

Math 5707 Spring 2017 (Darij Grinberg): homework set 5
Please hand in solutions to FOUR of the 7 problems.

[**Warning:** You are reading a rather obsolete version of this homework set. For an updated version, in which the basic properties of chip-firing are stated more generally and proven, see the solutions to this problem set.]

0.1. Reminders

See the lecture notes and also the handwritten notes for relevant material. See also the solutions to homework set 2 for various conventions and notations that are in use here.

0.2. Sandpiles: recapitulating definitions and results

Let me recall the definitions of the basic concepts on chipfiring done in class. Various sources on this material are [BjoLov92] (and, less directly, [BjLoSh91]), [HLMPPW13], [Musike09, Lectures 29–31] and [CorPet16]. (None of these is as readable as I would like to have it, but the whole subject is about 30 years old, with most activity very recent... Also, be aware of incompatible notations, as well as of the fact that some of the sources only consider undirected graphs.) The particular case of the “integer lattice” graph has attracted particular attention due to the mysterious pictures it generates; see <http://www.math.cmu.edu/~wes/sand.html#next-page> for some of these pictures, as well as <http://www.math.cornell.edu/~levine/apollonian-slides.pdf> for a talk with various illustrations.

Let me give a brief (proof-less) survey of what we did in class (and a bit of what we should have done).

Fix a loopless multidigraph $D = (V, A, \phi)$.

Definition 0.1. A *configuration* (on D) means a map $f : V \rightarrow \mathbb{N}$. (Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$.)

A configuration is also called a *chip configuration* or *sandpile*.

We like to think of a configuration as a way to place a finite number of game chips on the vertices of D : Namely, the configuration f corresponds to placing $f(v)$ chips on the vertex v for each $v \in V$. The chips are understood to be undistinguishable, so the only thing that matters is how many of them are placed on each given vertex. Sometimes, we speak of grains of sand instead of chips.

Definition 0.2. A \mathbb{Z} -*configuration* (on D) means a map $f : V \rightarrow \mathbb{Z}$. We shall regard each configuration as a \mathbb{Z} -configuration (since $\mathbb{N} \subseteq \mathbb{Z}$).

Definition 0.3. Let $f : V \rightarrow \mathbb{Z}$ be a \mathbb{Z} -configuration.

(a) A vertex $v \in V$ is said to be *active* in f if and only if $f(v) \geq \deg^+ v$. (Recall that $\deg^+ v$ is the outdegree of v .)

(b) The \mathbb{Z} -configuration f is said to be *stable* if no vertex $v \in V$ is active in f .

Notice that there are only finitely many stable configurations (because if f is a stable configuration, then, for each $v \in V$, the stability of f implies $f(v) \leq \deg^+ v$, whereas the fact that f is a configuration implies $f(v) \geq 0$; but these two inequalities combined leave only finitely many possible values for $f(v)$).

Definition 0.4. The set \mathbb{Z}^V of all \mathbb{Z} -configurations can be equipped with operations of addition and subtraction, defined as follows:

- For any two \mathbb{Z} -configurations $f : V \rightarrow \mathbb{Z}$ and $g : V \rightarrow \mathbb{Z}$, we define a \mathbb{Z} -configuration $f + g : V \rightarrow \mathbb{Z}$ by setting

$$(f + g)(v) = f(v) + g(v) \quad \text{for each } v \in V.$$

- For any two \mathbb{Z} -configurations $f : V \rightarrow \mathbb{Z}$ and $g : V \rightarrow \mathbb{Z}$, we define a \mathbb{Z} -configuration $f - g : V \rightarrow \mathbb{Z}$ by setting

$$(f - g)(v) = f(v) - g(v) \quad \text{for each } v \in V.$$

These operations of addition and subtraction satisfy the standard rules (e.g., we always have $(f + g) + h = f + (g + h)$ and $(f - g) - h = f - (g + h)$). Hence, we can write terms like $f + g + h$ or $f - g - h$ without having to explicitly place parentheses.

Also, we can define a “zero configuration” $0 : V \rightarrow \mathbb{Z}$, which is the configuration that sends each $v \in V$ to the number 0. (Hopefully, the dual use of the symbol 0 for both the number 0 and this zero configuration is not too confusing.)

Also, for each \mathbb{Z} -configuration $f : V \rightarrow \mathbb{Z}$ and each integer N , we define a \mathbb{Z} -configuration $Nf : V \rightarrow \mathbb{Z}$ by

$$(Nf)(v) = Nf(v) \quad \text{for each } v \in V.$$

Definition 0.5. Let $f : V \rightarrow \mathbb{Z}$ be any \mathbb{Z} -configuration. Then, $\sum f$ shall denote the integer $\sum_{v \in V} f(v)$.

This integer $\sum f$ is called the *degree* of f .

If f is a configuration, then $\sum f$ is the total number of chips in f .

Definition 0.6. Let $v \in V$ be a vertex. Then, a \mathbb{Z} -configuration Δv is defined by setting

$$(\Delta v)(w) = \begin{cases} \deg^+ v, & \text{if } w = v; \\ -a_{v,w}, & \text{if } w \neq v \end{cases} \quad \text{for all } w \in V,$$

where $a_{v,w}$ denotes the number of all arcs of D having source v and target w . (Note that $a_{v,w}$ might be > 1 , since D is a multidigraph.)

Definition 0.7. Let $v \in V$ be a vertex. Then, *firing* v is the operation on \mathbb{Z} -configurations (i.e., formally speaking, the mapping from \mathbb{Z}^V to \mathbb{Z}^V) that sends each \mathbb{Z} -configuration $f : V \rightarrow \mathbb{Z}$ to $f - \Delta v$.

We sometimes say “toppling v ” instead of “firing v ”.

If $f : V \rightarrow \mathbb{N}$ is a configuration, then the \mathbb{Z} -configuration $f - \Delta v$ obtained by firing v can be described as follows: The vertex v “donates” $\deg^+ v$ of its chips to its neighbors, by sending one chip along each of its outgoing arcs (i.e., for each arc having source v , the vertex v sends one chip along this arc to the target of this arc). Thus, the number of chips on v (weakly) decreases, while the number of chips on each other vertex (weakly) increases. Of course, the resulting \mathbb{Z} -configuration $f - \Delta v$ is not necessarily a configuration. (In fact, it is a configuration if and only if the vertex v is active in f .)

Notice that $\sum (\Delta v) = 0$ for each vertex v . Thus, $\sum (f - \Delta v) = \sum f - \underbrace{\sum (\Delta v)}_{=0} = \sum f$ for each \mathbb{Z} -configuration $f : V \rightarrow \mathbb{Z}$ and each vertex v . In other words, firing a vertex v does not change the degree of a \mathbb{Z} -configuration.

Definition 0.8. Let $f : V \rightarrow \mathbb{N}$ be a configuration.

Let (v_1, v_2, \dots, v_k) be a sequence of vertices of D .

(a) The sequence (v_1, v_2, \dots, v_k) is said to be *legal* for f if for each $i \in \{1, 2, \dots, k\}$, the vertex v_i is active in the \mathbb{Z} -configuration $f - \Delta v_1 - \Delta v_2 - \dots - \Delta v_{i-1}$.

(b) The sequence (v_1, v_2, \dots, v_k) is said to be *stabilizing* for f if the \mathbb{Z} -configuration $f - \Delta v_1 - \Delta v_2 - \dots - \Delta v_k$ is stable.

What is the rationale behind the notions of “legal” and “stabilizing”? A sequence of vertices provides a way to modify a configuration by first firing the first vertex in the sequence, then firing the second, and so on. The sequence is said to be *legal* (for f) if the configuration remains a configuration throughout this ordeal (i.e., at no point does a vertex have a negative number of chips). The sequence is said to be *stabilizing* (for f) if the \mathbb{Z} -configuration resulting from it at the very end is stable.

We notice some obvious consequences of the definitions:

- If a sequence (v_1, v_2, \dots, v_k) is legal for a configuration f , then all of the \mathbb{Z} -configurations $f - \Delta v_1 - \Delta v_2 - \dots - \Delta v_i$ for $i \in \{0, 1, \dots, k\}$ are actually configurations.
- If a sequence (v_1, v_2, \dots, v_k) is legal for a configuration f , then each prefix of this sequence (i.e., each sequence of the form (v_1, v_2, \dots, v_i) for some $i \in \{0, 1, \dots, k\}$) is legal for f as well.
- If a sequence (v_1, v_2, \dots, v_k) is stabilizing for a configuration f , then each permutation of this sequence (i.e., each sequence of the form $(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)})$ for a permutation σ of $\{1, 2, \dots, k\}$) is stabilizing for f as well.

- If (v_1, v_2, \dots, v_k) is a legal sequence for a configuration f , then (v_1, v_2, \dots, v_k) is stabilizing if and only if there exist no $v \in V$ such that the sequence $(v_1, v_2, \dots, v_k, v)$ is legal.

An important property of chipfiring is the following result (sometimes called the “least action principle”):

Theorem 0.9. Let $f : V \rightarrow \mathbb{N}$ be any configuration. Let ℓ and s be two sequences of vertices of D such that ℓ is legal for f while s is stabilizing for f . Then, ℓ is a subpermutation of s .

Here, we are using the following notation:

Definition 0.10. Let (p_1, p_2, \dots, p_u) and (q_1, q_2, \dots, q_v) be two finite sequences. Then, we say that (p_1, p_2, \dots, p_u) is a *subpermutation* of (q_1, q_2, \dots, q_v) if and only if, for each object t , the following holds: The number of $i \in \{1, 2, \dots, u\}$ satisfying $p_i = t$ is less or equal to the number of $j \in \{1, 2, \dots, v\}$ satisfying $q_j = t$.

Equivalently, the sequence (p_1, p_2, \dots, p_u) is a subpermutation of the sequence (q_1, q_2, \dots, q_v) if and only if you can obtain the former from the latter by removing some entries and permuting the remaining entries. (“Some” allows for the possibility of “zero”.)

Corollary 0.11. Let $f : V \rightarrow \mathbb{N}$ be any configuration. Let ℓ and ℓ' be two sequences of vertices of D that are both legal and stabilizing for f . Then:

(a) The sequence ℓ' is a permutation of ℓ .

In particular:

(b) The sequences ℓ and ℓ' have the same length.

(c) For each $t \in V$, the number of times t appears in ℓ' equals the number of times t appears in ℓ .

(d) The configuration obtained from f by firing all vertices in ℓ (one after the other) equals the configuration obtained from f by firing all vertices in ℓ' (one after the other).

Next we state some facts about legal sequences:

Lemma 0.12. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Let $h = \sum f$. Let ℓ be a legal sequence for f .

Let a be an arc of D . Let u be the source of a , and let v be the target of a .

(a) If u appears more than h times in the sequence ℓ , then v must appear at least once in the sequence ℓ .

(b) Fix $k \in \mathbb{N}$. If u appears more than kh times in the sequence ℓ , then v must appear at least k times in the sequence ℓ .

Lemma 0.13. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Let $h = \sum f$. Let ℓ be a legal sequence for f .

Let u and v be two vertices of D such that there exists a path of length d from u to v .

If u appears at least $\frac{h^{d+1} - 1}{h - 1}$ times in the sequence ℓ , then v must appear at least once in the sequence ℓ .

(The fraction $\frac{h^{d+1} - 1}{h - 1}$ should be interpreted as $d + 1$ when $h = 1$.)

Proposition 0.14. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Let $h = \sum f$. Let ℓ be a legal sequence for f . Let $n = |V|$.

Let q be a vertex of D such that for each vertex $u \in V$, there exists a path from u to q .

If the length of ℓ is $> (n - 1) \left(\frac{h^n - 1}{h - 1} - 1 \right)$, then q must appear at least once in the sequence ℓ .

(The fraction $\frac{h^n - 1}{h - 1}$ should be interpreted as n when $h = 1$.)

Proposition 0.15. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Let $h = \sum f$. Let $\ell = (\ell_1, \ell_2, \dots, \ell_k)$ be a legal sequence for f . Let $g = f - \Delta\ell_1 - \Delta\ell_2 - \dots - \Delta\ell_k$ be the configuration obtained from f by firing the vertices in ℓ (one after the other).

(a) We have $g \in \{0, 1, \dots, h\}^V$. (In other words, $g(v) \in \{0, 1, \dots, h\}$ for each $v \in V$.)

(b) Let $n = |V|$. If the sequence ℓ has length $\geq (h + 1)^n$, then there exist legal sequences (for f) of arbitrary length.

Definition 0.16. Let $f : V \rightarrow \mathbb{N}$ be a configuration.

We say that f is *finitary* if there exists a sequence of vertices that is stabilizing for f . Otherwise, we say that f is *infinitary*.

Theorem 0.17. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Then, exactly one of the following two statements holds:

- *Statement 1:* The configuration f is finitary.
 There exists a sequence s of vertices that is both legal and stabilizing for f .
 All such sequences are permutations of s .
 All legal sequences (for f) are subpermutations of s , and in particular are at most as long as s .
- *Statement 2:* The configuration f is infinitary.
 There exists no stabilizing sequence for f .

There exist legal sequences for f of arbitrary length. More precisely, each legal sequence for f can be extended to a longer legal sequence.

Definition 0.18. Let $f : V \rightarrow \mathbb{N}$ be a finitary configuration. Then, Statement 1 in Theorem 0.17 must hold. Therefore, there exists a sequence s of vertices that is both legal and stabilizing for f . The *stabilization* of f means the configuration obtained from f by firing all vertices in s (one after the other). (This does not depend on the choice of s , because of Corollary 0.11 (d).)

The stabilization of f is denoted by f° .

Something similar holds if we forbid firing a specific vertex:

Definition 0.19. Let $q \in V$.

Let $f : V \rightarrow \mathbb{N}$ be a configuration.

Let (v_1, v_2, \dots, v_k) be a sequence of vertices of D .

(a) The sequence (v_1, v_2, \dots, v_k) is said to be *q -legal* for f if it is legal and does not contain the vertex q .

(b) The sequence (v_1, v_2, \dots, v_k) is said to be *q -stabilizing* for f if the \mathbb{Z} -configuration $f - \Delta v_1 - \Delta v_2 - \dots - \Delta v_k$ has no active vertices except (possibly) q .

We can now define “ q -finitary” and “ q -infinitary” and obtain an analogue of Theorem 0.17. But the most commonly considered case is that when q is a “global sink” (a vertex with no outgoing arcs, and which is reachable from any vertex), and in this case **every** configuration is q -finitary. Let us state this as its own result:

Theorem 0.20. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Let $q \in V$. Assume that for each vertex $u \in V$, there exists a path $u \rightarrow q$. Then, there exists a sequence s of vertices that is both q -legal and q -stabilizing for f . All such sequences are permutations of s . All q -legal sequences (for f) are subpermutations of s , and in particular are at most as long as s .

Definition 0.21. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Let $q \in V$. Assume that for each vertex $u \in V$, there exists a path $u \rightarrow q$. Then, Theorem 0.20 shows that there exists a sequence s of vertices that is both q -legal and q -stabilizing for f . The *q -stabilization* of f means the configuration obtained from f by firing all vertices in s (one after the other). (This does not depend on the choice of s , because of the analogue of Corollary 0.11 (d) for q -legal and q -stabilizing sequences.)

0.3. Exercise 1: better bounds for legal sequences

The following exercise improves on the bound given in Proposition 0.15 **(b)** and also on the one given in Proposition 0.14¹. I don't know whether the improved bounds can be further improved.

Exercise 1. Fix a loopless multidigraph $D = (V, A, \phi)$. Let $f : V \rightarrow \mathbb{N}$ be a configuration. Let $h = \sum f$. Let $n = |V|$. Assume that $n > 0$.

Let $\ell = (\ell_1, \ell_2, \dots, \ell_k)$ be a legal sequence for f having length $k \geq \binom{n+h-1}{n-1}$.

Prove the following:

(a) There exist legal sequences (for f) of arbitrary length.

(b) Let q be a vertex of D such that for each vertex $u \in V$, there exists a path from u to q . Then, q must appear at least once in the sequence ℓ .

[**Hint:** For **(a)**, apply the same pigeonhole-principle argument as for Proposition 0.15 **(b)**.]

In the above exercise, you are allowed to use the fact² that the number of n -tuples (a_1, a_2, \dots, a_n) of nonnegative integers satisfying $a_1 + a_2 + \dots + a_n = h$ is $\binom{n+h-1}{n-1}$.

0.4. Exercise 2: examples of chip-firing

¹To see that Exercise 1 **(b)** improves on the bound given in Proposition 0.14, we need to check that $(n-1) \left(\frac{h^n-1}{h-1} - 1 \right) + 1 \geq \binom{n+h-1}{n-1}$. This is easy for $n \leq 1$ (in fact, the case $n = 0$ is impossible due to the existence of a $q \in V$, and the case $n = 1$ is an equality case). In the remaining case $n \geq 2$, the stronger inequality $\frac{h^n-1}{h-1} - 1 + 1 \geq \binom{n+h-1}{n-1}$ can be proven by a simple induction over n .

²See, for example, <https://math.stackexchange.com/questions/36250/number-of-monomials-of-certain-degree> for a proof of this fact (in the language of monomials). Or see [Stanle11, §1.2] (search for “weak composition” and read the first paragraph that comes up).

Exercise 2. (a) Let D be the following digraph:

$$u \longrightarrow v \longrightarrow q$$

(i.e., the digraph D with three vertices u, v, q and two arcs uv, vq).

Let k be a positive integer. Consider the configuration g_k on D which has k chips at u and 0 chips at each other vertex.

Find the q -stabilization of g_k .

(b) Let D be the following digraph:

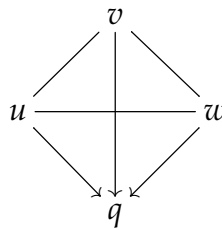
$$u \longleftrightarrow v \longrightarrow q$$

where a curve without an arrow stands for one arc in each direction. (Thus, formally speaking, the digraph D has three vertices u, v, q and three arcs uv, vu, vq .)

Let k be a positive integer. Consider the configuration g_k on D which has k chips at u and 0 chips at each other vertex.

Find the q -stabilization of g_k .

(c) Let D be the following digraph:



where a curve without an arrow stands for one arc in each direction. (Thus, formally speaking, the digraph D has four vertices u, v, w, q and nine arcs $uv, vu, vw, wv, wu, uw, uq, vq, wq$.)

Let $k \geq 2$ be an integer. Consider the configuration f_k on D which has k chips at each vertex (i.e., which has $f_k(v') = k$ for each $v' \in \{u, v, w, q\}$).

Find the q -stabilization of f_k .

0.5. Exercise 3: a lower bound on the degree of an infinitary configuration

Exercise 3. Assume that the multidigraph D is strongly connected. Let $f : V \rightarrow \mathbb{N}$ be an infinitary configuration.

(a) Prove that D cannot have more than $\sum f$ vertex-disjoint cycles. (A set of cycles is said to be *vertex-disjoint* if no two distinct cycles in the set have a vertex in common.)

(b) Prove that D cannot have more than $\sum f$ arc-disjoint cycles. (A set of cycles is said to be *arc-disjoint* if no two distinct cycles in the set have an arc in common.)

Exercise 3 (b) is [BjoLov92, Theorem 2.2], but the proof given there is vague and unrigorous.

0.6. Exercise 4: an associativity law for stabilizations

Recall Definition 0.18.

Exercise 4. Let $f : V \rightarrow \mathbb{N}$, $g : V \rightarrow \mathbb{N}$ and $h : V \rightarrow \mathbb{N}$ be three configurations such that both configurations f and $g + h$ are finitary, and such that the configuration $f + (g + h)^\circ$ is also finitary.

Prove the following:

(a) The configurations $f + g$ and h are also finitary.

(b) The configurations $f + g + h$ and $(f + g)^\circ + h$ are also finitary, and satisfy

$$(f + g + h)^\circ = (f + (g + h)^\circ)^\circ = ((f + g)^\circ + h)^\circ.$$

[Hint: The following piece of notation is useful: If k and k' are two configurations, then $k \rightarrow k'$ shall mean that there exists a legal sequence ℓ for k such that firing all vertices in ℓ (one after the other) transforms k into k' . This relation \rightarrow is reflexive and transitive. Show that if c , k and k' are three configurations satisfying $k \rightarrow k'$, then $c + k \rightarrow c + k'$.]

0.7. Exercise 5: chip-firing on the integer lattice

Now, we shall briefly discuss chip-firing on the integer lattice \mathbb{Z}^2 ; this is one of the most famous cases of chip-firing, leading to some of the pretty pictures. For examples and illustrations, check out [Ellenb15] as well as some of the links above.

We have not defined infinite graphs in class; the theory of infinite graphs involves some subtleties that would take us too far. However, for this particular exercise, we need only a specific infinite graph, which is fairly simple.

Definition 0.22. (a) A *locally finite multigraph* means a triple (V, E, ϕ) , where V and E are sets and $\phi : E \rightarrow \mathcal{P}_2(V)$ is a map having the following property:

(*) For each $v \in V$, there exist only finitely many $e \in E$ satisfying $v \in \phi(e)$.

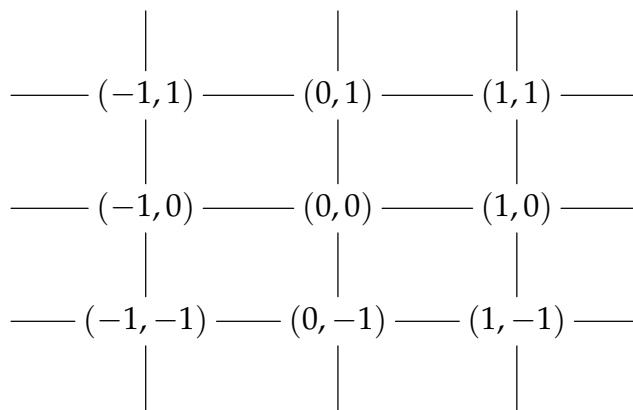
Most of the concepts defined for (usual) multigraphs still make sense for locally finite multigraphs. In particular, the elements of V are called the vertices, and the elements of E are called the edges. The property (*) says that each vertex is contained in only finitely many edges; this allows us to define the degree of a vertex.

(b) The *integer lattice* shall mean the locally finite multigraph defined as follows:

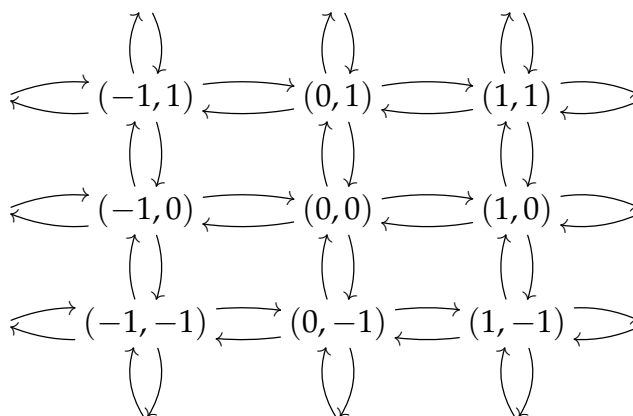
- The vertices of the integer lattice are the pairs (i, j) of two integers i and j . In other words, the vertex set of the integer lattice is $\mathbb{Z}^2 = \{(i, j) \mid i \in \mathbb{Z} \text{ and } j \in \mathbb{Z}\}$. We view these vertices as points in the plane, and draw the multigraph accordingly.
- Two vertices of the integer lattice are adjacent if and only if they have distance 1 (as points in the plane). In other words, a vertex (i, j) is adjacent to the four vertices $(i + 1, j)$, $(i, j + 1)$, $(i - 1, j)$, $(i, j - 1)$ and no others.

(c) You can guess how locally finite multidigraphs are defined. Each locally finite multigraph can be regarded as a locally finite multidigraph by replacing each edge by a pair of two arcs (directed in both possible directions).

Let us show a piece of the integer lattice, viewed as a locally finite multigraph:



And here is it again, viewed as a locally finite multidigraph:



Exercise 5. Let f be a configuration on the integer lattice (where we view the integer lattice as a locally finite multidigraph). (The notion of a configuration and related notions are defined in the same way as for usual, finite multidigraphs.)

Assume that only finitely many vertices $v \in \mathbb{Z}^2$ satisfy $f(v) \neq 0$. (Thus, the total number of chips $\sum f$ is finite.)

An edge e of the integer lattice is said to be *non-void* in f if and only if at least one of the endpoints of e has at least one chip in f .

Prove the following:

(a) If an edge of the integer lattice is non-void in f , then this edge remains non-void after firing any legal sequence of vertices. (“Firing a sequence” means firing all the vertices in the sequence, one after the other.)

(b) The total number of configurations that can be obtained from f by firing a legal sequence of vertices is finite.

(c) If we fire any vertex, then the sum $\sum_{(i,j) \in \mathbb{Z}^2} f((i,j)) \cdot (i+j)^2$ increases.

(d) The configuration f is finitary (so its stabilization is well-defined).

This exercise gives the reason why pictures such as the ones in [Ellenb15] exist (although it does not explain their shapes and patterns).

0.8. Exercise 6: acyclic orientations are determined by their score vectors

Now, we leave the chip-firing setting.

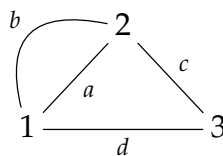
Roughly speaking, an *orientation* of a multigraph G is a way to assign to each edge of G a direction (thus making it an arc). If the resulting **digraph** has no cycles, then this orientation will be called *acyclic*. A rigorous way to state this definition is the following:

Definition 0.23. Let $G = (V, E, \psi)$ be a multigraph.

(a) An *orientation* of G is a map $\phi : E \rightarrow V \times V$ such that each $e \in E$ has the following property: If we write $\phi(e)$ in the form $\phi(e) = (u, v)$, then $\psi(e) = \{u, v\}$.

(b) An orientation ϕ of G is said to be *acyclic* if and only if the multidigraph (V, E, ϕ) has no cycles.

Example 0.24. Let $G = (V, E, \psi)$ be the following multigraph:

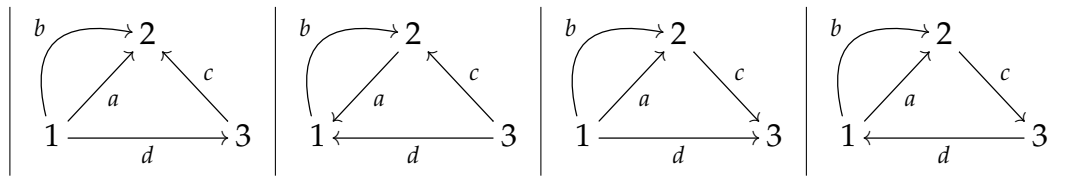


Then, the following four maps ϕ are orientations of G :

- the map sending a to $(1, 2)$, sending b to $(1, 2)$, sending c to $(3, 2)$, and sending d to $(1, 3)$;

- the map sending a to $(2,1)$, sending b to $(1,2)$, sending c to $(3,2)$, and sending d to $(3,1)$;
- the map sending a to $(1,2)$, sending b to $(1,2)$, sending c to $(2,3)$, and sending d to $(1,3)$;
- the map sending a to $(1,2)$, sending b to $(1,2)$, sending c to $(2,3)$, and sending d to $(3,1)$.

Here are the multidigraphs (V, E, ϕ) corresponding to these four maps (in the order mentioned):



Only the first and the third of these orientations ϕ are acyclic (since only the first and the third of these multidigraphs have no cycles).

Exercise 6. Let $G = (V, E, \psi)$ be a multigraph.

Prove the following:

(a) If ϕ is any acyclic orientation of G , and if $V \neq \emptyset$, then there exists a $v \in V$ such that no arc of the multidigraph (V, E, ϕ) has target v .

(b) If ϕ_1 and ϕ_2 are two acyclic orientations of G such that each $v \in V$ satisfies

$$\deg_{(V, E, \phi_1)}^+ v = \deg_{(V, E, \phi_2)}^+ v,$$

then $\phi_1 = \phi_2$.

0.9. Exercise 7: the lattice structure on minimum cuts

Let us recall some terminology from lecture 16:

- A *network* consists of:
 - a digraph (V, A) ;
 - two distinct vertices $s \in V$ and $t \in V$, called the *source* and the *sink*, respectively (although we do not require s to have indegree 0 or t to have outdegree 0);
 - a function $c : A \rightarrow \mathbb{Q}_+$, called the *capacity function*. (Here, \mathbb{Q}_+ means the set $\{x \in \mathbb{Q} \mid x \geq 0\}$.)
- Given a network consisting of a digraph (V, A) , a source $s \in V$ and a sink $t \in V$, and a capacity function $c : A \rightarrow \mathbb{Q}_+$, we define the following notations:

- For any subset S of V , we let \bar{S} denote the subset $V \setminus S$ of V .
- If P and Q are two subsets of V , then $[P, Q]$ shall mean the set of all arcs $a \in A$ whose source belongs to P and whose target belongs to Q . (In other words, $[P, Q] = A \cap (P \times Q)$.)
- If P and Q are two subsets of V , then the number $c(P, Q) \in \mathbb{Q}_+$ is defined by

$$c(P, Q) = \sum_{a \in [P, Q]} c(a).$$

We also refer to lecture 16 for the definition of a flow (which is not necessary for the following problem, but may be helpful).

Exercise 7. Consider a network consisting of a digraph (V, A) , a source $s \in V$ and a sink $t \in V$, and a capacity function $c : A \rightarrow \mathbb{Q}_+$ such that $s \neq t$.

An *s-t-cutting subset* shall mean a subset S of V satisfying $s \in S$ and $t \notin S$.

Let m denote the minimum possible value of $c(S, \bar{S})$ where S ranges over the *s-t-cutting subsets*. (Recall that this is the maximum value of a flow, according to the maximum-flow-minimum-cut theorem.)

An *s-t-cutting subset* S is said to be *cut-minimal* if it satisfies $c(S, \bar{S}) = m$.

Let X and Y be two cut-minimal *s-t-cutting subsets*. Prove that $X \cap Y$ and $X \cup Y$ also are cut-minimal *s-t-cutting subsets*.

References

- [BjoLov92] Anders Björner, László Lovász, *Chip-firing games on directed graphs*, J. Algebraic Combinatorics 1 (1991), pp. 305–328, revised version, July 1992.
<http://www.cs.elte.hu/~lovasz/morepapers/abacus.pdf>
- [BjLoSh91] Anders Björner, László Lovász, P. W. Shor, *Chip-firing games on graphs*, Europ. J. Comb. 12 (1991), pp. 283–291.
<http://www.cs.elte.hu/~lovasz/morepapers/chips.pdf>
- [CorPet16] Scott Corry, David Perkinson, *Divisors and Sandpiles*, draft of 20 November 2016.
http://people.reed.edu/~davidp/divisors_and_sandpiles/
- [Ellenb15] Jordan Ellenberg, *The Amazing, Autotuning Sandpile*, Nautilus, 2 April 2015.
<http://nautil.us/issue/23/dominoes/the-amazing-autotuning-sandpile>
- [HLMPPW13] Alexander E. Holroyd, Lionel Levine, Karola Mészáros, Yuval Peres, James Propp, David B. Wilson, *Chip-Firing and Rotor-Routing on Directed Graphs*, arXiv:0801.3306v4.
<https://arxiv.org/abs/0801.3306v4>

- [Musike09] Gregg Musiker, *18.312: Algebraic Combinatorics, Spring 2009*, MIT OpenCourseWare.
<https://ocw.mit.edu/courses/mathematics/18-312-algebraic-combinatorics-spring-2009/readings-and-lecture-notes/>
- [Stanle11] Richard P. Stanley, *Enumerative Combinatorics, volume 1*, Cambridge University Press, 2011.
<http://math.mit.edu/~rstan/ec/ec1/>
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