Math 5707 Spring 2017 (Darij Grinberg): homework set 4
Please hand in solutions to FIVE of the 7 problems.

0.1. Reminders

See the lecture notes and also the handwritten notes for relevant material. See also the solutions to homework set 2 for various conventions and notations that are in use here.

0.2. Exercise 1: Directed arc-disjoint version of Menger's theorem

In class, you have seen the following two theorems:

Theorem 0.1 (Menger's theorem, DVE (directed vertex-disjoint version))**.** Let $D = (V, A)$ be a digraph. Let S and T be two subsets of V.

An *S-T-path* in D means a path in D whose starting point belongs to S and whose ending point belongs to T.

Several paths are said to be *vertex-disjoint* if no two have a vertex in common.

A subset C of V is said to be *S-T-disconnecting* if each S-T-path contains at least one vertex in C.

The maximum number of vertex-disjoint S-T-paths equals the minimum size of an S-T-disconnecting subset of V.

Theorem 0.2 (Menger's theorem, DVI (directed internally vertex-disjoint version))**.** Let $D = (V, A)$ be a digraph. Let s and t be two vertices of D such that $(s, t) \notin A$.

An *s-t-path* in D means a path from s to t in D.

Several s-t-paths are said to be *internally vertex-disjoint* if no two have a vertex in common except for the vertices s and t.

A subset C of V is said to be an *s-t-vertex-cut* if it contains neither s nor t, and if each s-t-path contains at least one vertex in C.

The maximum number of internally vertex-disjoint s-t-paths equals the minimum size of an s-t-vertex-cut.

Here are two other versions of Menger's theorem:

Theorem 0.3 (Menger's theorem, DA (directed arc-disjoint version))**.** Let $D = (V, A, \phi)$ be a multidigraph. Let s and t be two distinct vertices of D.

An *s-t-path* in D means a path from s to t in D.

Several paths in D are said to be *arc-disjoint* if no two have an arc in common.

A subset C of A is said to be an *s-t-cut* if it has the form

$$C = \{a \in A \mid \text{the source of } a \text{ belongs to } U, \text{but the target of } a \text{ does not}\}$$

for some subset U of V satisfying $s \in U$ and $t \notin U$.

The maximum number of arc-disjoint s-t-paths equals the minimum size of an s-t-cut.

Remark 0.4. The assumption in Theorem 0.3 that s and t be distinct is not really necessary – you can omit it if you are willing to put up with the idea that the maximum number of arc-disjoint s-t-paths is ∞ (because you can count the trivial path (s) infinitely often, thanks to it being arc-disjoint from itself), and that the minimum size of an s-t-cut is ∞ as well (since there exists no s-t-cut, and thus you are taking the minimum of an empty set of integers, which according to one possible convention is ∞). Feel free to ignore these kinds of hairsplitting.

Theorem 0.5 (Menger's theorem, DAS (directed arc-disjoint set version))**.** Let $D = (V, A, \phi)$ be a multidigraph. Let S and T be two disjoint subsets of V.

An *S-T-path* in D means a path in D whose starting point belongs to S and whose ending point belongs to T.

Several paths in D are said to be *arc-disjoint* if no two have an arc in common.

A subset C of A is said to be an *S-T-cut* if it has the form

$$C = \{a \in A \mid \text{the source of } a \text{ belongs to } U, \text{but the target of } a \text{ does not}\}$$

for some subset U of V satisfying $S \subseteq U$ and $T \cap U = \varnothing$.

The maximum number of arc-disjoint S-T-paths equals the minimum size of an S-T-cut.

Exercise 1. Prove Theorem 0.3 and Theorem 0.5. (You are allowed to use Theorem 0.1 and Theorem 0.2.)

0.3. Exercise 2: Undirected Menger's theorems

Exercise 2. State and prove analogues of Theorem 0.1, Theorem 0.2, Theorem 0.3 and Theorem 0.5 for undirected graphs (simple graphs in the case of the first two theorems; multigraphs for the last two). (You can use the directed-graph versions in the proofs.)

0.4. Exercise 3: matchings in a Cartesian product

Recall that each simple graph $G = (V, E)$ can be viewed as a multigraph in a natural way (namely, as the multigraph (V, E, ι), where ι is the map from E to $\mathcal{P}_2(V)$ sending each edge e to e). Thus, everything we say about multigraphs can be applied to simple graphs.

Let me recall a definition from class in slightly greater generality:

> **Definition 0.6.** Let $G = (V, E, \phi)$ be a multigraph. A *matching* in G means a subset M of E such that no two distinct edges in M have a vertex in common.

In class, we have been discussing matchings in simple graphs; of course, this is a particular case of matchings in multigraphs. The difference between simple graphs and multigraphs does not really matter for the purpose of the existence or nonexistence of matchings (after all, a matching cannot use more than one edge through each vertex, so it cannot include two parallel edges); but it matters if you want to, e.g., count matchings.

> **Definition 0.7.** A *bipartite graph* is a triple (G, X, Y), where G is a multigraph, and where X and Y are two subsets of $\mathrm{V}(G)$ satisfying the following conditions:
>
> - We have $X \cap Y = \varnothing$ and $X \cup Y = \mathrm{V}(G)$. (In other words, each vertex of G lies in exactly one of the two sets X and Y.)
>
> - Each edge of G has exactly one endpoint in X and exactly one endpoint in Y.
>
> We shall usually write $(G; X, Y)$ instead of (G, X, Y) for a bipartite graph, putting a semicolon between the G and the X in order to stress the different roles that G plays and that X and Y play.

In class, we used simple graphs instead of multigraphs in the above definition (if I remember correctly). Again, the difference is not important, as far as the things done in class are concerned (i.e., comparing the sizes of matchings and vertex covers). Again, the difference starts creeping up when you start counting (matchings, cycles, paths, etc.), but this is not a class about counting.

> **Definition 0.8.** Let M be a matching in a multigraph G.
> **(a)** A vertex v of G is said to be *matched in M* if there exists an edge $e \in M$ such that v is an endpoint of e. In this case, this edge is unique (since M is a matching), and the other endpoint of this edge (i.e., the one distinct from v) is called the *M-partner* of v.
> **(b)** Let S be a subset of $\mathrm{V}(G)$. The matching M is said to be *S-complete* if each vertex $v \in S$ is matched in M.

> **Exercise 3.** Let $(G; X, Y)$ and $(H; U, V)$ be bipartite graphs.
> Assume that G is a simple graph and has an X-complete matching.
> Assume that H is a simple graph and has a U-complete matching.
> Consider the Cartesian product $G \times H$ of G and H defined in Exercise 1 of homework set 2.
> **(a)** Show that $(G \times H; (X \times V) \cup (Y \times U), (X \times U) \cup (Y \times V))$ is a bipartite graph.
> **(b)** Prove that the graph $G \times H$ has an $(X \times V) \cup (Y \times U)$-complete matching.

(We required that G and H be simple graphs just in order to not have to define $G \times H$ for multigraphs.)

0.5. Exercise 4: extending subsets

Exercise 4. Let S be a finite set. Let $k \in \mathbb{N}$ be such that $|S| \geq 2k + 1$. Prove that there exists an injective map $f : \mathcal{P}_k(S) \to \mathcal{P}_{k+1}(S)$ such that each $X \in \mathcal{P}_k(S)$ satisfies $f(X) \supseteq X$.

(In other words, prove that we can add to each k-element subset X of S an additional element from $S \setminus X$ such that the resulting $(k+1)$-element subsets are distinct.)

[**Hint:** First, reduce the problem to the case when $|S| = 2k + 1$. Then, in that case, restate it as a claim about matchings in a certain bipartite graph.]

0.6. Exercise 5: the self-grading problem

Exercise 5. Let S be a finite set, and let $k \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_k$ be k subsets of S such that each element of S lies in exactly one of these k subsets. Prove that the following statements are equivalent:

- *Statement 1:* There exists a bijection $\sigma : S \to S$ such that each $i \in \{1, 2, \ldots, k\}$ satisfies $\sigma(A_i) \cap A_i = \varnothing$.

- *Statement 2:* Each $i \in \{1, 2, \ldots, k\}$ satisfies $|A_i| \leq |S| / 2$.

(A restatement of Exercise 5: Let S be a finite set of students who have submitted homework. The students have been collaborating on the homework, forming k disjoint collaboration groups. A lazy professor wants the students to grade each other's homework, but he wants to avoid having a student grading the homework of a student from the same collaboration group. Prove that he can organize this (i.e., find a grader for each student) if and only if each collaboration group has size $\leq |S| / 2$.)

0.7. Exercise 6: unions and intersections of neighbor-critical sets

Definition 0.9. Let G be a multigraph. If S is a subset of $V(G)$, then $N(S)$ (or, to be more precise, $N_G(S)$) shall denote the subset $\{u \in V(G) \mid \text{at least one neighbor of } u \text{ belongs to } S\}$ of $V(G)$.

Exercise 6. Let $(G; X, Y)$ be a bipartite graph. Assume that each $S \subseteq X$ satisfies $|N(S)| \geq |S|$. (Thus, Hall's theorem shows that G has an X-complete matching.)

A subset S of X will be called *neighbor-critical* if $|N(S)| = |S|$.

Let A and B be two neighbor-critical subsets of X. Prove that the subsets $A \cup B$ and $A \cap B$ are also neighbor-critical.

0.8. Exercise 7: systems of common representatives

Exercise 7. Let S be a finite set. Let $k \in \mathbb{N}$.

Let $A_1, A_2, \ldots, A_k$ be k subsets of S.

Let $B_1, B_2, \ldots, B_k$ be k subsets of S.

A *system of common representatives* shall mean a choice of k distinct elements $t_1, t_2, \ldots, t_k$ of S as well as two bijections $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ and $g : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ such that each $i \in \{1, 2, \ldots, k\}$ satisfies

$$t_i \in A_{f(i)} \qquad \text{and} \qquad t_i \in B_{g(i)}.$$

Prove that a system of common representatives exists if and only if each two subsets I and J of $\{1, 2, \ldots, k\}$ satisfy

$$\left| \left(\bigcup_{i \in I} A_i \right) \cap \left(\bigcup_{j \in J} B_j \right) \right| \geq |I| + |J| - k.$$

[You are free to use any of Theorem 0.1, Theorem 0.2, Theorem 0.3 and Theorem 0.5 here.]