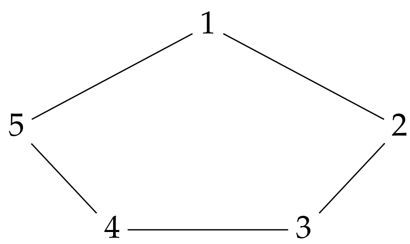
Math 5707 Spring 2017 (Darij Grinberg): homework set 3
Please hand in solutions to FIVE of the 8 problems.

See the lecture notes and also the handwritten notes for relevant material. Also, some definitions can be found in the solutions to hw2. If you reference results from the lecture notes, please **mention the date and time** of the version of the notes you are using (as the numbering changes during updates).

If v is a vertex of a simple graph $G = (V, E)$, then the *eccentricity* of v is defined to be $\max \{d(v,u) \mid u \in V\}$ (where $d(v,u)$ is the distance between v and u, as usual). A *center* of a simple graph G means a vertex whose eccentricity is minimum (among the eccentricities of all vertices).

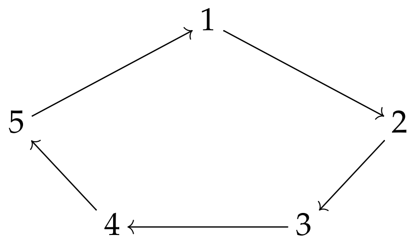
Exercise 1. Let T be a tree. Let $(v_0, v_1, \ldots, v_k)$ be a longest path of T. Prove that each center of T belongs to this path (i.e., is one of the vertices $v_0, v_1, \ldots, v_k$).

Exercise 2. (a) Consider the cycle graph C_n for some $n \geq 2$. Its vertices are $1, 2, \ldots, n$, and its edges are $12, 23, \ldots, (n-1)n, n1$. (Here is how it looks for $n = 5$:



) Find the number of spanning trees of C_n.

(b) Consider the directed cycle graph $\overrightarrow{C}_n$ for some $n \geq 2$. It is a digraph; its vertices are $1, 2, \ldots, n$, and its arcs are $12, 23, \ldots, (n-1)n, n1$. (Here is how it looks for $n = 5$:



) Find the number of oriented spanning trees of $\overrightarrow{C}_n$ with root 1.

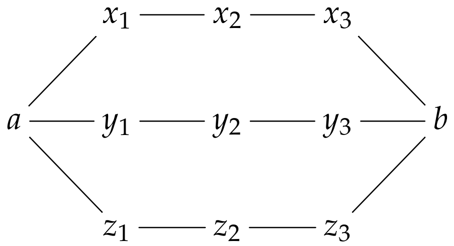
(c) Fix $m \geq 1$. Let G be the simple graph with $3m + 2$ vertices

$$a, b, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_m$$

and the following $3m + 3$ edges:

$$\begin{aligned} & ax_1, ay_1, az_1, \\ & x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1} \qquad \text{for all } i \in \{1, 2, \ldots, m-1\}, \\ & x_m b, y_m b, z_m b. \end{aligned}$$

(Thus, the graph consists of two vertices a and b connected by three paths, each of length $m+1$, with no overlaps between the paths except for their starting and ending points. Here is a picture for $m = 3$:



) Compute the number of spanning trees of G.

[To argue why your number is correct, a sketch of the argument in 1-2 sentences should be enough; a fully rigorous proof is not required.]

Exercise 3. If G is a multigraph, then $\operatorname{conn} G$ shall denote the number of connected components of G. (Note that this is 0 when G has no vertices, and 1 if G is connected.)

Let (V, H, ϕ) be a multigraph. Let E and F be two subsets of H.

(a) Prove that

$$\begin{aligned} &\operatorname{conn}(V, E, \phi \mid_E) + \operatorname{conn}(V, F, \phi \mid_F) \\ &\leq \operatorname{conn}(V, E \cup F, \phi \mid_{E \cup F}) + \operatorname{conn}(V, E \cap F, \phi \mid_{E \cap F}). \end{aligned} \tag{1}$$

[Feel free to restrict yourself to the case of a simple graph; in this case, E and F are two subsets of $\mathcal{P}_2(V)$, and you have to show that

$$\operatorname{conn}(V, E) + \operatorname{conn}(V, F) \leq \operatorname{conn}(V, E \cup F) + \operatorname{conn}(V, E \cap F).$$

This isn't any easier than the general case, but saves you the hassle of carrying the map ϕ around.

Also, feel free to take inspiration from the proof of the classical fact that $\dim X + \dim Y = \dim(X+Y) + \dim(X \cap Y)$ when X and Y are two subspaces of a finite-dimensional vector space U. That proof relies on choosing a basis of $X \cap Y$ and extending it to bases of X and Y, then merging the extended bases to a basis of $X + Y$. A "basis" of a multigraph G is a spanning forest: a spanning subgraph that is a forest and has the same number of connected components as G. More precisely, it is the set of the edges of a spanning forest.]

(b) Give an example where the inequality (1) does **not** become an equality.

Exercise 4. Let T be a tree having more than 1 vertex. Let L be the set of leaves of T. Prove that it is possible to add $|L| - 1$ new edges to T in such a way that the resulting multigraph has a Hamiltonian cycle.

If u and v are two vertices of a digraph G, then $d(u, v)$ denotes the *distance* from

u to v. This is defined to be the minimum length of a path from u to v if such a path exists; otherwise it is defined to be the symbol ∞. Notice that $d(u,v)$ is not usually the same as $d(v,u)$ (unlike for simple graphs).

Exercise 5. Let a, b and c be three vertices of a strongly connected digraph $G = (V, A)$ such that $|V| \geq 4$.

(a) Prove that $d(b,c) + d(c,a) + d(a,b) \leq 3|V| - 4$.

(b) For each $n \geq 5$, construct an example in which $|V| = n$ and $d(b,c) + d(c,a) + d(a,b) = 3|V| - 4$. (No proof required for the example.)

Recall that a *k-coloring* of a simple graph $G = (V, E)$ means a map $f : V \to \{1, 2, \ldots, k\}$. Such a k-coloring f is said to be *proper* if no two adjacent vertices u and v have the same color (i.e., satisfy $f(u) = f(v)$).

Exercise 6. We have learned that a simple graph G (or multigraph G) has a proper 2-coloring if and only if all cycles of G have even length.

(a) Is it true that if all cycles of a simple graph G (or multigraph G) have length divisible by 3, then G has a proper 3-coloring?

(b) Is it true that if a simple graph G has a proper 3-coloring, then all cycles of G have length divisible by 3 ?

Exercise 7. In class, we have proven the following fact: If $G = (V, E)$ is a simple graph, then G has an independent set of size $\geq \dfrac{n}{1+d}$, where $n = |V|$ and $d = \dfrac{1}{n} \sum_{v \in V} \deg v$. (Notice that d is simply the average degree of a vertex of G.)

Use this to prove Turán's theorem (Theorem 2.5.15 in the lecture notes).

Exercise 8. Extra credit:

In this exercise, "number" means (e.g.) a real number. (Feel free to restrict yourself to positive integers if it helps you. The most general interpretation would be "element of a commutative ring", but you don't need to work in this generality.)

For any n numbers $x_1, x_2, \ldots, x_n$, we define $v(x_1, x_2, \ldots, x_n)$ to be the number

$$\prod_{1 \leq i < j \leq n} (x_j - x_i) = \det \left(\left(x_j^{i-1} \right)_{1 \leq i \leq n,\ 1 \leq j \leq n} \right).$$

Let $x_1, x_2, \ldots, x_n$ be n numbers. Let t be a further number. Prove at least one of the following facts combinatorially (i.e., without using any properties of the determinant other than its definition as a sum over permutations):

(a) We have

$$\sum_{k=1}^{n} v(x_1, x_2, \ldots, x_{k-1}, x_k + t, x_{k+1}, x_{k+2}, \ldots, x_n) = n v(x_1, x_2, \ldots, x_n).$$

(b) For each $m \in \{0, 1, \ldots, n-1\}$, we have

$$\sum_{k=1}^{n} x_k^m v\left(x_1, x_2, \ldots, x_{k-1}, t, x_{k+1}, x_{k+2}, \ldots, x_n\right) = t^m v\left(x_1, x_2, \ldots, x_n\right).$$

(c) We have

$$\sum_{k=1}^{n} x_k v\left(x_1, x_2, \ldots, x_{k-1}, x_k + t, x_{k+1}, x_{k+2}, \ldots, x_n\right) = \left(\binom{n}{2} t + \sum_{k=1}^{n} x_k\right) v\left(x_1, x_2, \ldots, x_n\right).$$