbb{N}$ is defined by setting

$$\delta_v(w) = \begin{cases} 1, & \text{if } w = v; \\ 0, & \text{if } w \neq v \end{cases} \qquad \text{for all } w \in V.$$

(Roughly speaking, δ_v is the configuration having 1 chip at vertex v and no further chips.)

Thus, each $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ satisfies $f = \sum\limits_{v \in V} f(v)\, \delta_v$. Notice that $\sum \delta_v = 1$ for each $v \in V$.

Definition 0.49. Consider a multidigraph $D = (V, A, \phi)$.

(a) If c is a cycle of D, then $\mathrm{V}(c)$ shall mean the set of all vertices on c.

(b) Let K be a set of cycles of D. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration on D. We say that f is *K-captured* if and only if we can choose a vertex $v_c \in \mathrm{V}(c)$ for each cycle $c \in K$ such that $f \geq \sum\limits_{c \in K} \delta_{v_c}$.

The notion of "K-captured" $\mathbb{Z}$-configurations is somewhat subtle. Intuitively speaking, a $\mathbb{Z}$-configuration f is K-captured if and only if (pretending that the chips in f are distinguishable) we can assign to each cycle $c \in K$ a specific chip in f that lies on a vertex of c, without having to assign the same chip to two different cycles. Notice that the vertices v_c in Definition 0.49 **(b)** need not be distinct. (Intuitively speaking, this means that the chips assigned to different cycles may lie on the same vertices – but they must not be the same chip.)

We notice that if a $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ is K-captured for some set K of cycles of D, then f is a configuration (since the vertices v_c from Definition 0.49 **(b)** satisfy $f \geq \sum\limits_{c \in K} \delta_{v_c} \geq 0$). Also, we notice that a $\mathbb{Z}$-configuration $f : V \to \mathbb{Z}$ is $\varnothing$-captured if and only if f is a configuration.

Lemma 0.50. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration on a multidigraph $D = (V, A, \phi)$. Let K be a set of cycles of D such that f is K-captured. Then, $\sum f \geq |K|$.

Proof of Lemma 0.50. We know that f is K-captured. In other words, we can choose a vertex $v_c \in \mathrm{V}(c)$ for each cycle $c \in K$ such that $f \geq \sum\limits_{c \in K} \delta_{v_c}$. Consider these vertices v_c.

Now,

$$\begin{aligned}\sum f &= \sum_{p \in V} \underbrace{f(p)}_{\substack{\geq \left(\sum\limits_{c \in K} \delta_{v_c}\right)(p) \\ (\text{since } f \geq \sum\limits_{c \in K} \delta_{v_c})}} \geq \sum_{p \in V} \left(\sum_{c \in K} \delta_{v_c}\right)(p) = \sum_{p \in V} \sum_{c \in K} \delta_{v_c}(p) \\ &= \sum_{c \in K} \underbrace{\sum_{p \in V} \delta_{v_c}(p)}_{\substack{=\sum \delta_{v_c} = 1 \\ (\text{since } \sum \delta_v = 1 \text{ for each } v \in V)}} = \sum_{c \in K} 1 = |K|.\end{aligned}$$

This proves Lemma 0.50. □

The following lemma will be crucial:

Lemma 0.51. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration on a multidigraph $D = (V, A, \phi)$. Let $w \in V$ be a vertex such that w is active in f. Let C be a set of arc-disjoint cycles of D. Let $C_w = \{c \in C \mid w \in \mathrm{V}(c)\}$. Let $K \subseteq C$ be such that f is K-captured. Then, the $\mathbb{Z}$-configuration $f - \Delta w$ is $K \cup C_w$-captured.

Proof of Lemma 0.51. We know that f is K-captured. In other words, we can choose a vertex $v_c \in \mathrm{V}(c)$ for each cycle $c \in K$ such that $f \geq \sum\limits_{c \in K} \delta_{v_c}$. Consider these vertices v_c.

Notice that $K \cup C_w \subseteq C$ (since $K \subseteq C$ and $C_w \subseteq C$).

For each $c \in K \cup C_w$, we define a vertex $u_c \in \mathrm{V}(c)$ by

$$u_c = \begin{cases} v_c, & \text{if } w \notin \mathrm{V}(c); \\ (\text{the vertex succeeding } w \text{ on the cycle } c), & \text{if } w \in \mathrm{V}(c) \end{cases}.$$

(This definition makes sense, because v_c is a well-defined element of $\mathrm{V}(c)$ whenever $w \notin \mathrm{V}(c)$ [20].)

Now, we claim that

$$(f - \Delta w)(p) \geq \left(\sum_{c \in K \cup C_w} \delta_{u_c}\right)(p) \qquad \text{for every } p \in V. \tag{10}$$

[*Proof of (10):* Let $p \in V$. We must prove the inequality (10).

[20] *Proof.* Assume that $w \notin \mathrm{V}(c)$. Then, $c \notin C_w$ (by the definition of C_w). Combining this with $c \in K \cup C_w$, we obtain $c \in (K \cup C_w) \setminus C_w \subseteq K$. Hence, v_c is a well-defined element of $\mathrm{V}(c)$.

Notice that

$$\left(\sum_{c\in K\cup C_w}\delta_{u_c}\right)(p) = \sum_{c\in K\cup C_w}\underbrace{\delta_{u_c}(p)}_{=\begin{cases}1, & \text{if } p=u_c;\\ 0, & \text{if } p\neq u_c\end{cases}} = \sum_{c\in K\cup C_w}\begin{cases}1, & \text{if } p=u_c;\\ 0, & \text{if } p\neq u_c\end{cases}$$
$$= \left|\{c\in K\cup C_w \mid p=u_c\}\right|. \tag{11}$$

The definition of Δw yields

$$(\Delta w)(p) = [p=w]\deg^+ w - a_{w,p}. \tag{12}$$

Let $Q = \{c\in K\setminus C_w \mid p=v_c\}$. Then, it is not hard to see that

$$f(p) - [p=w]\deg^+ w \geq |Q| \tag{13}$$

[21].

[21] *Proof of (13):* We are in one of the following two cases:

Case 1: We have $p=w$.

Case 2: We have $p\neq w$.

First, let us consider Case 1. In this case, we have $p=w$. Recall that w is active in f. Thus, $f(w)\geq \deg^+ w$.

But there is no $c\in K\setminus C_w$ satisfying $p=v_c$. (In fact, if such a c would exist, then it would satisfy $w=p=v_c\in \mathrm{V}(c)$, which (in view of $c\in K\setminus C_w\subseteq K\subseteq C$) would yield $c\in C_w$ (by the definition of C_w), which would contradict $c\in K\setminus C_w$.)

Now, $Q=\{c\in K\setminus C_w \mid p=v_c\}=\varnothing$ (since there is no $c\in K\setminus C_w$ satisfying $p=v_c$). Hence, $|Q|=|\varnothing|=0$. Now,

$$f\left(\underbrace{p}_{=w}\right) - \underbrace{[p=w]}_{\substack{=1\\ (\text{since } p=w)}}\deg^+ w = f(w)-\deg^+ w\geq 0 \qquad \left(\text{since } f(w)\geq \deg^+ w\right)$$
$$= |Q|.$$

Thus, (13) is proven in Case 1.

Let us now consider Case 2. In this case, we have $p\neq w$. But $Q=\left\{c\in \underbrace{K\setminus C_w}_{\subseteq K} \mid p=v_c\right\}\subseteq \{c\in K \mid p=v_c\}$, so that $|Q|\leq |\{c\in K \mid p=v_c\}|$.

Recall that $f\geq \sum_{c\in K}\delta_{v_c}$. Hence,

$$f(p)\geq \left(\sum_{c\in K}\delta_{v_c}\right)(p) = \sum_{c\in K}\underbrace{\delta_{v_c}(p)}_{=\begin{cases}1, & \text{if } p=v_c;\\ 0, & \text{if } p\neq v_c\end{cases}} = \sum_{c\in K}\begin{cases}1, & \text{if } p=v_c;\\ 0, & \text{if } p\neq v_c\end{cases}$$
$$= |\{c\in K \mid p=v_c\}| \geq |Q| \qquad \left(\text{since } |Q|\leq |\{c\in K \mid p=v_c\}|\right).$$

On the other hand, let $R = \{c \in K \cup C_w \mid p = u_c\} \setminus Q$. Then, the sets Q and R are disjoint, and satisfy $\{c \in K \cup C_w \mid p = u_c\} \subseteq Q \cup R$. Hence,

$$|\{c \in K \cup C_w \mid p = u_c\}| \leq |Q \cup R| = |Q| + |R| \tag{14}$$

(since Q and R are disjoint). Moreover, $R \subseteq C$ [22]. Thus, any two cycles in R are arc-disjoint[23].

But if c is an element of R, then the arc with source w that belongs to the cycle c is a well-defined element of $\{\text{arcs of } D \text{ having source } w \text{ and target } p\}$ [24]. Hence, the map

$$\begin{aligned} R &\to \{\text{arcs of } D \text{ having source } w \text{ and target } p\}, \\ c &\mapsto (\text{the arc with source } w \text{ that belongs to the cycle } c) \end{aligned}$$

is well-defined. This map is furthermore injective[25]. Hence,

$$|R| \leq |\{\text{arcs of } D \text{ having source } w \text{ and target } p\}| = a_{w,p}$$

Now,

$$f(p) - \underbrace{[p = w]}_{\substack{=0 \\ (\text{since } p \neq w)}} \deg^+ w = f(p) \geq |Q|.$$

Thus, (13) is proven in Case 2.

Hence, we have proven (13) in both possible cases 1 and 2. This completes the proof of (13).

[22] *Proof.* We have $R = \{c \in K \cup C_w \mid p = u_c\} \setminus Q \subseteq \{c \in K \cup C_w \mid p = u_c\} \subseteq K \cup C_w \subseteq C$.

[23] *Proof.* We have $R \subseteq C$. Hence, any two cycles in R are arc-disjoint (since C is a set of arc-disjoint cycles).

[24] *Proof.* Recall that

$$R = \{c \in K \cup C_w \mid p = u_c\} \setminus Q = \{c \in K \cup C_w \mid p = u_c \text{ and } c \notin Q\}. \tag{15}$$

Now, let c be an element of R. Thus, c is an element of $K \cup C_w$ satisfying $p = u_c$ and $c \notin Q$ (by (15)).

Assume (for the sake of contradiction) that $w \notin \mathrm{V}(c)$. Hence, $c \notin C_w$ (by the definition of C_w, since $c \in K \cup C_w \subseteq C$). Combining this with $c \in K \cup C_w$, we obtain $c \in (K \cup C_w) \setminus C_w = K \setminus C_w$.

But the definition of u_c yields $u_c = v_c$ (since $w \notin \mathrm{V}(c)$). Thus, $p = u_c = v_c$. Thus, $c \in K \setminus C_w$ and $p = v_c$. Hence, $c \in Q$ (by the definition of Q). This contradicts $c \notin Q$. This contradiction shows that our assumption (that $w \notin \mathrm{V}(c)$) was false. Hence, we must have $w \in \mathrm{V}(c)$. In other words, the vertex w lies on the cycle c. Thus, the arc with source w that belongs to the cycle c is well-defined.

It remains to prove that this arc is an element of $\{\text{arcs of } D \text{ having source } w \text{ and target } p\}$. In other words, it remains to prove that this arc has source w and target p. Since it clearly has source w, we thus only need to show that it has target p.

We have $w \in \mathrm{V}(c)$. The definition of u_c thus yields $u_c = (\text{the vertex succeeding } w \text{ on the cycle } c)$.

Hence, $p = u_c = (\text{the vertex succeeding } w \text{ on the cycle } c)$. Therefore, the arc with source w that belongs to the cycle c has target p. This completes our proof.

[25] *Proof.* This is because any two cycles in R are arc-disjoint.

(by the definition of $a_{w,p}$). Thus, (14) becomes

$$\begin{aligned}
|\{c \in K \cup C_w \mid p = u_c\}| &\leq \underbrace{|Q|}_{\substack{\leq f(p) - [p=w]\deg^+ w \\ \text{(by (13))}}} + \underbrace{|R|}_{\leq a_{w,p}} \leq f(p) - [p = w]\deg^+ w + a_{w,p} \\
&= f(p) - \underbrace{\left([p = w]\deg^+ w - a_{w,p}\right)}_{\substack{=(\Delta w)(p) \\ \text{(by (12))}}} = f(p) - (\Delta w)(p) \\
&= (f - \Delta w)(p).
\end{aligned}$$

Hence,

$$(f - \Delta w)(p) \geq |\{c \in K \cup C_w \mid p = u_c\}| = \left(\sum_{c \in K \cup C_w} \delta_{u_c}\right)(p)$$

(by (11)). Hence, (10) is proven.]

From (10), we obtain $f - \Delta w \geq \sum_{c \in K \cup C_w} \delta_{u_c}$.

Thus, the $\mathbb{Z}$-configuration $f - \Delta w$ is $K \cup C_w$-captured (since $u_c \in \mathrm{V}(c)$ for each $c \in K \cup C_w$). This proves Lemma 0.51. □

Corollary 0.52. Let $f : V \to \mathbb{Z}$ be a $\mathbb{Z}$-configuration on a multidigraph $D = (V, A, \phi)$. Let $(\ell_1, \ell_2, \ldots, \ell_k)$ be a legal sequence for f. Let C be a set of arc-disjoint cycles of D. Let $C_w = \{c \in C \mid w \in \mathrm{V}(c)\}$ for each $w \in V$. Let $K \subseteq C$ be such that f is K-captured. Then, the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_k$ is $K \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k}$-captured.

Proof of Corollary 0.52. We will show that for each $i \in \{0, 1, \ldots, k\}$,

$$\text{the } \mathbb{Z}\text{-configuration } f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i \text{ is } K \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_i}\text{-captured.} \tag{16}$$

[*Proof of (16):* We shall prove (16) by induction over i:

Induction base: If $i = 0$, then (16) simply states that the $\mathbb{Z}$-configuration f is K-captured. This is true, because it was an assumption. This completes the induction base.

Induction step: Let $j \in \{0, 1, \ldots, k\}$ be positive. Assume that (16) holds for $i = j - 1$. We must show that (16) holds for $i = j$.

We have assumed that (16) holds for $i = j - 1$. In other words,

$$\text{the } \mathbb{Z}\text{-configuration } f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{j-1} \text{ is } K \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_{j-1}}\text{-captured.}$$

But the vertex ℓ_j is active in the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{j-1}$ (since the sequence $(\ell_1, \ell_2, \ldots, \ell_k)$ is legal for f). Hence, Lemma 0.51 (applied to $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{j-1}$, ℓ_j and $K \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_{j-1}}$ instead of f, w and K) shows that the $\mathbb{Z}$-configuration $\left(f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_{j-1}\right) - \Delta\ell_j$ is $\left(K \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_{j-1}}\right) \cup$

C_{ℓ_j}-captured[26]. In other words, the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_j$ is $K \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_j}$-captured. In other words, (16) holds for $i = j$. This completes the induction step. Thus, (16) is proven by induction.]

Now, applying (16) to $i = k$, we obtain the claim of Corollary 0.52. □

Solution to Exercise 3 ***(b)*** *(sketched).* Assume the contrary. Thus, D has more than $\sum f$ arc-disjoint cycles. In other words, there exists a set C of arc-disjoint cycles of D such that $|C| > \sum f$. Consider this C.

Set $h = \sum f$. Thus, $|C| > \sum f = h$.

Set $n = |V|$. From $|C| > h \geq 0$, we conclude that the set C is nonempty, so that D has at least one cycle. Thus, D has at least one vertex. In other words, $n > 0$.

Let $C_w = \{c \in C \mid w \in \mathrm{V}(c)\}$ for each $w \in V$.

The configuration f is infinitary. Hence, there exist legal sequences (for f) of arbitrary length. In particular, this shows that there exists a legal sequence $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ for f having length $k \geq \binom{n+h-1}{n-1}$. Consider such an ℓ.

Each vertex q of D must appear at least once in the sequence ℓ [27]. Thus, $C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k} = C$ [28].

But the configuration f is $\varnothing$-captured (since any configuration is $\varnothing$-captured). Hence, Corollary 0.52 (applied to $K = \varnothing$) shows that the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_k$ is $\varnothing \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k}$-captured. In other words, the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_k$ is C-captured (since $\varnothing \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k} = C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k} = C$). Hence, Lemma 0.50 (applied to $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_k$ and C instead of f and K) yields

$$\sum (f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_k) \geq |C| > h.$$

[26] since $K \cup C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_{j-1}} \subseteq C$ (since both K and each C_w are subsets of C)

[27] *Proof.* Let q be a vertex of D. Then, for each vertex $u \in V$, there exists a path from u to q (since the multidigraph D is strongly connected). Hence, Exercise 1 **(b)** shows that q must appear at least once in the sequence ℓ. Qed.

[28] *Proof.* Clearly, $C_w \subseteq C$ for each $w \in V$. Thus, $C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k} \subseteq C$.

On the other hand, let $c \in C$. Thus, c is a cycle of D. Hence, there exists some $v \in \mathrm{V}(c)$. Consider this v. Then, v must appear at least once in the sequence ℓ (since each vertex q of D must appear at least once in the sequence ℓ). In other words, there exists some $i \in \{1, 2, \ldots, k\}$ such that $v = \ell_i$. Consider this i. Now, from $\ell_i = v \in \mathrm{V}(c)$, we obtain $c \in C_{\ell_i}$ (by the definition of C_{ℓ_i}). Therefore, $c \in C_{\ell_i} \subseteq C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k}$.

Now, forget that we fixed c. We thus have proven that $c \in C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k}$ for each $c \in C$. In other words, $C \subseteq C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k}$. Combining this with $C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k} \subseteq C$, we obtain $C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_k} = C$.

Since

$$\sum \underbrace{(f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_k)}_{=f-\sum\limits_{i=1}^{k}\Delta\ell_i}$$

$$= \sum \left(f - \sum_{i=1}^{k} \Delta\ell_i \right) = \sum f - \sum_{i=1}^{k} \underbrace{\sum \Delta\ell_i}_{\substack{=0 \\ (\text{since } \sum \Delta v = 0 \text{ for each } v \in V)}} = \sum f - \underbrace{\sum_{i=1}^{k} 0}_{=0} = \sum f,$$

this rewrites as $\sum f > h$. This contradicts $h = \sum f$. This contradiction completes the solution of Exercise 3 **(b)**. □

Now that we have solved Exercise 3 **(b)**, we can obtain a second solution to Exercise 3 **(a)**:

*Second solution to Exercise 3 **(a)** (sketched).* Assume the contrary. Thus, D has more than $\sum f$ vertex-disjoint cycles. Hence, D has more than $\sum f$ arc-disjoint cycles (since vertex-disjoint cycles are automatically arc-disjoint). But this contradicts Exercise 3 **(b)**. This contradiction completes the solution of Exercise 3 **(a)**. □

0.6. Exercise 4: an associativity law for stabilizations

Recall Definition 0.33.

Exercise 4. Let $f : V \to \mathbb{N}$, $g : V \to \mathbb{N}$ and $h : V \to \mathbb{N}$ be three configurations such that both configurations f and $g + h$ are finitary, and such that the configuration $f + (g + h)^\circ$ is also finitary.

Prove the following:

(a) The configurations $f + g$ and h are also finitary.

(b) The configurations $f + g + h$ and $(f + g)^\circ + h$ are also finitary, and satisfy

$$(f + g + h)^\circ = \left(f + (g + h)^\circ\right)^\circ = \left((f + g)^\circ + h\right)^\circ.$$

In order to solve Exercise 4, we shall use the notations from Definition 0.41. We also state a few simple lemmas:

Lemma 0.53. Let $c : V \to \mathbb{N}$ and $d : V \to \mathbb{N}$ be two configurations such that $c + d$ is finitary. Then, c and d are finitary.

Proof of Lemma 0.53 (sketched). The configuration $c + d$ is finitary. Thus, there exists a stabilizing sequence s for $c + d$. Consider this s.

Firing the vertices in s (one after the other) transforms the configuration $c + d$ into a stable $\mathbb{Z}$-configuration z (since s is stabilizing). Consider this z.

The $\mathbb{Z}$-configuration z is stable. In other words, no vertex of V is active in z. In other words, $z(v) < \deg^+ v$ for each $v \in V$. Hence,

$$(z-d)(v) = z(v) - \underbrace{d(v)}_{\geq 0} \leq z(v) < \deg^+ v$$

for each $v \in V$. In other words, no vertex of V is active in $z-d$. In other words, the $\mathbb{Z}$-configuration $z-d$ is stable.

But firing the vertices in s (one after the other) transforms the configuration c into the $\mathbb{Z}$-configuration $z-d$ (since firing the vertices in s (one after the other) transforms the configuration $c+d$ into the $\mathbb{Z}$-configuration z). Thus, the sequence s is stabilizing for c (since the $\mathbb{Z}$-configuration $z-d$ is stable). Hence, there exists a stabilizing sequence for c. In other words, c is finitary. Similarly, d is finitary. This proves Lemma 0.53. $\square$

Lemma 0.54. Let $c : V \to \mathbb{N}$ and $d : V \to \mathbb{N}$ be two configurations such that $c \overset{*}{\to} d$.

(a) If c is finitary, then d is finitary and satisfies $c^\circ = d^\circ$.

(b) If d is finitary, then c is finitary and satisfies $c^\circ = d^\circ$.

Proof of Lemma 0.54 (sketched). From $c \overset{*}{\to} d$, we conclude that there exists a legal sequence ℓ_1 for c such that firing all vertices in ℓ_1 transforms c into d. Consider this ℓ_1.

(b) Assume that d is finitary. Hence, the stabilization d° is well-defined, and is a stable configuration (by the definition of stabilization). Also, d° is obtained from d by firing vertices from a legal sequence (by the definition of d°). Hence, $d \overset{*}{\to} d^\circ$. From $c \overset{*}{\to} d \overset{*}{\to} d^\circ$, we obtain $c \overset{*}{\to} d^\circ$ (since the relation $\overset{*}{\to}$ is transitive). Since d° is stable, this shows that c is finitary, and that d° is the stabilization of c.

Now, c° is well-defined (since c is finitary) and satisfies $d^\circ = c^\circ$ (since d° is the stabilization of c). In other words, $c^\circ = d^\circ$. This proves Lemma 0.54 **(b)**.

(a) Assume that c is finitary.

Let us first show that d is finitary.

Indeed, assume the contrary. Thus, d is infinitary. Hence, there exist arbitrarily long legal sequences for d. In other words, for each $N \in \mathbb{N}$, there exists a legal sequence for d having length $\geq N$.

Fix $N \in \mathbb{N}$. Then, there exists a legal sequence for d having length $\geq N$ (as we have just seen). Fix such a legal sequence, and denote it by ℓ_2.

Write the sequences ℓ_1 and ℓ_2 as $\ell_1 = (v_1, v_2, \ldots, v_p)$ and $\ell_2 = (w_1, w_2, \ldots, w_q)$. Let ℓ be the sequence $\ell_1 * \ell_2 = (v_1, v_2, \ldots, v_p, w_1, w_2, \ldots, w_q)$. (We are using the notation from Definition 0.19 here.) This sequence $\ell = \ell_1 * \ell_2$ is legal for c (by Lemma 0.20, applied to c, ℓ_1, d and ℓ_2 instead of f, u, g and v). Moreover, the length of ℓ is $\underbrace{p}_{\geq 0} + q \geq q =$ (the length of ℓ_2) $\geq N$ (since ℓ_2 has length $\geq N$). Thus, there exists a legal sequence for c having length $\geq N$ (namely, ℓ).

Now, forget that we fixed N. We thus have showed that for each $N \in \mathbb{N}$, there exists a legal sequence for c having length $\geq N$ (namely, ℓ). In other words, there exist arbitrarily long legal sequences for c. In other words, c is infinitary. This contradicts the fact that c is finitary. This contradiction shows that our assumption was false. Thus, we have proven that d is finitary.

Lemma 0.54 **(b)** thus yields $c^\circ = d^\circ$. This proves Lemma 0.54 **(a)**. $\square$

Hints to Exercise 4. For each finitary configuration k, we have

$$k \overset{*}{\to} k^\circ \tag{17}$$

(because k° is obtained from k by firing vertices from a legal sequence). Hence, $g + h \overset{*}{\to} (g+h)^\circ$. Thus, Proposition 0.42 **(b)** (applied to $c = f$ and $k = g + h$ and $k' = (g+h)^\circ$) yields $f + g + h \overset{*}{\to} f + (g+h)^\circ$.

But the configuration $f + (g+h)^\circ$ is also finitary. Thus, (17) (applied to $k = f + (g+h)^\circ$) yields $f + (g+h)^\circ \overset{*}{\to} \left(f + (g+h)^\circ\right)^\circ$. Thus,

$$f + g + h \overset{*}{\to} f + (g+h)^\circ \overset{*}{\to} \left(f + (g+h)^\circ\right)^\circ.$$

Since the configuration $\left(f + (g+h)^\circ\right)^\circ$ is stable, we thus conclude that $f + g + h$ is finitary, and furthermore that the configuration $\left(f + (g+h)^\circ\right)^\circ$ is the stabilization of $f + g + h$. Thus,

$$(f + g + h)^\circ = \left(f + (g+h)^\circ\right)^\circ. \tag{18}$$

The configuration $f + g + h$ is finitary. Hence, Lemma 0.53 (applied to $c = f + g$ and $d = h$) shows that the configurations $f + g$ and h are also finitary. This solves Exercise 4 **(a)**.

(b) Applying (17) to $k = f + g$, we obtain $f + g \overset{*}{\to} (f+g)^\circ$. Thus, Proposition 0.42 **(b)** (applied to $c = h$ and $k = f + g$ and $k' = (f+g)^\circ$) yields $h + f + g \overset{*}{\to} h + (f+g)^\circ$. Thus, $f + g + h = h + f + g \overset{*}{\to} h + (f+g)^\circ = (f+g)^\circ + h$. Hence, Lemma 0.54 **(a)** (applied to $c = f + g + h$ and $d = (f+g)^\circ + h$) shows that the configuration $(f+g)^\circ + h$ is finitary and satisfies

$$(f + g + h)^\circ = \left((f+g)^\circ + h\right)^\circ \tag{19}$$

(since the configuration $f + g + h$ is finitary). Combining (18) and (19), we obtain

$$(f + g + h)^\circ = \left(f + (g+h)^\circ\right)^\circ = \left((f+g)^\circ + h\right)^\circ.$$

This completes the solution of Exercise 4 **(b)**. $\square$

0.7. Exercise 5: chip-firing on the integer lattice

Now, we shall briefly discuss chip-firing on the integer lattice $\mathbb{Z}^2$; this is one of the most famous cases of chip-firing, leading to some of the pretty pictures. For examples and illustrations, check out [Ellenb15] as well as some of the links above.

We have not defined infinite graphs in class; the theory of infinite graphs involves some subtleties that would take us too far. However, for this particular exercise, we need only a specific infinite graph, which is fairly simple.

Definition 0.55. (a) A *locally finite multigraph* means a triple (V, E, ϕ), where V and E are sets and $\phi : E \to \mathcal{P}_2(V)$ is a map having the following property:

(*) For each $v \in V$, there exist only finitely many $e \in E$ satisfying $v \in \phi(e)$.

Most of the concepts defined for (usual) multigraphs still make sense for locally finite multigraphs. In particular, the elements of V are called the vertices, and the elements of E are called the edges. The property (*) says that each vertex is contained in only finitely many edges; this allows us to define the degree of a vertex.

(b) The *integer lattice* shall mean the locally finite multigraph defined as follows:

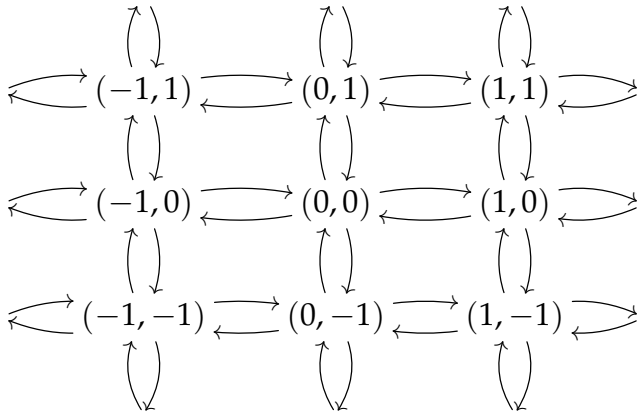
- The vertices of the integer lattice are the pairs (i, j) of two integers i and j. In other words, the vertex set of the integer lattice is $\mathbb{Z}^2 = \{(i, j) \mid i \in \mathbb{Z} \text{ and } j \in \mathbb{Z}\}$. We view these vertices as points in the plane, and draw the multigraph accordingly.
- Two vertices of the integer lattice are adjacent if and only if they have distance 1 (as points in the plane). In other words, a vertex (i, j) is adjacent to the four vertices $(i+1, j), (i, j+1), (i-1, j), (i, j-1)$ and no others.

(c) You can guess how locally finite multidigraphs are defined. Each locally finite multigraph can be regarded as a locally finite multidigraph by replacing each edge by a pair of two arcs (directed in both possible directions).

Let us show a piece of the integer lattice, viewed as a locally finite multigraph:

$$
\begin{array}{ccccccc}
 & | & & | & & | & \\
\text{——} & (-1,1) & \text{——} & (0,1) & \text{——} & (1,1) & \text{——} \\
 & | & & | & & | & \\
\text{——} & (-1,0) & \text{——} & (0,0) & \text{——} & (1,0) & \text{——} \\
 & | & & | & & | & \\
\text{——} & (-1,-1) & \text{——} & (0,-1) & \text{——} & (1,-1) & \text{——} \\
 & | & & | & & |
\end{array}
$$

And here is it again, viewed as a locally finite multidigraph:



Exercise 5. Let f be a configuration on the integer lattice (where we view the integer lattice as a locally finite multidigraph). (The notion of a configuration and related notions are defined in the same way as for usual, finite multidigraphs.)

Assume that only finitely many vertices $v \in \mathbb{Z}^2$ satisfy $f(v) \neq 0$. (Thus, the total number of chips $\sum f$ is finite.)

An edge e of the integer lattice is said to be *non-void* in f if and only if at least one of the endpoints of e has at least one chip in f.

Prove the following:

(a) If an edge of the integer lattice is non-void in f, then this edge remains non-void after firing any legal sequence of vertices. ("Firing a sequence" means firing all the vertices in the sequence, one after the other.)

(b) The total number of configurations that can be obtained from f by firing a legal sequence of vertices is finite.

(c) If we fire any vertex, then the sum $\sum\limits_{(i,j)\in\mathbb{Z}^2} f((i,j)) \cdot (i+j)^2$ increases.

(d) The configuration f is finitary (so its stabilization is well-defined).

This exercise gives the reason why pictures such as the ones in [Ellenb15] exist (although it does not explain their shapes and patterns).

Hints to Exercise 5. **(a)** Recall that we view the integer lattice as a locally finite multidigraph. Thus, each edge e of the integer lattice is not an actual edge, but really is two arcs (u,v) and (v,u) (where u and v are its endpoints). These two arcs form a cycle, which we denote by c_e (we choose arbitrarily which vertex to begin this cycle at). Clearly, e is non-void in a configuration f if and only if the cycle c_e is non-void in f (where the notion of "non-void" for a cycle is understood as in Definition 0.45). Thus, Exercise 5 **(a)** follows from Lemma 0.46 (at least if we extend the latter lemma to locally finite multidigraphs).

(b) For each configuration g, we shall denote by S_g the set $\{v \in \mathbb{Z}^2 \mid g(v) \neq 0\}$. Clearly, if $\sum g$ is finite, then the set S_g is finite. Thus, in particular, the set S_g is finite whenever the configuration g is obtained from f by repeatedly firing vertices.

Now, we make the following observation:

> *Observation 1:* If a configuration g' is obtained from a configuration g by firing an active vertex, then each $v \in S_{g'}$ either belongs to S_g or has at least one neighbor in S_g.

Observation 1 is easy to check.

Using Observation 1, Exercise 5 **(a)**, and straightforward induction (over the length of the legal sequence), we can argue the following observation:

> *Observation 2:* If a configuration g' is obtained from a configuration g by firing a legal sequence of vertices, then for each $v \in S_{g'}$, there exists a vertex $w \in S_g$ and a walk p from w to v such that each edge of p is non-void in g'.

Now, let $h = \sum f$. Define a subset T of $\mathbb{Z}^2$ by

$$T = \left\{ v \in \mathbb{Z}^2 \ \mid \ \text{there exists some } w \in S_f \text{ such that } d(w, v) \leq 4h \right\}.$$

We call this set T the *$4h$-neighborhood of S_f*. This set T is finite (since S_f is finite).

Let f' be a configuration obtained from the configuration f by firing a legal sequence of vertices. Let $v \in S_{f'}$. Then, Observation 2 (applied to $g = f$ and $g' = f'$) shows that there exists a vertex $w \in S_f$ and a walk p from w to v such that each edge of p is non-void in f'. Consider these w and p. We WLOG assume that the walk p is a path (since otherwise, we can simply remove cycles from p until p becomes a path). Since $\sum f' = \sum f = h$, there cannot be more than $4h$ edges that are non-void in f' (since each chip makes only 4 edges non-void, and there can be overlap). Thus, the path p cannot have more than $4h$ edges (since each edge of p is non-void in f'). In other words, the length of the path p is $\leq 4h$. Thus, there exists a path of length $\leq 4h$ from w to v (namely, the path p). Hence, $d(w, v) \leq 4h$. Hence, $v \in T$ (by the definition of T, because $w \in S_f$).

Now, forget that we fixed v. We thus have shown that $v \in T$ for each $v \in S_{f'}$. In other words, $S_{f'} \subseteq T$. Hence, $f'(v) = 0$ for each $v \in \mathbb{Z}^2 \setminus T$ (because of the definition of $S_{f'}$). Moreover, $f'(v) \in \{0, 1, \ldots, h\}$ for each $v \in T$ (since $f'(v) \leq \sum f' = h$).

Now, forget that we fixed f'. We thus have proven that each configuration f' obtained from the configuration f by firing a legal sequence of vertices satisfies

$$\begin{aligned} f'(v) &= 0 \qquad \text{for each } v \in \mathbb{Z}^2 \setminus T, \qquad \text{and} \\ f'(v) &\in \{0, 1, \ldots, h\} \qquad \text{for each } v \in T. \end{aligned}$$

Thus, all the configurations f' obtained from the configuration f by firing a legal sequence of vertices can be regarded as maps from T to $\{0, 1, \ldots, h\}$. Of course, there are only finitely many such maps; thus, there are only finitely many configurations that can be obtained from f by firing a legal sequence of vertices. This solves Exercise 5 **(b)**.

(c) Firing a given vertex (p,q) results in four chips disappearing from this vertex and reappearing on its four adjacent vertices $(p+1,q), (p-1,q), (p,q+1), (p,q-1)$. Thus, the value of $\sum_{(i,j)\in\mathbb{Z}^2} f((i,j)) \cdot (i+j)^2$ increases by

$$((p+1)+q)^2 + ((p-1)+q)^2 + (p+(q+1))^2 + (p+(q-1))^2 - 4(p+q)^2 = 4.$$

In particular, it increases. This solves Exercise 5 **(c)**.

(d) Let $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ be a legal sequence for f. Then, for each $i \in \{0,1,\ldots,k\}$, the $\mathbb{Z}$-configuration $f - \Delta\ell_1 - \Delta\ell_2 - \cdots - \Delta\ell_i$ is a configuration; denote this configuration by f_i. For each $i \in \{0,1,\ldots,k\}$, the configuration f_i is obtained from f by firing a legal sequence of vertices.

But each $p \in \{1,2,\ldots,k\}$ satisfies

$$\sum_{(i,j)\in\mathbb{Z}^2} f_{p-1}((i,j)) \cdot (i+j)^2 < \sum_{(i,j)\in\mathbb{Z}^2} f_p((i,j)) \cdot (i+j)^2$$

(by Exercise 5 **(c)**, because the configuration f_p is obtained from f_{p-1} by firing the vertex ℓ_p). Thus, the sequence

$$\left(\sum_{(i,j)\in\mathbb{Z}^2} f_p((i,j)) \cdot (i+j)^2\right)_{p\in\{0,1,\ldots,k\}}$$

is strictly increasing. Hence, its entries are distinct. Thus, the configurations $f_0, f_1, \ldots, f_k$ are distinct.

Recall that for each $i \in \{0,1,\ldots,k\}$, the configuration f_i is obtained from f by firing a legal sequence of vertices. But Exercise 5 **(b)** shows that the total number of such configurations is finite. Hence, if k is large enough, then two of the configurations $f_0, f_1, \ldots, f_k$ are equal. This contradicts the fact that the configurations $f_0, f_1, \ldots, f_k$ are distinct. Hence, k cannot be large enough.

Now, forget that we fixed k. We thus have shown that if $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ is a legal sequence for f, then k cannot be arbitrarily large. In other words, f does not have arbitrarily long legal sequences. In other words, f is finitary. This solves Exercise 5 **(d)**. □

0.8. Exercise 6: acyclic orientations are determined by their score vectors

Now, we leave the chip-firing setting.

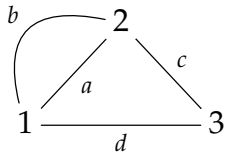
Roughly speaking, an *orientation* of a multigraph G is a way to assign to each edge of G a direction (thus making it an arc). If the resulting **digraph** has no cycles, then this orientation will be called *acyclic*. A rigorous way to state this definition is the following:

Definition 0.56. Let $G = (V, E, \psi)$ be a multigraph.

(a) An *orientation* of G is a map $\phi : E \to V \times V$ such that each $e \in E$ has the following property: If we write $\phi(e)$ in the form $\phi(e) = (u, v)$, then $\psi(e) = \{u, v\}$.

(b) An orientation ϕ of G is said to be *acyclic* if and only if the multidigraph (V, E, ϕ) has no cycles.

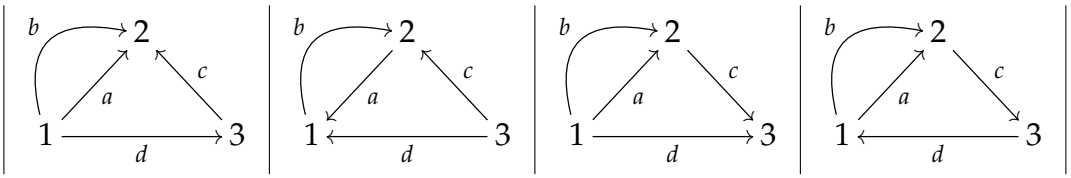
Example 0.57. Let $G = (V, E, \psi)$ be the following multigraph:



Then, the following four maps ϕ are orientations of G:

- the map sending a to $(1, 2)$, sending b to $(1, 2)$, sending c to $(3, 2)$, and sending d to $(1, 3)$;
- the map sending a to $(2, 1)$, sending b to $(1, 2)$, sending c to $(3, 2)$, and sending d to $(3, 1)$;
- the map sending a to $(1, 2)$, sending b to $(1, 2)$, sending c to $(2, 3)$, and sending d to $(1, 3)$;
- the map sending a to $(1, 2)$, sending b to $(1, 2)$, sending c to $(2, 3)$, and sending d to $(3, 1)$.

Here are the multidigraphs (V, E, ϕ) corresponding to these four maps (in the order mentioned):



Only the first and the third of these orientations ϕ are acyclic (since only the first and the third of these multidigraphs have no cycles).

Exercise 6. Let $G = (V, E, \psi)$ be a multigraph.

Prove the following:

(a) If ϕ is any acyclic orientation of G, and if $V \neq \varnothing$, then there exists a $v \in V$ such that no arc of the multidigraph (V, E, ϕ) has target v.

(b) If ϕ_1 and ϕ_2 are two acyclic orientations of G such that each $v \in V$ satisfies

$$\deg^+_{(V,E,\phi_1)} v = \deg^+_{(V,E,\phi_2)} v,$$

then $\phi_1 = \phi_2$.

Our solution to Exercise 6 uses the following fact:

Proposition 0.58. Let $D = (V, A, \phi)$ be a multidigraph with $|V| > 0$. Assume that each vertex $v \in V$ satisfies $\deg^- v > 0$. Then, D has at least one cycle.

Proof of Proposition 0.58. Proposition 0.58 is simply the obvious generalization of Exercise 3 on Midterm 1 to multidigraphs; it is proven in precisely the same way. □

Hints to Exercise 6. **(a)** Let ϕ be any acyclic orientation of G. Thus, the multidigraph (V, E, ϕ) has no cycles.

Assume that $V \neq \varnothing$. Then, $|V| > 0$.

If each vertex $v \in V$ satisfies $\deg^-_{(V,E,\phi)} v > 0$, then the multidigraph (V, E, ϕ) has at least one cycle[29]. Therefore, not each vertex $v \in V$ satisfies $\deg^-_{(V,E,\phi)} v > 0$ (because the multidigraph (V, E, ϕ) has no cycles). In other words, there exists a vertex $v \in V$ satisfying $\deg^-_{(V,E,\phi)} v = 0$. In other words, there exists a $v \in V$ such that no arc of the multidigraph (V, E, ϕ) has target v. This solves Exercise 6 **(a)**.

(b) We shall solve Exercise 6 **(b)** by induction over $|V|$.

The *induction base* (the case when $|V| = 0$) is obvious (because in this case there are no edges, and thus there is only one possible orientation).

Now, we come to the *induction step*: Let $N \in \mathbb{N}$ be positive. Assume (as the induction hypothesis) that Exercise 6 **(b)** holds whenever $|V| = N - 1$. We must now prove that Exercise 6 **(b)** also holds whenever $|V| = N$.

So let G, V, E and ψ be as in Exercise 6 **(b)**, and assume that $|V| = N$. Let ϕ_1 and ϕ_2 be two acyclic orientations of G such that each $v \in V$ satisfies

$$\deg^+_{(V,E,\phi_1)} v = \deg^+_{(V,E,\phi_2)} v. \tag{20}$$

We must then prove that $\phi_1 = \phi_2$.

We have $|V| = N > 0$, thus $V \neq \varnothing$. Hence, Exercise 6 **(a)** (applied to $\phi = \phi_1$) shows that there exists a $v \in V$ such that no arc of the multidigraph (V, E, ϕ_1) has target v. Fix such a v, and denote it by w. Thus, no arc of the multidigraph (V, E, ϕ_1) has target w. Hence, $\deg^-_{(V,E,\phi_1)} w = 0$.

But ϕ_1 is an orientation of G. Hence, some of the edges of G that contain w become arcs with source w in ϕ_1, whereas the remaining edges of G that contain w become arcs with target w in ϕ_1. Therefore,

$$\deg^-_{(V,E,\phi_1)} w + \deg^+_{(V,E,\phi_1)} w = \deg_G w. \tag{21}$$

Similarly,

$$\deg^-_{(V,E,\phi_2)} w + \deg^+_{(V,E,\phi_2)} w = \deg_G w. \tag{22}$$

[29] This follows from Proposition 0.58 (applied to (V, E, ϕ) and E instead of D and A).

Now, (21) yields

$$\deg_G w = \underbrace{\deg^-_{(V,E,\phi_1)} w}_{=0} + \deg^+_{(V,E,\phi_1)} w = \deg^+_{(V,E,\phi_1)} w = \deg^+_{(V,E,\phi_2)} w$$

(by (20), applied to $v = w$). Hence,

$$\deg^+_{(V,E,\phi_2)} w = \deg_G w = \deg^-_{(V,E,\phi_2)} w + \deg^+_{(V,E,\phi_2)} w \qquad \text{(by (22))}.$$

Subtracting $\deg^+_{(V,E,\phi_2)} w$ from both sides of this equality, we find $0 = \deg^-_{(V,E,\phi_2)} w$. Hence, no arc of the multidigraph (V, E, ϕ_2) has target w.

Now, we see that

$$\text{each edge } e \text{ of } G \text{ satisfying } w \in \psi(e) \text{ satisfies } \phi_1(e) = \phi_2(e) \tag{23}$$

[30].

On the other hand, let $G' = (V', E', \psi')$ be the multigraph obtained from G by removing the vertex w and all edges containing w. (Thus, $V' = V \setminus \{w\}$, $E' = \{e \in E \mid w \notin \psi(e)\}$ and $\psi' = \psi\mid_{E'}$.) Notice that $|V'| = |V \setminus \{w\}| = \underbrace{|V|}_{=N} - 1 = N - 1$.

Let ϕ_1' and ϕ_2' be the two orientations of the multigraph G' obtained by restricting ϕ_1 and ϕ_2. (Thus, $\phi_1' = \phi_1\mid_{E'}$ and $\phi_2' = \phi_2\mid_{E'}$.) Clearly, the orientation ϕ_1' is acyclic (since each cycle of the multidigraph (V', E', ϕ_1') would be a cycle of the multidigraph (V, E, ϕ_1), but the latter multidigraph has no cycles since ϕ_1 is acyclic). Similarly, the orientation ϕ_2' is acyclic.

Each $v \in V'$ satisfies

$$\deg^+_{(V',E',\phi_1')} v = \deg^+_{(V',E',\phi_2')} v$$

[31]. Hence, the induction hypothesis (applied to G', V', E', ψ', ϕ_1' and ϕ_2' instead of G, V, E, ψ, ϕ_1 and ϕ_2) shows that $\phi_1' = \phi_2'$ (since $|V'| = N - 1$).

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[30] *Proof of (23):* Let e be an edge of G satisfying $w \in \psi(e)$. Write $\psi(e)$ in the form $\psi(e) = \{u, w\}$ for some $u \in V$. Then, $\phi_1(e)$ is either (u, w) or (w, u). But since no arc of the multidigraph (V, E, ϕ_1) has target w, we cannot have $\phi_1(e) = (u, w)$. Hence, we have $\phi_1(e) = (w, u)$. Similarly, $\phi_2(e) = (w, u)$ (since no arc of the multidigraph (V, E, ϕ_2) has target w). Hence, $\phi_1(e) = (w, u) = \phi_2(e)$. This proves (23).

[31] *Proof.* Let $v \in V'$. Thus, $v \in V' = V \setminus \{w\}$, so that $v \in V$ and $v \neq w$.

Now, recall that no arc of the multidigraph (V, E, ϕ_1) has target w. Hence, each arc of the multidigraph (V, E, ϕ_1) having source v is also an arc of the multidigraph (V', E', ϕ_1'). Therefore, the arcs of the multidigraph (V, E, ϕ_1) having source v are precisely the arcs of the multidigraph (V', E', ϕ_1') having source v. Therefore, $\deg^+_{(V',E',\phi_1')} v = \deg^+_{(V,E,\phi_1)} v$. Similarly, $\deg^+_{(V',E',\phi_2')} v = \deg^+_{(V,E,\phi_2)} v$. Now,

$$\begin{aligned}\deg^+_{(V',E',\phi_1')} v &= \deg^+_{(V,E,\phi_1)} v = \deg^+_{(V,E,\phi_2)} v \qquad \text{(by (20))} \\ &= \deg^+_{(V',E',\phi_2')} v,\end{aligned}$$

qed.

Thus,

$$\text{each edge } e \text{ of } G \text{ satisfying } w \notin \psi(e) \text{ satisfies } \phi_1(e) = \phi_2(e) \tag{24}$$

[32]. Combining this with (23), we conclude that each edge e of G satisfies $\phi_1(e) = \phi_2(e)$. In other words, we have $\phi_1 = \phi_2$.

Thus, we have proven that Exercise 6 **(b)** also holds whenever $|V| = N$. This completes the induction step. Hence, Exercise 6 **(b)** is solved. □

[*Remark:* Exercise 6 **(b)** remains valid even if we don't require the orientation ϕ_2 to be acyclic. The above solution to this exercise still works in this generality (mutatis mutandis).]

0.9. Exercise 7: the lattice structure on minimum cuts

Let us recall some terminology from [Grinbe17b]:

- A *network* consists of:
  - a simple digraph (V, A);
  - two distinct vertices $s \in V$ and $t \in V$, called the *source* and the *sink*, respectively (although we do not require s to have indegree 0 or t to have outdegree 0);
  - a function $c : A \to \mathbb{Q}_+$, called the *capacity function*. (Here, $\mathbb{Q}_+$ means the set $\{x \in \mathbb{Q} \mid x \geq 0\}$.)
- Given a network consisting of a digraph (V, A), a source $s \in V$ and a sink $t \in V$, and a capacity function $c : A \to \mathbb{Q}_+$, we define the following notations:
  - For any subset S of V, we let $\overline{S}$ denote the subset $V \setminus S$ of V.
  - If P and Q are two subsets of V, then $[P, Q]$ shall mean the set of all arcs $a \in A$ whose source belongs to P and whose target belongs to Q. (In other words, $[P, Q] = A \cap (P \times Q)$.)
  - If P and Q are two subsets of V, then the number $c(P, Q) \in \mathbb{Q}_+$ is defined by
$$c(P, Q) = \sum_{a \in [P, Q]} c(a).$$

We also refer to lecture 16 [Grinbe17b] for the definition of a flow (which is not necessary for the following problem, but may be helpful).

[32] *Proof of (24):* Let e be an edge of G satisfying $w \notin \psi(e)$. Then, e is an edge of the multigraph G' (by the definition of G'). Hence, $\phi_1'(e)$ and $\phi_2'(e)$ are well-defined. Moreover, ϕ_1' is a restriction of ϕ_1; hence, $\phi_1'(e) = \phi_1(e)$. Similarly, $\phi_2'(e) = \phi_2(e)$. Now, $\phi_1(e) = \underbrace{\phi_1'}_{=\phi_2'}(e) = \phi_2'(e) = \phi_2(e)$.

This proves (24).

Exercise 7. Consider a network consisting of a simple digraph (V, A), a source $s \in V$ and a sink $t \in V$, and a capacity function $c : A \to \mathbb{Q}_+$ such that $s \neq t$.

An *s-t-cutting subset* shall mean a subset S of V satisfying $s \in S$ and $t \notin S$.

Let m denote the minimum possible value of $c\left(S, \overline{S}\right)$ where S ranges over the s-t-cutting subsets. (Recall that this is the maximum value of a flow, according to the maximum-flow-minimum-cut theorem.)

An s-t-cutting subset S is said to be *cut-minimal* if it satisfies $c\left(S, \overline{S}\right) = m$.

Let X and Y be two cut-minimal s-t-cutting subsets. Prove that $X \cap Y$ and $X \cup Y$ also are cut-minimal s-t-cutting subsets.

Exercise 7 is not new; it appears, e.g., in [PicQue80, Corollary 3][33].

In our solution of Exercise 7, we shall use some further material from [Grinbe17b]. Namely, we shall use the concept of flows, and the following fact:

Lemma 0.59. Consider a network consisting of a simple digraph (V, A), a source $s \in V$ and a sink $t \in V$, and a capacity function $c : A \to \mathbb{Q}_+$ such that $s \neq t$. Let f be a flow on this network.

Let S be a subset of V satisfying $s \in S$ and $t \notin S$.

(a) We have $|f| \leq c\left(S, \overline{S}\right)$.

(b) We have $|f| = c\left(S, \overline{S}\right)$ if and only if

$$\left(\text{each } a \in \left[\overline{S}, S\right] \text{ satisfies } f(a) = 0\right) \tag{25}$$

and

$$\left(\text{each } a \in \left[S, \overline{S}\right] \text{ satisfies } f(a) = c(a)\right). \tag{26}$$

Proof of Lemma 0.59 (sketched). **(a)** This is precisely [Grinbe17b, Proposition 1.7 **(c)**].

(b) This is precisely [Grinbe17b, Proposition 1.7 **(d)**]. □

Hints to Exercise 7. From the max-flow-min-cut-theorem (specifically, [Grinbe17b, Theorem 1.10]), we know that

$$\begin{aligned}
\max\left\{|f| \mid f \text{ is a flow}\right\} &= \min\left\{c\left(S, \overline{S}\right) \mid \underbrace{S \subseteq V;\ s \in S;\ t \notin S}_{\iff (S \text{ is an } s\text{-}t\text{-cutting subset of } V)}\right\} \\
&= \min\left\{c\left(S, \overline{S}\right) \mid S \text{ is an } s\text{-}t\text{-cutting subset of } V\right\} \\
&= m \qquad (\text{by the definition of } m).
\end{aligned}$$

Hence, there exists a flow f with $|f| = m$. Consider such an f.

The subset X is s-t-cutting; thus, X is a subset of V satisfying $s \in X$ and $t \notin X$.

The subset Y is s-t-cutting; thus, Y is a subset of V satisfying $s \in Y$ and $t \notin Y$.

[33] To be fully precise, the version in [PicQue80, Corollary 3] differs by having c be a map $A \to \mathbb{R}_{>0}$ (where $\mathbb{R}_{>0}$ is the set of all positive reals) instead of having c be a map $A \to \mathbb{Q}_+$. But the difference is unsubstantial, and roughly the same proofs work for both results.

We have $c\left(X,\overline{X}\right) = m$ (since X is cut-minimal) and $c\left(Y,\overline{Y}\right) = m$ (since Y is cut-minimal). Thus, $|f| = m = c\left(X,\overline{X}\right)$ and $|f| = m = c\left(Y,\overline{Y}\right)$.

We have $s \in X \cap Y$ (since $s \in X$ and $s \in Y$) and $s \in X \cup Y$ (likewise). We have $t \notin X \cup Y$ (since $t \notin X$ and $t \notin Y$) and $t \notin X \cap Y$ (likewise). The subset $X \cap Y$ of V is s-t-cutting (since $s \in X \cap Y$ and $t \notin X \cap Y$). Similarly, the subset $X \cup Y$ of V is s-t-cutting.

Lemma 0.59 **(b)** (applied to $S = X$) shows that $|f| = c\left(X,\overline{X}\right)$ if and only if

$$\left(\text{each } a \in \left[\overline{X}, X\right] \text{ satisfies } f(a) = 0\right) \tag{27}$$

and

$$\left(\text{each } a \in \left[X, \overline{X}\right] \text{ satisfies } f(a) = c(a)\right). \tag{28}$$

Therefore, (27) and (28) hold (since $|f| = c\left(X,\overline{X}\right)$).

Lemma 0.59 **(b)** (applied to $S = Y$) shows that $|f| = c\left(Y,\overline{Y}\right)$ if and only if

$$\left(\text{each } a \in \left[\overline{Y}, Y\right] \text{ satisfies } f(a) = 0\right) \tag{29}$$

and

$$\left(\text{each } a \in \left[Y, \overline{Y}\right] \text{ satisfies } f(a) = c(a)\right). \tag{30}$$

Therefore, (29) and (30) hold (since $|f| = c\left(Y,\overline{Y}\right)$).

Now,

$$\left(\text{each } a \in \left[\overline{X \cap Y}, X \cap Y\right] \text{ satisfies } f(a) = 0\right) \tag{31}$$

[34] and

$$\left(\text{each } a \in \left[X \cap Y, \overline{X \cap Y}\right] \text{ satisfies } f(a) = c(a)\right) \tag{32}$$

[35].

But Lemma 0.59 **(b)** (applied to $S = X \cap Y$) shows that $|f| = c\left(X \cap Y, \overline{X \cap Y}\right)$ if and only if (31) and (32) hold. Hence, $|f| = c\left(X \cap Y, \overline{X \cap Y}\right)$ (since (31) and (32)

[34] *Proof of (31):* Let $a \in \left[\overline{X \cap Y}, X \cap Y\right]$. We must prove that $f(a) = 0$.

We have $a \in \left[\overline{X \cap Y}, X \cap Y\right]$. Thus, $a = (u,v)$ for some $u \notin X \cap Y$ and $v \in X \cap Y$. Consider these u and v.

We have $v \in X \cap Y \subseteq Y$. Hence, if $u \notin Y$, then $(u,v) \in \left[\overline{Y}, Y\right]$. Thus, if $u \notin Y$, then $a = (u,v) \in \left[\overline{Y}, Y\right]$. Therefore, if $u \notin Y$, then (29) shows that $f(a) = 0$. Thus, for the rest of this proof, we WLOG assume that $u \notin Y$ does not hold. Hence, $u \in Y$. Now, $u \in Y$ but $u \notin X \cap Y$. Hence, $u \in Y \setminus (X \cap Y) = Y \setminus X$, so that $u \notin X$. But $v \in X \cap Y \subseteq X$. Hence, $(u,v) \in \left[\overline{X}, X\right]$, so that $a = (u,v) \in \left[\overline{X}, X\right]$. Thus, (27) shows that $f(a) = 0$. This proves (31).

[35] *Proof of (32):* Let $a \in \left[X \cap Y, \overline{X \cap Y}\right]$. We must prove that $f(a) = c(a)$.

We have $a \in \left[X \cap Y, \overline{X \cap Y}\right]$. Thus, $a = (u,v)$ for some $u \in X \cap Y$ and $v \notin X \cap Y$. Consider these u and v.

We have $u \in X \cap Y \subseteq Y$. Hence, if $v \notin Y$, then $(u,v) \in \left[Y, \overline{Y}\right]$. Thus, if $v \notin Y$, then $a = (u,v) \in \left[Y, \overline{Y}\right]$. Therefore, if $v \notin Y$, then $f(a) = c(a)$ follows immediately from (30). Thus, for the rest of this proof, we WLOG assume that $v \notin Y$ does not hold. Hence, $v \in Y$. Now, $v \in Y$ but $v \notin X \cap Y$. Hence, $v \in Y \setminus (X \cap Y) = Y \setminus X$, so that $v \notin X$. But $u \in X \cap Y \subseteq X$. Hence, $(u,v) \in \left[X, \overline{X}\right]$. Thus, $a = (u,v) \in \left[X, \overline{X}\right]$. Hence, (28) shows that $f(a) = c(a)$. This proves (32).

hold). Thus, $c\left(X \cap Y, \overline{X \cap Y}\right) = |f| = m$. In other words, the s-t-cutting subset $X \cap Y$ of V is cut-minimal.

Furthermore,

$$\left(\text{each } a \in \left[\overline{X \cup Y}, X \cup Y\right] \text{ satisfies } f(a) = 0\right) \tag{33}$$

[36] and

$$\left(\text{each } a \in \left[X \cup Y, \overline{X \cup Y}\right] \text{ satisfies } f(a) = c(a)\right) \tag{34}$$

[37].

But Lemma 0.59 **(b)** (applied to $S = X \cup Y$) shows that $|f| = c\left(X \cup Y, \overline{X \cup Y}\right)$ if and only if (33) and (34) hold. Hence, $|f| = c\left(X \cup Y, \overline{X \cup Y}\right)$ (since (33) and (34) hold). Thus, $c\left(X \cup Y, \overline{X \cup Y}\right) = |f| = m$. In other words, the s-t-cutting subset $X \cup Y$ of V is cut-minimal. Exercise 7 is thus shown. □

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[36] *Proof of (33):* Let $a \in \left[\overline{X \cup Y}, X \cup Y\right]$. We must prove that $f(a) = 0$.

We have $a \in \left[\overline{X \cup Y}, X \cup Y\right]$. In other words, $a = (u, v)$ for some $u \notin X \cup Y$ and $v \in X \cup Y$. Consider these u and v.

From $u \notin X \cup Y$, we obtain $u \notin X$ and $u \notin Y$. Hence, if $v \in X$, then $(u, v) \in \left[\overline{X}, X\right]$. Thus, if $v \in X$, then $a = (u, v) \in \left[\overline{X}, X\right]$. Hence, if $v \in X$, then $f(a) = 0$ follows immediately from (27). Thus, for the rest of this proof, we WLOG assume that $v \in X$ does not hold. Hence, $v \notin X$. Now, $v \in X \cup Y$ but $v \notin X$. Hence, $v \in (X \cup Y) \setminus X = Y \setminus X \subseteq Y$. But $u \notin Y$. Thus, $(u, v) \in \left[\overline{Y}, Y\right]$. Therefore, $a = (u, v) \in \left[\overline{Y}, Y\right]$. Hence, (29) shows that $f(a) = 0$. This proves (33).

[37] *Proof of (34):* Let $a \in \left[X \cup Y, \overline{X \cup Y}\right]$. We must prove that $f(a) = c(a)$.

We have $a \in \left[X \cup Y, \overline{X \cup Y}\right]$. In other words, $a = (u, v)$ for some $u \in X \cup Y$ and $v \notin X \cup Y$. Consider these u and v.

From $v \notin X \cup Y$, we obtain $v \notin X$ and $v \notin Y$. Hence, if $u \in X$, then $(u, v) \in \left[X, \overline{X}\right]$. Thus, if $u \in X$, then $a = (u, v) \in \left[X, \overline{X}\right]$. Therefore, if $u \in X$, then $f(a) = c(a)$ follows immediately from (28). Thus, for the rest of this proof, we WLOG assume that $u \in X$ does not hold. Hence, $u \notin X$. Now, $u \in X \cup Y$ but $u \notin X$. Hence, $u \in (X \cup Y) \setminus X = Y \setminus X \subseteq Y$. But $v \notin Y$. Thus, $(u, v) \in \left[Y, \overline{Y}\right]$. Now, $a = (u, v) \in \left[Y, \overline{Y}\right]$. Hence, (30) shows that $f(a) = c(a)$. This proves (34).

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