Mathematics 5707 Homework 4

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Exercise 3. Let $(G; X, Y)$ and $(H; U, V)$ be bipartite graphs. Assume that G is a simple graph and has an X-complete matching. Assume that H is a simple graph and has a U-complete matching. Consider the Cartesian product $G \times H$ of G and H defined in Exercise 1 of homework set 2.

(a) Show that $(G \times H; (X \times V) \cup (Y \times U), (X \times U) \cup (Y \times V))$ is a bipartite graph.

Proof: The vertex set of G is $X \cup Y$, while the vertex set of H is $U \cup V$. Hence, the vertex set of $G \times H$ is

$$\begin{aligned} & (X \cup Y) \times (U \cup V) \\ & = (X \times (U \cup V)) \cup (Y \times (U \cup V)) \\ & \qquad \text{(by distributivity for set union and Cartesian product)} \\ & = ((X \times U) \cup (X \times V)) \cup ((Y \times U) \cup (Y \times V)) \\ & \qquad \text{(by distributivity for set union and Cartesian product)} \\ & = ((X \times V) \cup (Y \times U)) \cup ((X \times U) \cup (Y \times V)) . \end{aligned}$$

Moreover, the two sets $(X \times V) \cup (Y \times U)$ and $(X \times U) \cup (Y \times V)$ are disjoint, because the four sets $X \times U$, $X \times V$, $Y \times U$ and $Y \times V$ are disjoint (which, in turn, follows from the disjointness of X and Y and the disjointness of U and V).

Hence, in order to show that $(G \times H; (X \times V) \cup (Y \times U), (X \times U) \cup (Y \times V))$ is a bipartite graph, we only need to prove that each edge of $G \times H$ has an endpoint in $(X \times V) \cup (Y \times U)$ and an endpoint in $(X \times U) \cup (Y \times V)$. So let e be an edge of $G \times H$. By the definition of $G \times H$, this means that we are in one of the following two cases:

> *Case 1:* The edge e connects the vertex (p, a) with the vertex (p, b), where p is some vertex of G and where a and b are two vertices of H such that ab is an edge of H.

Case 2: The edge e connects the vertex (a,q) with the vertex (b,q), where q is some vertex of H and where a and b are two vertices of G such that ab is an edge of G.

Let us only study Case 1 (as Case 2 is similar). In this case, consider the edge ab. Since H is bipartite, one of its endpoints a and b belongs to U, while the other belongs to V. Thus, depending on whether p belongs to X or to Y, the edge e either has an endpoint in $X \times U$ and an endpoint in $X \times V$, or has an endpoint in $Y \times U$ and an endpoint in $Y \times V$. In either case, the edge e thus has an endpoint in $(X \times V) \cup (Y \times U)$ and an endpoint in $(X \times U) \cup (Y \times V)$. This proves what we wanted to prove in Case 1. (As we said, Case 2 is similar.) ■

(b) Prove that the graph $G \times H$ has an $(X \times V) \cup (Y \times U)$-complete matching.

Proof: The graph G has an X-complete matching. Fix such a matching and denote it by M. For each vertex p of G that is matched in M, we denote the M-partner of p by p'. Note that each $x \in X$ is matched in M (since the matching M is X-complete) and satisfies $x' \in Y$ (since xx' is an edge of G, but $(G; X, Y)$ is a bipartite graph).

The graph H has a U-complete matching. Fix such a matching and denote it by N. For each vertex p of H that is matched in N, we denote the N-partner of p by p'. Note that each $u \in U$ is matched in N (since the matching N is U-complete) and satisfies $u' \in V$ (since $(H; U, V)$ is a bipartite graph). Also, each $v \in V$ that is matched in N must satisfy $v' \in U$ (since $(H; U, V)$ is a bipartite graph).

Now, it is straightforward to verify that

$$\begin{aligned} &\{\{(x,v),(x,v')\} \mid v \in V \text{ is matched in } N, \text{ and } x \in X\} \\ &\quad \cup \{\{(x,v),(x',v)\} \mid v \in V \text{ is not matched in } N, \text{ and } x \in X\} \\ &\quad \cup \{\{(y,u),(y,u')\} \mid u \in U \text{ and } y \in Y\} \end{aligned}$$

is an $(X \times V) \cup (Y \times U)$-complete matching of $G \times H$. Thus, such a matching exists. ■

Exercise 4. Let S be a finite set. Let $k \in \mathbb{N}$ be such that $|S| \geq 2k+1$. Prove that there exists an injective map $f : \mathcal{P}_k(S) \to \mathcal{P}_{k+1}(S)$ such that each $X \in \mathcal{P}_k(S)$ satisfies $f(X) \supseteq X$.

(In other words, prove that we can add to each k-element subset X of S an additional element from $S \setminus X$ such that the resulting $(k+1)$-element subsets are distinct.)

Proof: Define the bipartite graph $(G; \mathcal{P}_k(S), \mathcal{P}_{k+1}(S))$ as follows:

$$\mathbf{V}(G) = \mathcal{P}_k(S) \cup \mathcal{P}_{k+1}(S),$$
$$\mathbf{E}(G) = \{\{X, Y\} \mid X \in \mathcal{P}_k(S),\ Y \in \mathcal{P}_{k+1}(S),\ X \subset Y\}.$$

Thus, a $\mathcal{P}_k(S)$-complete matching of G corresponds to an injective map $f : \mathcal{P}_k(S) \to \mathcal{P}_{k+1}(S)$ such that each $X \in \mathcal{P}_k(S)$ satisfies $f(X) \supseteq X$, where each edge in the matching is of the form $\{X, f(X)\}$. We must show that such a matching exists. By Hall's theorem, we have such a matching if for every subset $A \subseteq \mathcal{P}_k(S)$, $|N(A)| \geq |A|$. Observe that every vertex in $\mathcal{P}_k(S)$ has degree $|S| - k \geq k + 1$. Thus, any subset A has $|A|(|S| - k)$ edges leaving it, so $N(A)$ has at least $|A|(|S| - k)$ edges incident upon it. But, each vertex in $N(A)$ has degree $k + 1$, so $|N(A)|(k + 1) \geq |A|(|S| - k) \geq |A|(k + 1)$. This implies that $|N(A)| \geq |A|$ for all subsets A. Therefore, we have a $\mathcal{P}_k(S)$-complete matching of G, which implies that there exists an injective map $f : \mathcal{P}_k(S) \to \mathcal{P}_{k+1}(S)$ such that each $X \in \mathcal{P}_k(S)$ satisfies $f(X) \supseteq X$. ■

Exercise 5. Let S be a finite set, and let $k \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_k$ be k subsets of S such that each element of S lies in exactly one of these k subsets. Prove that the following statements are equivalent:

Statement 1: There exists a bijection $\sigma : S \to S$ such that each $i \in \{1, 2, \ldots, k\}$ satisfies $\sigma(A_i) \cap A_i = \emptyset$.

Statement 2: Each $i \in \{1, 2, \ldots, k\}$ satisfies $|A_i| \leq \frac{|S|}{2}$.

Proof: To prove Statement 1 implies Statement 2, suppose a bijection $\sigma : S \to S$ such that each $i \in 1, 2, \ldots, k$ satisfies $\sigma(A_i) \cap A_i = \emptyset$ exists and that there exists an $i \in \{1, 2, \ldots, k\}$ such that $|A_i| > \frac{|S|}{2}$. Then, $|\sigma(A_i)| \leq |S \setminus A_i| < \frac{|S|}{2}$, so $|\sigma(A_i)| \neq |A_i|$ which contradicts the assumption that σ is a bijection. Therefore Statement 1 implies Statement 2.

To prove the converse, define the bipartite graph $(G; S, \mathcal{P}_1(S))$ where $\mathbf{V}(G) = S \cup \mathcal{P}_1(S)$ and $\mathbf{E}(G) = \{\{s_1, \{s_2\}\} \mid s_1, s_2 \in S,$ if $s_1 \in A_i,$ then $s_2 \notin A_i\}$. Thus, an S-complete matching of G corresponds to a bijection $\sigma : S \to S$ such that each $i \in 1, 2, \ldots, k$ satisfies $\sigma(A_i) \cap A_i = \emptyset$. Such a matching exists if for every subset $B \subset S$, $|N(B)| \geq |B|$. Clearly, this is the case, since if $B \subseteq A_i$, then $|B| \leq \frac{|S|}{2}$ and $|N(B)| = |\{\{s\} \mid s \in S \setminus A_i\}| = |S \setminus A_i| \geq \frac{|S|}{2}$, and if B contains elements from more than one of the A_i, then $N(B) = \mathcal{P}_1(S)$, which has as many elements as S itself. Therefore, Hall's theorem implies that there exists an S-complete matching of G, and from such a matching we can construct a bijection $\sigma : S \to S$ such that each $i \in 1, 2, \ldots, k$ satisfies $\sigma(A_i) \cap A_i = \emptyset$. Hence, Statement 1 and Statement 2 are equivalent. ■

Exercise 6. Let $(G; X, Y)$ be a bipartite graph. Assume that each $S \subseteq X$ satisfies $|N(S)| \geq |S|$. (Thus, Hall's theorem shows that G has an X-complete matching.)

A subset S of X will be called neighbor-critical if $|N(S)| = |S|$.

Let A and B be two neighbor-critical subsets of X. Prove that the subsets $A \cup B$ and $A \cap B$ are also neighbor-critical.

Proof: Let M be an X-complete matching of G. (This exists, according to the parenthetical statement in the exercise.) Consider the following lemma:

Lemma: S is a neighbor-critical subset of X if and only if $N(S) = \{y \in Y \mid \text{there exists } x \in S \text{ such that } xy \in M\}$.

Proof of Lemma: Suppose S is a neighbor critical subset of X. Then, $|N(S)| = |S|$, and M matches $|S|$ elements of Y to the elements of S. Therefore, these elements can be the only elements of $N(S)$, so $N(S) = \{y \in Y \mid \text{there exists } x \in S \text{ such that } xy \in M\}$. Now, let S be an arbitrary subset of X such that $N(S) = \{y \in Y \mid \text{there exists } x \in S \text{ such that } xy \in M\}$. Clearly, since M matches each element of $N(S)$ to an element of S, $|N(S)| = |S|$. □

Let A and B be neighbor-critical subsets of X. Then,

$$\begin{aligned} N(A \cup B) &= N(A) \cup N(B) \\ &= \{y \in Y \mid \text{there exists } x \in A \text{ such that } xy \in M\} \cup \{y \in Y \mid \\ &\quad \text{there exists } x \in B \text{ such that } xy \in M\} \\ &= \{y \in Y \mid \text{there exists } x \in A \cup B \text{ such that } xy \in M\}. \end{aligned}$$

Therefore, the lemma implies that $A \cup B$ is neighbor-critical. Now, since G has an X-complete matching, $|N(A \cap B)| \geq |A \cap B|$. By way of contradiction, suppose this inequality is strict. Then, $N(A \cap B)$ contains a vertex v that is not matched to a vertex in $A \cap B$ by M. Now, since A and B are neighbor-critical, v is matched to either a vertex in $A \setminus B$ or $B \setminus A$. Assume without loss of generality that v is matched to a vertex in $A \setminus B$. Then, $N(B)$ includes the $|B|$ vertices that are matched to vertices in B and v, so $|N(B)| > |B|$, a contradiction. Therefore, $A \cap B$ is neighbor-critical. ■