

Mathematics 5707 Homework 3

Jacob Ogden

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Exercise 1. Let T be a tree. Let (v_0, v_1, \dots, v_k) be a longest path of T . Prove that each center of T belongs to this path.

Proof: The path (v_0, v_1, \dots, v_k) is a longest path in T and has length k . Thus, v_1 and v_k have eccentricity k . Similarly, v_1 and v_{k-1} have eccentricity $k - 1$. The eccentricity continues to decrease as we move along the path (v_0, v_1, \dots, v_k) towards its middle, and by symmetry we can see that it attains its minimum value at the center(s) (plural if the path is of odd length) of the path. We must show that there is no vertex in T with eccentricity less than that of the centers of the path, i.e. that the centers of the path are the centers of T . To show this, note that any vertex not on the path (v_0, v_1, \dots, v_k) is a greater distance from the ends of the path than the nearest vertex to it that lies on the path, since the path between any two vertices of T is unique. Therefore, for any vertex $u \notin \{v_0, v_1, \dots, v_k\}$ there exists $v_i \in \{v_0, v_1, \dots, v_k\}$ such that $\text{eccentricity}(u) > \text{eccentricity}(v_i)$. Hence, the vertices of T with minimum eccentricity, which are the centers of T , lie on the path (v_0, v_1, \dots, v_k) . ■

Exercise 2. (a) Consider the cycle graph C_n for some $n \geq 2$. Its vertices are $1, 2, \dots, n$, and its edges are $12, 23, \dots, (n-1)n, n1$. Find the number of spanning trees of C_n .

There are n spanning trees of C_n .

Proof: Observe that a tree is formed by removing any edge from the cycle graph C_n and there are n edges that can be removed giving n trees. Each of these trees is a spanning tree, since removing a single edge from a cycle preserves connectedness. Now, there are no additional spanning trees of C_n since removing any edge from a tree makes it disconnected. Note that all spanning trees of C_n are isomorphic and that in the case $n = 2$ the two spanning trees differ only in the identities of their edge. ■

(b) Consider the directed graph \vec{C}_n for some $n \geq 2$. It is a digraph with vertices

1, 2, ..., n and arcs 12, 23, ..., (n - 1)n, n1. Find the number of spanning trees of \vec{C}_n with root 1.

There is a unique spanning tree of \vec{C}_n with root 1.

Proof: As above, spanning trees of \vec{C}_n are constructed by removing a single edge of \vec{C}_n . If such a spanning tree is to have root 1, the edge 12 must be removed, so there is only one way to construct such a tree. ■

(c) Fix $m \geq 1$. Let G be the simple graph with $3m + 2$ vertices

$$a, b, x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_m$$

and the following $3m + 3$ edges:

$$\begin{aligned} & ax_1, ay_1, az_1, \\ & x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1} \quad \text{for all } i \in \{1, 2, \dots, m-1\}, \\ & x_m b, y_m b, z_m b. \end{aligned}$$

Compute the number of spanning trees of G .

To begin, notice that G has three cycles, so we construct spanning trees of G by removing edges so as to disrupt all of these cycles while maintaining the connectedness of G . Also note that every pair of cycles shares one of the paths from a to b . Thus, removing a single edge from two of the three paths from a to b gives us a spanning tree, since removing two edges from the same path from a to b gives a disconnected graph and removing only one edge leaves a cycle intact. Therefore, the number of spanning trees of G can be calculated by finding the number of ways to choose the two paths from which we remove an edge and multiplying by the number of pairs of edges (one in each path) we can remove. Clearly, there are 3 ways to choose two paths given a set of three. Now, each path has $m + 1$ edges, so there are $(m + 1)^2$ pairs of edges we can remove to obtain a spanning tree for any pair of paths. Hence, G has $3(m + 1)^2$ spanning trees.

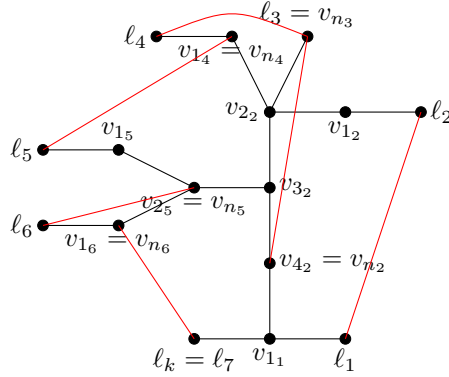
Exercise 4. Let T be a tree with more than one vertex. Let L be the set of leaves of T . Prove that it is possible to add $|L| - 1$ new edges to T in such a way that the resulting multigraph has a Hamiltonian cycle.

Proof: To begin, let $|L| = k$ and select two leaves, $\ell_1, \ell_k \in L$. There exists a unique path $P_1 = (\ell_k, v_{11}, v_{21}, \dots, v_{n1}, \ell_1)$ from ℓ_k to ℓ_1 , and it includes none of the other leaves of T . We must add at most $k - 1$ edges to T to construct a cycle that includes all vertices of T . We do this in the following way:

1. Select a leaf ℓ_2 , add the edge $\ell_1\ell_2$ to T .
2. Find a path (possibly of length 0) from ℓ_2 to a vertex that has a neighbor that lies on P_1 but shares no vertices with the latter. Denote this path $P_2 = (\ell_2, v_{1_2}, v_{2_2}, \dots, v_{n_2})$. (We use the subscript on the subscript to note the leaf at which this path originated and to distinguish the length of this path from future paths of this form, since they will not in general be of the same length.)
3. Select another leaf ℓ_3 . Add the edge $v_{n_2}\ell_3$ to T .
4. Find a path (possibly of length 0) from ℓ_3 to a vertex that has a neighbor on P_1 or P_2 , but shares no vertices with these paths. Call it $P_3 = (\ell_3, v_{1_3}, v_{2_3}, \dots, v_{n_3})$.
5. Select a leaf ℓ_4 . Add the edge $v_{n_3}\ell_4$ to T .
6. Continue this process until the edge $v_{n_{k-1}}\ell_k$ has been added to T .

We have thus added $k - 1$ edges, and we have the Hamiltonian cycle given by $(P_1, P_2, \dots, P_k, \ell_k)$, since we have added precisely the edges that connect the ends of these paths. ■

Example: T and its edges are in black, edges we construct are in red.



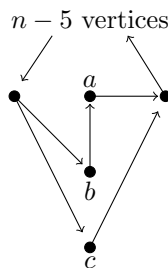
Exercise 5. Let a , b , and c be three vertices of a strongly connected digraph $G = (V, A)$ such that $|V| \geq 4$.

- (a) Prove that $d(a, b) + d(b, c) + d(c, a) \leq 3|V| - 4$.

Proof: Choose a shortest path from a to b , a shortest path from b to c , and a shortest path from c to a .

Observe that the greatest possible distance between any two vertices is $|V| - 1$, in which case the path between these two vertices passes through every vertex. Then, we easily conclude that $d(a, b) + d(b, c) + d(c, a) \leq 3|V| - 3$. Now suppose the case of equality. Then, the vertex c lies on the shortest path from a to b , the vertex a lies on the shortest path from b to c , and the vertex b lies on the shortest path from c to a . Now, suppose there exists a vertex v between c and b on the path from a to b . Then the path from b to c must visit v . However, if the path from b to c visits v before visiting a , then we have a path from c to a which does not visit b , a contradiction. Also, if the path from b to c visits v after visiting a , then we have a shorter path from a to b , since there is an edge that allows us to skip c , another contradiction. Therefore, c is the penultimate vertex on the path from a to b . Hence, $cb \in A$. By symmetry, the same argument shows that $ac \in A$. Hence, (c, b, a) is a path in G ; therefore, $d(c, a) \leq 2$. By symmetry, we similarly get $d(b, c) \leq 2$ and $d(a, b) \leq 2$. Adding up, we find $d(a, b) + d(b, c) + d(c, a) \leq 2 + 2 + 2 = 6 < 3|V| - 3$ (since $|V| \geq 4 > 3$), which contradicts $d(a, b) + d(b, c) + d(c, a) = 3|V| - 3$. Therefore, $d(a, b) + d(b, c) + d(c, a) < 3|V| - 3$ or equivalently, $d(a, b) + d(b, c) + d(c, a) \leq 3|V| - 4$. ■

(b) For each $n \geq 5$, construct an example in which $|V| = n$ and $d(a, b) + d(b, c) + d(c, a) = 3|V| - 4$.



Exercise 6. We have learned that a simple graph G (or multigraph G) has a proper 2-coloring if and only if all cycles of G have even length.

(a) Is it true that if all cycles of a simple graph G (or multigraph G) have length divisible by 3, then G has a proper 3-coloring?

If all cycles of G have length divisible by 3, then G has a proper 3-coloring.

[No solution to this part submitted.]

(b) Is it true that if a simple graph G has a proper 3-coloring, then all cycles of G have length divisible by 3?

It is not true that if G has a proper 3-coloring, then all cycles of G have length divisible by 3.

Counterexample: Consider the cycle graph C_4 . It has a proper 3-coloring since it has a proper 2-coloring (which is trivially a proper 3-coloring). However, it has a cycle of length 4, not divisible by 3. Any graph whose cycles are all of even length and has a cycle of length $\equiv 2$ or $\equiv 4 \pmod{6}$ is a counterexample. Another counterexample is the Petersen graph, which has cycles of length 5 and a proper 3-coloring.