University of Minnesota

Math 5707: Graph Theory, Spring 2017 Homework 5

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1 Exercise 1

1.1 Problem

Fix a loopless multidigraph $D = (V, A, \phi)$. Let $f : V \to \mathbb{N}$ be a configuration. Let $h = \sum f$. Let $n = |V|$. Assume that $n > 0$.

Let $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ be a legal sequence for f having length $k \geq \binom{n+h-1}{n-1}$.

Prove the following:

(a) There exist legal sequences (for f) of arbitrary length.

(b) Let q be a vertex of D such that for each vertex $u \in V$, there exists a path from u to q. Then, q must appear at least once in the sequence ℓ.

1.2 Solution

Proof of part (a): For $i = 0, 1, \ldots, k$, let $f_i = f - \Delta\ell_1 - \ldots - \Delta\ell_i$. Thus, in particular, $f_0 = f$.

Since h is unchanged by any legal sequence, $\sum\limits_{v\in V} f_i(v) = h$ for all $i \in \{0, 1, \ldots, k\}$. Now, per the fact referenced in the problem set, the total number of possible configurations f_i for which $\sum\limits_{v\in V} f_i(v) = h$ is given by $\dbinom{n+h-1}{n-1}$ (because these configurations are in bijection with the n-tuples $(a_1, a_2, \ldots, a_n)$ of nonnegative integers satisfying $a_1 + a_2 + \cdots + a_n = h$, but the number of the latter n-tuples is $\dbinom{n+h-1}{n-1}$). But in the course of firing the sequence ℓ on f, at least $\dbinom{n+h-1}{n-1} + 1$ configurations will appear (including f). Then by the pigeonhole principle, at least one pair of configurations must be identical; that is, for some $i, j \in \{0, 1, \ldots, k\}$ with $i < j$, we have $f_i = f_j$. Since the sequence $(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)$ is legal on configuration f_i, it is also legal on configuration f_j. Thus, starting at f, we could fire the sequence $(\ell_1, \ell_2, \ldots, \ell_i)$, followed by arbitrarily many repetitions of the sequence $(\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j)$. □

Proof of part (b): Assume the contrary. Thus, q never appears in ℓ. As we have seen in our proof of part (a), there exist legal sequences (for f) of arbitrary length, and furthermore, there exist such sequences having the form $\left(\ell_1, \ell_2, \ldots, \ell_i, \underbrace{\ell_{i+1}, \ell_{i+2}, \ldots, \ell_j}_{\text{periodically repeated}}\right)$. These latter sequences do not contain q (since q never appears in ℓ), and therefore are q-legal. Hence, q-legal sequences of arbitrary length exist.

By Theorem 0.20 on the homework set, there exists a sequence s that is q-legal and q-stabilizing for f. Furthermore, all q-legal sequences (for f) are at most as long as s. This contradicts the fact that q-legal sequences of arbitrary length exist. This contradiction shows that our assumption was false; hence, part (b) is proven. □

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3 Exercise 3

3.1 Problem

Assume that the multidigraph D is strongly connected. Let $f : V \to \mathbb{N}$ be an infinitary configuration.

(a) Prove that D cannot have more than $\sum f$ vertex-disjoint cycles. (A set of cycles is said to be *vertex-disjoint* if no two distinct cycles in the set have a vertex in common.)

(b) Prove that D cannot have more than $\sum f$ arc-disjoint cycles. (A set of cycles is said to be *arc-disjoint* if no two distinct cycles in the set have an arc in common.)

3.2 Solution to part (a)

Proof of part (a): Let $h = \sum f$. A cycle will be called *non-void* in a configuration g if at least one vertex on the cycle contains a chip in g. Given a configuration g and a sequence of vertices $\ell = (v_1, v_2, \ldots, v_k)$, the configuration $g - \Delta v_1 - \Delta v_2 - \cdots - \Delta v_k$ shall be denoted $g - \ell$. I begin with a claim:

> *Claim 1:* If a cycle in D is non-void in a configuration g, it will remain non-void after firing any legal sequence of vertices.

Proof of Claim 1. Let $c = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = v_0)$ be a cycle in D. Suppose c is non-void in a configuration g. Consider the outcome of firing a vertex $w \in V$ that is active in g. There are two cases to consider:

- *Case 1:* $w = v_i$ for some $i \in \{0, 1, \ldots, k-1\}$. In this case, there is at least one arc from w to v_{i+1}, so $(g - \Delta w)(v_{i+1}) \geq g(v_{i+1}) + 1 \geq 1$. That is, vertex v_{i+1} contains at least one chip in configuration $(g - \Delta w)$. Hence c is non-void in $(g - \Delta w)$.
- *Case 2:* $w \notin \{v_0, v_1, \ldots, v_{k-1}\}$. In this case, since c is non-void in g, we know that $g(v_i) > 0$ for some $i \in \{0, 1, \ldots, k-1\}$. Firing the vertex w cannot decrease the number of chips on v_i (since $w \neq v_i$), so $(g - \Delta w)(v_i) \geq g(v_i) > 0$. That is, vertex v_i contains at least one chip in configuration $(g - \Delta w)$. Hence c is non-void in $(g - \Delta w)$.

In either case, c remains non-void after firing any active vertex. Then by induction, c will remain non-void after firing any legal sequence of vertices. This proves Claim 1. □

Now, suppose (for the sake of contradiction) that D has at least $h+1$ vertex-disjoint cycles. Name these cycles $c_1, c_2, \ldots, c_{h+1}$. For $i = 1, 2, \ldots, h+1$, pick a vertex q_i that is on the cycle c_i.

Now, we shall define $h+1$ sequences $\ell_1, \ell_2, \ldots, \ell_{h+1}$ of vertices of D with the property that each ℓ_i is legal for the configuration $f - \ell_1 - \ell_2 - \cdots - \ell_{i-1}$ (so that the concatenation of all the $h+1$ sequences is legal for f) and that each vertex q_i has at least one chip in $f - \ell_1 - \ell_2 - \cdots - \ell_i$. These sequences are defined by recursion over i:

- Fix an $i \in \{1, 2, \ldots, h+1\}$, and assume that the sequences $\ell_1, \ell_2, \ldots, \ell_{i-1}$ are already constructed. Since D is strongly connected, Theorem 0.20 on the homework set tells us that there is a legal sequence ℓ_i that is q_i-stabilizing for configuration $f - \ell_1 - \cdots - \ell_{i-1}$. The sequence ℓ_i cannot be stabilizing for $f - \ell_1 - \cdots - \ell_{i-1}$ (since f is infinitary), so the vertex q_i is active in configuration $f - \ell_1 - \cdots - \ell_i$. Thus c_i is non-void in configuration $f - \ell_1 - \cdots - \ell_i$, and by Claim 1 it will remain non-void after firing any further legal sequence of vertices.

Hence there is a legal sequence (namely, the concatenation of $\ell_1, \ell_2, \ldots, \ell_{h+1}$) that will render each of the $h+1$ cycles non-void. But this implies that there is at least one vertex in each cycle with at least one chip on it. Since the cycles are vertex-disjoint, there must be at least $h+1$ chips, a contradiction. Therefore, D cannot have more than h vertex-disjoint cycles. □

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6 Exercise 6

6.1 Problem

Let $G = (V, E, \psi)$ be a multigraph.

Prove the following:

(a) If ϕ is any acyclic orientation of G, and if $|V| > 0$, then there exists a $v \in V$ such that no arc of the multidigraph (V, E, ϕ) has target v.

(b) If ϕ_1 and ϕ_2 are two acyclic orientations of G such that each $v \in V$ satisfies

$$\deg^+_{(V,E,\phi_1)} v = \deg^+_{(V,E,\phi_2)} v,$$

then $\phi_1 = \phi_2$.

6.2 Solution

Proof of part (a): Suppose the contrary: ϕ is an acyclic orientation of G such that for each $v \in V$, there exist $u \in V$ and $e \in E$ such that $\phi(e) = (u, v)$. Fix a longest path $\rho = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$ in the multidigraph (V, E, ϕ). By supposition, there exists an edge $e_0 \in E$ such that $\phi(e_0) = (u, v_0)$ for some $u \in V$. If u were distinct from each v_i in ρ, then ρ would not be a longest path. Hence $u = v_i$ for some $i \in \{0, 1, \ldots, k\}$. But then $(u, e_0, v_0, e_1, v_1, \ldots, e_i, v_i = u)$ is a cycle in (V, E, ϕ), a contradiction (since ϕ is acyclic). Therefore, there exists a $v \in V$ such that no arc of (V, E, ϕ) has target v. □

Proof of part (b): Let ϕ_1 and ϕ_2 be two acyclic orientations of G such that each $v \in V$ satisfies

$$\deg^+_{(V,E,\phi_1)} v = \deg^+_{(V,E,\phi_2)} v. \tag{1}$$

Suppose (for the sake of contradiction) there is an edge $e_1 \in E$ such that $\phi_1(e_1) \neq \phi_2(e_1)$, that is, $\phi_1(e_1) = (u, v)$ and $\phi_2(e_1) = (v, u)$ for some distinct $u, v \in V$. Then there must be a distinct $e_2 \in E$ such that $\phi_1(e_2) = (v, w)$ and $\phi_2(e_2) = (w, v)$ for some $w \in V$ distinct from v. (If not, we would have $\deg^+_{(V,E,\phi_1)} v < \deg^+_{(V,E,\phi_2)} v$, which would contradict (1).) By induction then, we can construct a walk in (V, E, ϕ_1) by following an arbitrarily long sequence of these edges (i.e. $(u, e_1, v, e_2, w, \ldots)$). Since $|V|$ is finite, this walk must eventually return to a previously visited vertex, implying that there is a cycle in (V, E, ϕ_1), a contradiction (since ϕ_1 is acyclic). Therefore, there is no edge $e \in E$ such that $\phi_1(e) \neq \phi_2(e)$. Then $\phi_1 = \phi_2$. □

[*Remark:* In our above proof of part (b), we only used that the assumption that ϕ_1 is acyclic, but not that ϕ_2 is acyclic. So the problem can be generalized.]

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7 Exercise 7

7.1 Problem

Consider a network consisting of a digraph (V, A), a source $s \in V$ and a sink $t \in V$, and a capacity function $c : A \to \mathbb{Q}_+$ such that $s \neq t$.

An *s-t-cutting subset* shall mean a subset S of V satisfying $s \in S$ and $t \notin S$.

Let m denote the minimum possible value of $c\left(S, \overline{S}\right)$ where S ranges over the s-t-cutting subsets. (Recall that this is the maximum value of a flow, according to the maximum-flow-minimum-cut theorem.)

An s-t-cutting subset S is said to be *cut-minimal* if it satisfies $c\left(S, \overline{S}\right) = m$.

Let X and Y be two cut-minimal s-t-cutting subsets. Prove that $X \cap Y$ and $X \cup Y$ also are cut-minimal s-t-cutting subsets.

7.2 Solution

Proof. I will use the notation $\mathrm{s}(a)$ to denote the source of an arc a, and the notation $\mathrm{t}(a)$ to denote the target of an arc a. To begin, note that since X and Y are s-t-cutting subsets, we know $s \in X$, $s \in Y$, $t \notin X$, and $t \notin Y$. Thus it follows that $s \in X \cup Y$, $s \in X \cap Y$, $t \notin X \cup Y$, and $t \notin X \cap Y$. Hence $X \cup Y$ and $X \cap Y$ are s-t-cutting subsets. We want to prove that $X \cup Y$ and $X \cap Y$ are both also cut-minimal. We shall achieve this by showing that $c(X \cup Y, \overline{X \cup Y}) + c(X \cap Y, \overline{X \cap Y}) \leq 2m$.

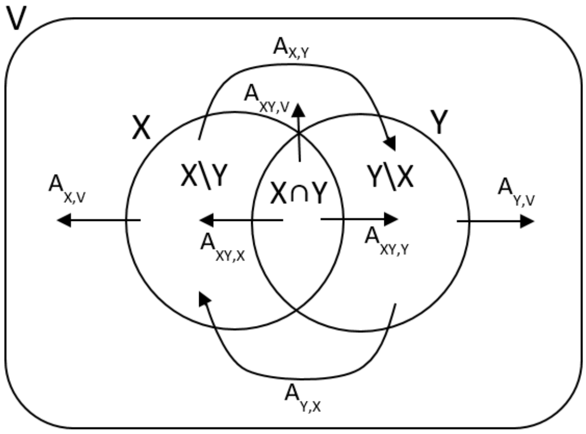
We begin with the following tedious set definitions (illustrated by the perhaps not so helpful figure):

$$\begin{aligned}
A_{X,Y} &= \{a \in A \mid \mathrm{s}(a) \in X \backslash Y \text{ and } \mathrm{t}(a) \in Y \backslash X\}, \\
A_{X,V} &= \{a \in A \mid \mathrm{s}(a) \in X \backslash Y \text{ and } \mathrm{t}(a) \in V \backslash (X \cup Y)\}, \\
A_{XY,Y} &= \{a \in A \mid \mathrm{s}(a) \in X \cap Y \text{ and } \mathrm{t}(a) \in Y \backslash X\}, \\
A_{XY,V} &= \{a \in A \mid \mathrm{s}(a) \in X \cap Y \text{ and } \mathrm{t}(a) \in V \backslash (X \cup Y)\}, \\
A_{Y,X} &= \{a \in A \mid \mathrm{s}(a) \in Y \backslash X \text{ and } \mathrm{t}(a) \in X \backslash Y\}, \\
A_{Y,V} &= \{a \in A \mid \mathrm{s}(a) \in Y \backslash X \text{ and } \mathrm{t}(a) \in V \backslash (X \cup Y)\}, \\
A_{XY,X} &= \{a \in A \mid \mathrm{s}(a) \in X \cap Y \text{ and } \mathrm{t}(a) \in X \backslash Y\}.
\end{aligned}$$

Note that the above sets are pairwise disjoint. Using these sets, we can express the following sets (which are clearly not disjoint in general):

$$\begin{aligned}
[X, \overline{X}] &= A_{X,Y} \cup A_{X,V} \cup A_{XY,Y} \cup A_{XY,V}, \\
[Y, \overline{Y}] &= A_{Y,X} \cup A_{Y,V} \cup A_{XY,X} \cup A_{XY,V}, \\
[X \cup Y, \overline{X \cup Y}] &= A_{X,V} \cup A_{Y,V} \cup A_{XY,V}, \\
[X \cap Y, \overline{X \cap Y}] &= A_{XY,X} \cup A_{XY,Y} \cup A_{XY,V}.
\end{aligned}$$

Figure 1: Schematic representation of set definitions



Set

$$P := [X, \overline{X}] \cup [Y, \overline{Y}],$$
$$Q := [X \cup Y, \overline{X \cup Y}] \cup [X \cap Y, \overline{X \cap Y}].$$

From the above we see that $Q \subseteq P$, and specifically we have

$$P = Q \cup A_{X,Y} \cup A_{Y,X}. \tag{2}$$

Since Q, $A_{X,Y}$ and $A_{Y,X}$ are disjoint, this yields

$$\sum_{a \in P} c(a) = \sum_{a \in Q} c(a) + \sum_{a \in A_{X,Y}} c(a) + \sum_{a \in A_{Y,X}} c(a). \tag{3}$$

Using these set definitions, we have

$$2m = c(X, \overline{X}) + c(Y, \overline{Y}) = \sum_{a \in [X, \overline{X}]} c(a) + \sum_{a \in [Y, \overline{Y}]} c(a) = \sum_{a \in P} c(a) + \sum_{a \in A_{XY,V}} c(a), \tag{4}$$

with the last equality holding because the arcs in $A_{XY,V}$ (and only these arcs) appear in both $[X, \overline{X}]$ and $[Y, \overline{Y}]$, but of course appear only once in P. Similarly,

$$\begin{aligned} c(X \cup Y, \overline{X \cup Y}) + c(X \cap Y, \overline{X \cap Y}) &= \sum_{a \in [X \cup Y, \overline{X \cup Y}]} c(a) + \sum_{a \in [X \cap Y, \overline{X \cap Y}]} c(a) \\ &= \sum_{a \in Q} c(a) + \sum_{a \in A_{XY,V}} c(a), \end{aligned} \tag{5}$$

again with the last equality holding because the arcs in $A_{XY,V}$ appear twice in the original sum, but appear only once in Q. Now, combining (3), (4), and (5), we get

$$\begin{aligned} 2m &= \sum_{a \in Q} c(a) + \sum_{a \in A_{X,Y}} c(a) + \sum_{a \in A_{Y,X}} c(a) + \sum_{a \in A_{XY,V}} c(a) \\ &= c(X \cup Y, \overline{X \cup Y}) + c(X \cap Y, \overline{X \cap Y}) + \sum_{a \in A_{X,Y}} c(a) + \sum_{a \in A_{Y,X}} c(a). \end{aligned}$$

Since $c(a)$ is nonnegative for each $a \in A$, it follows that $c(X \cup Y, \overline{X \cup Y}) + c(X \cap Y, \overline{X \cap Y}) \leq 2m$. But since m is the minimum possible value of $c\left(S, \overline{S}\right)$ where S is an s-t-cutting subset, we also have $c(X \cup Y, \overline{X \cup Y}) \geq m$ and $c(X \cap Y, \overline{X \cap Y}) \geq m$. Hence $c(X \cup Y, \overline{X \cup Y}) = c(X \cap Y, \overline{X \cap Y}) = m$. Both $X \cup Y$ and $X \cap Y$ are therefore cut-minimal s-t-cutting subsets. $\square$