Math 5707 Spring 2017 (Darij Grinberg): homework set 2
Solution sketches.

Contents

0.1. Notations and conventions

See the lecture notes and also the handwritten notes for relevant material. If you reference results from the lecture notes, please **mention the date and time** of the version of the notes you are using (as the numbering changes during updates).

Let me recall a few notations:

- A *simple graph* is a pair (V, E), where V is a finite set and where E is a subset of $\mathcal{P}_2(V)$. The set V is called the *vertex set* of the simple graph (and the elements of V are called the *vertices* of the simple graph), whereas the set E is called the *edge set* of the simple graph (and the elements of E are called the *edges* of the simple graph).

- A *multigraph* is a triple (V, E, ϕ), where V and E are finite sets and where ϕ is a map from E to $\mathcal{P}_2(V)$. The set V is called the *vertex set* of the multigraph (and the elements of V are called the *vertices* of the multigraph); the set E is called the *edge set* of the multigraph (and the elements of E are called the *edges* of the multigraph); the map ϕ is called the *endpoint map* of the multigraph. If e is an edge of the multigraph, then the two elements of $\phi(e)$ are called the *endpoints* of e.

- A *digraph* is a pair (V, A), where V is a finite set and where A is a subset of $V \times V$. The set V is called the *vertex set* of the digraph (and the elements of V are called the *vertices* of the digraph); the set A is called the *arc set* of the digraph (and the elements of A are called the *arcs* of the digraph). If (v, w) is an arc of the digraph, then v is called the *source* of this arc, and w is called the *target* of this arc. If the source of an arc equals its target, then this arc is said to be a *loop*.

- A *multidigraph* is a triple (V, A, ϕ), where V and A are finite sets and where ϕ is a map from A to $V \times V$. The set V is called the *vertex set* of the multidigraph (and the elements of V are called the *vertices* of the multidigraph); the set A is called the *arc set* of the multidigraph (and the elements of A are called the *arcs* of the multidigraph). If a is an arc of the multidigraph, and if $(v, w) = \phi(a)$, then v is called the *source* of this arc, and w is called the *target* of this arc.

- If u and v are two vertices of a simple graph G, then we use the shorthand notation uv for the set $\{u, v\}$ (even if this set is not an arc of G, and even if it is not a two-element set). If u and v are two vertices of a digraph G, then we use the shorthand notation uv for the pair (u, v). We hope the two notations will not be confused.

- A *walk* in a simple graph G (or in a digraph G) is defined to be a list of the form $(v_0, v_1, \ldots, v_k)$, where $v_0, v_1, \ldots, v_k$ are vertices of G, and where $v_i v_{i+1}$ is an edge of G (or an arc of G, respectively) for each $i \in \{0, 1, \ldots, k-1\}$. Here, the meaning of $v_i v_{i+1}$ depends on whether G is a simple graph or a digraph (namely, it means a set in the former case, and a pair in the latter).
  A *walk* in a multigraph G is defined to be a list of the form $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are vertices of G, and where e_i is an edge of G having endpoints v_{i-1} and v_i for each $i \in \{1, 2, \ldots, k\}$.
  A *walk* in a multidigraph G is defined to be a list of the form $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are vertices of G, and where a_i is an arc of G having source v_{i-1} and target v_i for each $i \in \{1, 2, \ldots, k\}$.
  In each of these cases, the *vertices* of the walk are defined to be $v_0, v_1, \ldots, v_k$. Moreover, v_0 is called the *starting point* of the walk, and v_k is called the *ending point* of the walk. The *edges* of the walk are defined to be $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ (if G is a simple graph) or $e_1, e_2, \ldots, e_k$ (if G is a multigraph). The *arcs* of the walk are defined to be $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ (if G is a digraph) or $a_1, a_2, \ldots, a_k$ (if G is a multidigraph).

- A *path* in a simple graph G (or in a digraph G, or in a multigraph G, or in a multidigraph G) means a walk in G such that the vertices of the walk are distinct.

- A *circuit* in a simple graph G (or in a digraph G, or in a multigraph G, or in a multidigraph G) means a walk in G such that the starting point of the walk equals the ending point of the walk.
  If $v_0, v_1, \ldots, v_k$ are the vertices of a circuit $\mathbf{c}$, then $v_0, v_1, \ldots, v_{k-1}$ are called the *non-ultimate vertices* of $\mathbf{c}$.

- A *cycle* in a simple graph G means a circuit $(v_0, v_1, \ldots, v_k)$ in G such that the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct and such that $k \geq 3$.
  A *cycle* in a digraph G means a circuit $(v_0, v_1, \ldots, v_k)$ in G such that the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct and such that $k \geq 1$. (In particular, each loop (v, v) gives rise to a cycle.)

A *cycle* in a multigraph G means a circuit $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ such that the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct, the edges $e_1, e_2, \ldots, e_k$ are also distinct, and such that $k \geq 2$.
A *cycle* in a multidigraph G means a circuit $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ in G such that the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct and such that $k \geq 1$.

- A simple graph G (or multigraph G, or digraph G, or multidigraph G) is said to be *strongly connected* if its vertex set is nonempty[1] and it has the property that for any two vertices u and v of G, there exists at least one walk from u to v in G. (Note that if there exists a walk from u to v in G, then there also exists a path from u to v in G.)

  When G is a simple graph or a multigraph, we usually say "G is connected" instead of "G is strongly connected".

- A *Hamiltonian path* in a simple graph G (or multigraph G, or digraph G, or multidigraph G) means a path $\mathbf{p}$ in G such that each vertex of G appears exactly once among the vertices of $\mathbf{p}$.

- A *Hamiltonian cycle* in a simple graph G (or multigraph G, or digraph G, or multidigraph G) means a cycle $\mathbf{c}$ of G such that each vertex of G appears exactly once among the non-ultimate vertices of $\mathbf{c}$.

- A walk $\mathbf{w}$ in a simple graph G (or multigraph G) is said to be *Eulerian* if each edge of G appears exactly once among the edges of $\mathbf{w}$.
  A walk $\mathbf{w}$ in a digraph G (or multidigraph G) is said to be *Eulerian* if each arc of G appears exactly once among the arcs of $\mathbf{w}$.
  Notice that this automatically defines the notion of an Eulerian circuit (namely, an Eulerian circuit is just a circuit that is Eulerian).

0.2. Exercise 1: Hamiltonian paths in Cartesian product graphs

Exercise 1. Let G and H be two simple graphs. The *Cartesian product* of G and H is a new simple graph, denoted $G \times H$, which is defined as follows:

- The vertex set $\operatorname{V}(G \times H)$ of $G \times H$ is the Cartesian product $\operatorname{V}(G) \times \operatorname{V}(H)$.
- A vertex (g, h) of $G \times H$ is adjacent to a vertex (g', h') of $G \times H$ if and only if we have
  - **either** $g = g'$ and $hh' \in \operatorname{E}(H)$,
  - **or** $h = h'$ and $gg' \in \operatorname{E}(G)$.

  (In particular, exactly one of the two equalities $g = g'$ and $h = h'$ has to hold when (g, h) is adjacent to (g', h').)

[1] Do not forget this requirement!

(a) Recall the n-dimensional cube graph Q_n defined for each $n \in \mathbb{N}$. (Its vertices are n-tuples $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$, and two such vertices are adjacent if and only if they differ in exactly one entry.) Prove that $Q_n \cong Q_{n-1} \times Q_1$ for each positive integer n. (Thus, Q_n can be obtained from Q_1 by repeatedly forming Cartesian products; i.e., it is a "Cartesian power" of Q_1.)

(b) Assume that each of the graphs G and H has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian path.

(c) Assume that both numbers $|\operatorname{V}(G)|$ and $|\operatorname{V}(H)|$ are > 1, and that at least one of them is even. Assume again that each of the graphs G and H has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian cycle.

Solution sketch to Exercise 1. **(a)** It suffices to check that the map

$$\{0,1\}^n \to \{0,1\}^{n-1} \times \{0,1\}, \qquad (a_1, a_2, \ldots, a_n) \mapsto ((a_1, a_2, \ldots, a_{n-1}), a_n)$$

is a graph isomorphism from Q_n to $Q_{n-1} \times Q_1$. The proof of this is straightforward; the main step is to check that two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ in $\{0,1\}^n$ differ in exactly one entry (i.e., are adjacent as vertices of Q_n) if and only if

- **either** we have $(a_1, a_2, \ldots, a_{n-1}) = (b_1, b_2, \ldots, b_{n-1})$ and $a_n \neq b_n$,
- **or** the $(n-1)$-tuples $(a_1, a_2, \ldots, a_{n-1})$ and $(b_1, b_2, \ldots, b_{n-1})$ differ in exactly one entry (i.e., are adjacent as vertices of Q_{n-1}) and we have $a_n = b_n$.

This is obvious.

(b) By assumption, there exists a Hamiltonian path $(g_1, g_2, \ldots, g_n)$ of G, and there exists a Hamiltonian path $(h_1, h_2, \ldots, h_m)$ of H. Use these two paths to construct the Hamiltonian path

$$\begin{aligned}
&((g_1, h_1), (g_1, h_2), \ldots, (g_1, h_m), \\
&(g_2, h_m), (g_2, h_{m-1}), \ldots, (g_2, h_1), \\
&(g_3, h_1), (g_3, h_2), \ldots, (g_3, h_m), \\
&(g_4, h_m), (g_4, h_{m-1}), \ldots, (g_4, h_1), \\
&\cdots)
\end{aligned} \tag{1}$$

in $G \times H$. (This Hamiltonian path first traverses all vertices of the form (g_1, h_i) in the order of increasing i, then traverses all vertices of the form (g_2, h_i) in the order of decreasing i, then traverses all vertices of the form (g_3, h_i) in the order of increasing i, and so on, always alternating between increasing and decreasing i. It ends at the vertex (g_n, h_m) if n is odd, and at the vertex (g_n, h_1) if n is even. Here is how it looks like:

$$\begin{array}{ccccccc}
(g_1, h_1) & \longrightarrow & (g_1, h_2) & \longrightarrow \cdots \longrightarrow & (g_1, h_m) \\
 & & & & \downarrow \\
(g_2, h_1) & \longleftarrow & (g_2, h_2) & \longleftarrow \cdots \longleftarrow & (g_2, h_m) \\
\downarrow & & & & \\
(g_3, h_1) & \longrightarrow & (g_3, h_2) & \longrightarrow \cdots \longrightarrow & (g_3, h_m) \\
 & & & & \downarrow \\
(g_4, h_1) & \longleftarrow & (g_4, h_2) & \longleftarrow \cdots \longleftarrow & (g_4, h_m) \\
\downarrow & & & & \\
\vdots & & & &
\end{array}$$

where the arrows merely signify the order in which the vertices are traversed by the path (the edges are still undirected).)

(c) At least one of the integers $|\mathrm{V}(G)|$ and $|\mathrm{V}(H)|$ is even. Since $G \times H \cong H \times G$ (in fact, there is a graph isomorphism $G \times H \to H \times G$ sending each vertex (v, w) of $G \times H$ to (w, v)), we can WLOG assume that $|\mathrm{V}(G)|$ is even (because otherwise we can simply switch G with H). Assume this, and recall furthermore that $|\mathrm{V}(H)| > 1$.

By assumption, there exists a Hamiltonian path $(g_1, g_2, \ldots, g_n)$ of G, and there exists a Hamiltonian path $(h_0, h_1, \ldots, h_m)$ of H. (Note that I am indexing the vertices of the former path from 1, but indexing the vertices of the latter path from 0.) Thus, $n = |\mathrm{V}(G)|$ is even. Also, $n = |\mathrm{V}(G)| > 1$. Also, $m + 1 = |\mathrm{V}(H)| > 1$, so that $m > 0$.

Now, consider the path (1). It is not a Hamiltonian path, since it misses the vertices of the form (g_i, h_0). But it is a path from (g_1, h_1) to (g_n, h_1) (since n is even) that uses each vertex **not** of this form exactly once; thus, we can extend it to a Hamiltonian cycle of $G \times H$ by appending the following vertices at its end:

$$(g_n, h_0), (g_{n-1}, h_0), \ldots, (g_1, h_0), (g_1, h_1).$$

(In other words, we close the path by going back to its starting point along the missing vertices (g_i, h_0).) Hence, we have found a Hamiltonian cycle of $G \times H$.

(Here is how this Hamiltonian cycle looks like:

$$\begin{tikzcd}
(g_1, h_0) \arrow[r] & (g_1, h_1) \arrow[r] & (g_1, h_2) \arrow[r] & \cdots \arrow[r] & (g_1, h_m) \arrow[d] \\
(g_2, h_0) \arrow[u] & (g_2, h_1) \arrow[d] & (g_2, h_2) \arrow[l] & \cdots \arrow[l] & (g_2, h_m) \arrow[l] \\
(g_3, h_0) \arrow[u] & (g_3, h_1) \arrow[r] & (g_3, h_2) \arrow[r] & \cdots \arrow[r] & (g_3, h_m) \arrow[d] \\
(g_4, h_0) \arrow[u] & (g_4, h_1) \arrow[d] & (g_4, h_2) \arrow[l] & \cdots \arrow[l] & (g_4, h_m) \arrow[l] \\
\vdots \arrow[u] & \vdots \arrow[d] & & & \\
(g_{n-1}, h_0) \arrow[u] & (g_{n-1}, h_1) \arrow[r] & (g_{n-1}, h_2) \arrow[r] & \cdots \arrow[r] & (g_{n-1}, h_m) \arrow[d] \\
(g_n, h_0) \arrow[u] & (g_n, h_1) \arrow[l] & (g_n, h_2) \arrow[l] & \cdots \arrow[l] & (g_n, h_m) \arrow[l]
\end{tikzcd}$$

where the arrows merely signify the order in which the vertices are traversed by the cycle (the edges are still undirected).) $\square$

0.3. Exercise 2: Eulerian circuits in K_3, K_5 and K_7

> **Exercise 2.** Let n be a positive integer. Recall that K_n denotes the complete graph on n vertices. This is the graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $\mathcal{P}_2(V)$ (so each two distinct vertices are connected).
>
> Find Eulerian circuits for the graphs K_3, K_5, and K_7.

Solution sketch to Exercise 2. An Eulerian circuit of K_3 is $(1, 2, 3, 1)$.

An Eulerian circuit of K_5 is $(1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1)$.

An Eulerian circuit of K_7 is $(1, 2, 3, 4, 5, 6, 7, 1, 3, 5, 7, 2, 4, 6, 1, 4, 7, 3, 6, 2, 5, 1)$.

[*Remark:* Of course, other choices are possible. For each odd positive integer n, the complete graph K_n has an Eulerian circuit (because it is connected, and each of its vertices has even degree $n - 1$), and so it has at least one Eulerian circuit; but in truth, there are many. (How many? See OEIS entry A007082. There doesn't seem to be an explicit formula.)

When n is an odd prime[2], there is actually a simple way to construct an Eulerian circuit in K_n: For each $k \in \{1, 2, \ldots, (n-1)/2\}$, let c_k be the cycle $(a_0, a_1, \ldots, a_n)$,

[2]Everyone gets confused by the notion of an "odd prime" at least once in their life. But it means exactly what it says: a prime that is odd. In other words, a prime that is distinct from 2.

where a_i denotes the unique element of $(1, 2, \ldots, n)$ that is congruent to $ki + 1$ modulo n. Then, the cycles $c_1, c_2, \ldots, c_{(n-1)/2}$ can be combined to a single Eulerian circuit. Finding Eulerian circuits on K_n for non-prime n is harder, but of course the algorithm done in class still works.] $\square$

0.4. Exercise 3: de Bruijn sequences exist

Exercise 3. Let n be a positive integer, and K be a nonempty finite set. Let $k = |K|$. A *de Bruijn sequence* of order n on K means a list $(c_0, c_1, \ldots, c_{k^n - 1})$ of k^n elements of K such that

(1) for each n-tuple $(a_1, a_2, \ldots, a_n) \in K^n$ of elements of K, there exists a **unique** $r \in \{0, 1, \ldots, k^n - 1\}$ such that $(a_1, a_2, \ldots, a_n) = (c_r, c_{r+1}, \ldots, c_{r+n-1})$.

Here, the indices are understood to be cyclic modulo k^n; that is, c_q (for $q \geq k^n$) is defined to be $c_{q \% k^n}$, where $q \% k^n$ denotes the remainder of q modulo k^n.

(Note that the condition (1) can be restated as follows: If we write the elements $c_0, c_1, \ldots, c_{k^n - 1}$ on a circular necklace (in this order), so that the last element $c_{k^n - 1}$ is followed by the first one, then each n-tuple of elements of K appears as a string of n consecutive elements on the necklace, and the position at which it appears on the necklace is unique.)

For example, $(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) = (1, 1, 2, 2, 3, 3, 1, 3, 2)$ is a de Bruijn sequence of order 2 on the set $\{1, 2, 3\}$, because for each 2-tuple $(a_1, a_2) \in \{1, 2, 3\}^2$, there exists a unique $r \in \{0, 1, \ldots, 8\}$ such that $(a_1, a_2) = (c_r, c_{r+1})$. Namely:

$$\begin{array}{lll}
(1,1) = (c_0, c_1); & (1,2) = (c_1, c_2); & (1,3) = (c_6, c_7); \\
(2,1) = (c_8, c_9); & (2,2) = (c_2, c_3); & (2,3) = (c_3, c_4); \\
(3,1) = (c_5, c_6); & (3,2) = (c_7, c_8); & (3,3) = (c_4, c_5).
\end{array}$$

Prove that there exists a de Bruijn sequence of order n on K (no matter what n and K are).

Hint: Let D be the digraph with vertex set K^{n-1} and an arc from $(a_1, a_2, \ldots, a_{n-1})$ to $(a_2, a_3, \ldots, a_n)$ for each $(a_1, a_2, \ldots, a_n) \in K^n$ (and no other arcs). Prove that D has an Eulerian circuit.

Solution sketch to Exercise 3. The hint suggests defining a digraph. I shall use a multidigraph instead, as this is slightly simpler and cleaner.

Recall that a *multidigraph* means a triple (V, A, ϕ), where V and A are two finite sets and where ϕ is a map from A to $V \times V$. We define a multidigraph D to be $\left(K^{n-1}, K^n, f\right)$, where the map $f : K^n \to K^{n-1} \times K^{n-1}$ is given by the formula

$$f(a_1, a_2, \ldots, a_n) = ((a_1, a_2, \ldots, a_{n-1}), (a_2, a_3, \ldots, a_n)).$$

(As usual, we write $f(a_1, a_2, \ldots, a_n)$ for $f((a_1, a_2, \ldots, a_n))$, since the extra parentheses do not add any clarity.) Thus, in the multidigraph D, there is an arc from

$(a_1, a_2, \ldots, a_{n-1})$ to $(a_2, a_3, \ldots, a_n)$ for each $(a_1, a_2, \ldots, a_n) \in K^n$. (Note that if $n = 1$, then there is only one vertex, namely the empty 0-tuple $()$, but there are $|K|$ many arcs from it to itself. This is why we are using a multidigraph instead of a digraph. Of course, you are free to throw the $n = 1$ case aside, seeing how easy it is to handle separately.)

Recall that the *indegree* of a vertex v of a multidigraph (V, A, ϕ) is defined to be the number of all arcs $a \in A$ whose target is v (that is, which satisfy $\phi(a) = (x, v)$ for some $x \in V$). This indegree is denoted by $\deg^- v$. Also, the *outdegree* of a vertex v of a multidigraph (V, A, ϕ) is defined to be the number of all arcs $a \in A$ whose source is v (that is, which satisfy $\phi(a) = (v, x)$ for some $x \in V$). This outdegree is denoted by $\deg^+ v$.

Recall that a multidigraph (V, A, ϕ) is said to be *strongly connected* if $V \neq \varnothing$ and if, for any $u \in V$ and $v \in V$, there is at least one walk from u to v in the multidigraph. The multidigraph D is strongly connected[3]. Moreover, each vertex $v \in K^{n-1}$ satisfies $\deg^- v = \deg^+ v$ (where both indegree and outdegree are taken respective to the multidigraph D) [4].

Recall that a multidigraph has an Eulerian circuit if and only if it is strongly connected and each vertex v satisfies $\deg^- v = \deg^+ v$. Hence, the multidigraph D has an Eulerian circuit (since it is strongly connected and each vertex v satisfies $\deg^- v = \deg^+ v$). Consider such an Eulerian circuit $\mathbf{c}$. It contains each arc of D exactly once, and thus has k^n arcs (since the number of arcs of D is $|K^n| = |K|^n = k^n$). Let $p_0, p_1, \ldots, p_{k^n-1}$ be these arcs (listed in the order in which they appear on the Eulerian circuit). We extend the indexing of these arcs modulo k^n; in other words, we set $p_i = p_{i\%k^n}$ for each $i \in \mathbb{Z}$. (This, of course, does not conflict with the already introduced notations $p_0, p_1, \ldots, p_{k^n-1}$, since each $i \in \{0, 1, \ldots, k^n - 1\}$ satisfies $i\%k^n = i$.)

Let me now explain what I intend to do before I go into the technical details. We want to construct a de Bruijn sequence $(c_0, c_1, \ldots, c_{k^n-1})$. I claim that the sequence

[3] *Proof.* Let $u \in K^{n-1}$ and $v \in K^{n-1}$. We must prove that there is at least one walk from u to v in D. Write the $(n-1)$-tuples u and v as $u = (u_1, u_2, \ldots, u_{n-1})$ and $v = (v_1, v_2, \ldots, v_{n-1})$, respectively. Then, the walk

$$\begin{aligned}
&((u_1, u_2, u_3, u_4, \ldots, u_{n-1}), (u_1, u_2, u_3, u_4, \ldots, u_{n-1}, v_1), \\
&\quad (u_2, u_3, u_4, \ldots, u_{n-1}, v_1), (u_2, u_3, u_4, \ldots, u_{n-1}, v_1, v_2), \\
&\quad\quad (u_3, u_4, \ldots, u_{n-1}, v_1, v_2), (u_3, u_4, \ldots, u_{n-1}, v_1, v_2, v_3), \\
&\quad\quad\quad \ldots, \\
&\quad\quad\quad\quad (u_{n-1}, v_1, v_2, \ldots, v_{n-2}), (u_{n-1}, v_1, v_2, \ldots, v_{n-2}, v_{n-1}), \\
&\quad\quad\quad\quad\quad (v_1, v_2, \ldots, v_{n-2}, v_{n-1}))
\end{aligned}$$

is a walk from u to v in D. Hence, such a walk exists.

[4] *Proof.* Let $v \in K^{n-1}$. Write the $(n-1)$-tuple v in the form $v = (v_1, v_2, \ldots, v_{n-1})$. Now, $\deg^- v$ is the number of arcs of D with target v. But these arcs are exactly the n-tuples of the form $(k, v_1, v_2, \ldots, v_{n-1}) \in K^n$ with $k \in K$ (by the definition of the multidigraph D). Hence, there are exactly $|K|$ of them. Therefore, $\deg^- v = |K|$ (since $\deg^- v$ is the number of these arcs). Similarly, $\deg^+ v = |K|$. Thus, $\deg^- v = |K| = \deg^+ v$.

of the first entries of the n-tuples $p_0, p_1, \ldots, p_{k^n-1}$ is such a de Bruijn sequence. Once this is proven, the exercise will clearly be solved.

For each n-tuple h and each $j \in \{1, 2, \ldots, n\}$, we will denote the j-th entry of h by $h[j]$. So each n-tuple h has the form $h = (h[1], h[2], \ldots, h[n])$. Now, I claim that

$$p_i[j+1] = p_{i+1}[j] \qquad \text{for each } i \in \mathbb{Z} \text{ and } j \in \{1, 2, \ldots, n-1\} \tag{2}$$

[5]. Hence,

$$p_i[j+g] = p_{i+g}[j] \qquad \text{for each } i \in \mathbb{Z} \text{ and } g \in \mathbb{N} \text{ and } j \in \{1, 2, \ldots, n-g\} \tag{3}$$

[6]. From this, we can easily obtain

$$(p_i[1], p_{i+1}[1], \ldots, p_{i+n-1}[1]) = p_i \qquad \text{for each } i \in \mathbb{Z} \tag{4}$$

[7].

Now, recall that $p_0, p_1, \ldots, p_{k^n-1}$ are the arcs of an Eulerian circuit of D. Hence, each arc of D appears exactly once in the list $(p_0, p_1, \ldots, p_{k^n-1})$. In other words, for each arc a of D, there exists a unique $r \in \{0, 1, \ldots, k^n - 1\}$ such that $a = p_i$. Since the arcs of D are the n-tuples in K^n, we can rewrite this as follows: For each n-tuple $a \in K^n$, there exists a unique $r \in \{0, 1, \ldots, k^n - 1\}$ such that $a = p_r$. Since each $r \in \mathbb{Z}$ satisfies $(p_r[1], p_{r+1}[1], \ldots, p_{r+n-1}[1]) = p_r$ (by (4), applied to $i = r$), this further rewrites as follows: For each n-tuple $a \in K^n$, there exists a unique $r \in \{0, 1, \ldots, k^n - 1\}$ such that $a = (p_r[1], p_{r+1}[1], \ldots, p_{r+n-1}[1])$. Renaming a as $(a_1, a_2, \ldots, a_n)$ in this result, we obtain the following: For each n-tuple $(a_1, a_2, \ldots, a_n) \in K^n$ of elements of K, there exists a unique $r \in \{0, 1, \ldots, k^n - 1\}$ such that $(a_1, a_2, \ldots, a_n) = (p_r[1], p_{r+1}[1], \ldots, p_{r+n-1}[1])$. In other words, the list $(p_0[1], p_1[1], \ldots, p_{k^n-1}[1])$ is a de Bruijn sequence of order n on K (because the indices are cyclic modulo k^n, so the result in the previous sentence is precisely what is required in the definition of a de Bruijn sequence). Therefore, there exists a de Bruijn sequence of order n on K.

[*Remark:* The underlying philosophy of this solution was to reduce the question of the existence of a de Bruijn sequence to the existence of an Eulerian circuit in a multidigraph. At a first glance, this appears unexpected, since it seems more natural to model a de Bruijn sequence by a Hamiltonian cycle (in a different multidigraph) instead. However, there are few good criteria for the existence of a

[5] *Proof of* (2). Let $i \in \mathbb{Z}$. The arc p_{i+1} immediately follows the arc p_i on the Eulerian circuit $\mathbf{c}$. Hence, the target of the arc p_i is the source of the arc p_{i+1}. But by the definition of the multidigraph D, the former target is $(p_i[2], p_i[3], \ldots, p_i[n])$, whereas the latter source is $(p_{i+1}[1], p_{i+1}[2], \ldots, p_{i+1}[n-1])$. Hence, we have shown that $(p_i[2], p_i[3], \ldots, p_i[n]) = (p_{i+1}[1], p_{i+1}[2], \ldots, p_{i+1}[n-1])$. In other words, $p_i[j+1] = p_{i+1}[j]$ for each $j \in \{1, 2, \ldots, n-1\}$. This proves (2).

[6] Indeed, (3) can easily be proven by induction on g, using (2) in the induction step.

[7] *Proof.* Let $i \in \mathbb{Z}$. Then, (3) (applied to $j = 1$) shows that $p_i[1+g] = p_{i+g}[1]$ for each $g \in \{0, 1, \ldots, n-1\}$. In other words, $(p_i[1], p_i[2], \ldots, p_i[n]) = (p_i[1], p_{i+1}[1], \ldots, p_{i+n-1}[1])$. Therefore, $(p_i[1], p_{i+1}[1], \ldots, p_{i+n-1}[1]) = (p_i[1], p_i[2], \ldots, p_i[n]) = p_i$.

Hamiltonian cycle, whereas the existence of an Eulerian circuit is easy to check. This is why modelling de Bruijn sequences by Eulerian circuits proves to be the more useful approach.

This exercise is merely the starting point of a theory. For example, it is generalized by the following theorem:

> **Theorem 0.1.** Let d and m be two positive integers such that $d \mid m$ and $d > 1$. Then, there exists a permutation $(x_1, x_2, \ldots, x_m)$ of the list $(0, 1, \ldots, m-1)$ with the following property: For each $i \in \{1, 2, \ldots, m\}$, we have $x_{i+1} \equiv dx_i + r_i \mod m$ for some $r_i \in \{0, 1, \ldots, d-1\}$. (Here, x_{m+1} should be understood as x_1.)

Why does Theorem 0.1 generalize Exercise 3? Well, if $m = d^n$ is a power of d, then we can identify the integers $0, 1, \ldots, d-1$ with n-tuples of elements of $\{0, 1, \ldots, d-1\}$ (by representing them in base d, including just enough leading zeroes to ensure that they all have n digits). Thus, Theorem 0.1 turns into Exercise 3 in this case. With some work, the solution of Exercise 3 can be extended to a proof of Theorem 0.1. (Some work is required to prove that the digraph is still strongly connected.) Note that IMO Shortlist 2002 problem C6 is equivalent to the $d = 2$ particular case of Theorem 0.1.

For more variations on the notion of a de Bruijn sequence, see [ChDiGr92]. There are several questions left open in that paper, some of which are apparently still unsolved.

On the other hand, we can also ask ourselves: How many de Bruijn sequences of order n exist for a given n and K ? Interestingly, the answer is very explicit: The number of all de Bruijn sequences of order n is

$$k!^{k^{n-1}}, \qquad \text{where } k = |K|.$$

[8] This is proven in the case of $K = \{0, 1\}$ in [Stanle13, Corollary 10.11]. The general case can be proven along the same lines.] $\square$

0.5. Exercise 4: Indegrees and outdegrees in digraphs

Recall that the *indegree* of a vertex v of a digraph $D = (V, A)$ is defined to be the number of all arcs $a \in A$ whose target is v. This indegree is denoted by $\deg^- (v)$ or by $\deg_D^- (v)$ (whenever the graph D is not clear from the context).

Likewise, the *outdegree* of a vertex v of a digraph $D = (V, A)$ is defined to be the number of all arcs $a \in A$ whose source is v. This outdegree is denoted by $\deg^+ (v)$ or by $\deg_D^+ (v)$ (whenever the graph D is not clear from the context).

> **Exercise 4.** Let D be a digraph. Show that $\sum\limits_{v \in \operatorname{V}(D)} \deg^- (v) = \sum\limits_{v \in \operatorname{V}(D)} \deg^+ (v)$.

[8] Some authors treat two de Bruijn sequences as equal if one of them is obtained from the other by cyclic rotation. With that convention, the number has to be divided by k^n.

Solution to Exercise 4. Let us actually prove a somewhat more general fact:

Fact 1. Let (V, A, ϕ) be a multidigraph. Then, $\sum\limits_{v \in V} \deg^- v = \sum\limits_{v \in V} \deg^+ v$.

Fact 1 generalizes Exercise 4, because each digraph $D = (V, A)$ gives rise to a multidigraph (V, A, id_A), and the indegrees and the outdegrees of the vertices of the former digraph are exactly the same as in the latter multidigraph. Hence, proving Fact 1 will suffice.

Proof of Fact 1. For each arc $a \in A$, let $s(a)$ denote the source of a, and let $t(a)$ denote the target of a. (Thus, each $a \in A$ satisfies $\phi(a) = (s(a), t(a))$.)

Now, let us count the number of arcs $a \in A$ of our multidigraph in two different ways:

- Each arc $a \in A$ has a unique source $s(a)$. Thus, we can obtain the number $|A|$ of all arcs $a \in A$ by computing, for each $v \in V$, the number of all arcs $a \in A$ satisfying $s(a) = v$, and then adding up these numbers over all $v \in V$. Thus, we obtain

$$|A| = \sum_{v \in V} \underbrace{(\text{the number of all } a \in A \text{ satisfying } s(a) = v)}_{\substack{= \deg^+ v \\ \text{(by the definition of the outdegree } \deg^+ v \text{ of } v)}} = \sum_{v \in V} \deg^+ v. \quad (5)$$

- Each arc $a \in A$ has a unique target $t(a)$. Thus, we can obtain the number $|A|$ of all arcs $a \in A$ by computing, for each $v \in V$, the number of all arcs $a \in A$ satisfying $t(a) = v$, and then adding up these numbers over all $v \in V$. Thus, we obtain

$$|A| = \sum_{v \in V} \underbrace{(\text{the number of all } a \in A \text{ satisfying } t(a) = v)}_{\substack{= \deg^- v \\ \text{(by the definition of the indegree } \deg^- v \text{ of } v)}} = \sum_{v \in V} \deg^- v. \quad (6)$$

Comparing (6) with (5), we obtain $\sum\limits_{v \in V} \deg^- v = \sum\limits_{v \in V} \deg^+ v$. This proves Fact 1. □

□

0.6. Exercise 5: Counting 3-cycles in tournaments

The next few exercises are about *tournaments*. A *tournament* is a loopless[9] digraph $D = (V, A)$ with the following property: For any two distinct vertices $u \in V$ and $v \in V$, **precisely** one of the two pairs (u, v) and (v, u) belongs to A. (In other

[9] A digraph (V, A) is said to be *loopless* if it has no loops. (A *loop* means an arc of the form (v, v) for some $v \in V$.)

words, any two distinct vertices are connected by an arc in one direction, but not in both.)

A 3-*cycle*[10] in a tournament $D = (V, A)$ means a triple (u, v, w) of vertices in V such that all three pairs (u, v), (v, w) and (w, u) belong to A.

Exercise 5. Let $D = (V, A)$ be a tournament. Set $n = |V|$ and $m = \sum_{v \in V} \binom{\deg^- (v)}{2}$.

(a) Show that $m = \sum_{v \in V} \binom{\deg^+ (v)}{2}$.

(b) Show that the number of 3-cycles in D is $3 \left(\binom{n}{3} - m \right)$.

Solution sketch to Exercise 5. Let us introduce some convenient notations for this exercise:

- A *3-set* shall mean a 3-element subset of V. Clearly, the number of all 3-sets is $\binom{n}{3}$ (since $|V| = n$).
- A 3-set $\{u, v, w\}$ is said to be *cyclic* if its elements in **some** order form a 3-cycle (i.e., if one of the triples (u, v, w), (u, w, v), (v, u, w), (v, w, u), (w, u, v) and (w, v, u) is a 3-cycle).
- A 3-set $\{u, v, w\}$ is said to be *acyclic* if it is not cyclic.
- We say that a 3-set S is *sourced* at a vertex $u \in V$ if this vertex u belongs to S and if the arcs uv and uw are in A, where v and w are the two vertices of S distinct from u.
- We say that a 3-set S is *targeted* at a vertex $u \in V$ if this vertex u belongs to S and if the arcs vu and wu are in A, where v and w are the two vertices of S distinct from u.

(a) Let us count the number of all acyclic 3-sets in two different ways.

First, we make a few observations:

Observation 1: Let u, v and w be three distinct vertices in V such that the arcs uv and uw are in A. Then, $\{u, v, w\}$ is an acyclic 3-set sourced at u.

Proof of Observation 1. Trivial. □

[10] Note that our notions of 3-cycles and of cycles are somewhat different in nature: A 3-cycle is a triple of distinct vertices, whereas a cycle of length k is a $(k+1)$-tuple of vertices with its first and its last entry being the same vertex. Thus, a 3-cycle (u, v, w) is not in itself a cycle, but rather corresponds to the cycle (u, v, w, u). But, of course, the 3-cycles are in bijection with the cycles of length 3; thus, the difference between these two notions is merely notational.

> *Observation 2:* Let $u \in V$ be a vertex. Then, the number of all acyclic 3-sets sourced at u is $\dbinom{\deg^+ u}{2}$.

Proof of Observation 2. First of all, we introduce one more notion: An *out-neighbor* of u shall mean a vertex $x \in V$ such that the arc ux is in A. Clearly, the out-neighbors of u are in bijection with the arcs of D whose source is u [11]. Hence, the number of the former out-neighbors equals the number of the latter arcs. Since the number of the latter arcs is $\deg^+ u$ (indeed, this is how $\deg^+ u$ was defined), we can thus conclude that the number of the former out-neighbors is $\deg^+ u$. In other words, the number of all out-neighbors of u is $\deg^+ u$.

Observation 2 now easily follows: A 3-set is an acyclic 3-set sourced at u if and only if it consists of u and two distinct out-neighbors of u. Hence, choosing an acyclic 3-set sourced at u is tantamount to choosing two distinct out-neighbors of u (without specifying the order[12]). But the latter can be done in exactly $\dbinom{\deg^+ u}{2}$ ways (since the number of all out-neighbors of u is $\deg^+ u$). Hence, the number of all acyclic 3-sets sourced at u is $\dbinom{\deg^+ u}{2}$. [13] This proves Observation 2. $\square$

---

[11] *Proof.* The two maps

$$\begin{aligned} (\text{the set of all out-neighbors of } u) &\to (\text{the set of all arcs of } D \text{ whose source is } u), \\ x &\mapsto ux \end{aligned}$$

and

$$\begin{aligned} (\text{the set of all arcs of } D \text{ whose source is } u) &\to (\text{the set of all out-neighbors of } u), \\ a &\mapsto (\text{the target of } a) \end{aligned}$$

are mutually inverse (this can be checked in a straightforward manner). Note that we are here relying on the fact that D is a digraph, not a multidigraph!

[12] See the next footnote for a more rigorous way to write up this argument.

[13] Here is a more rigorous way to present this argument:

Let U be the set of all out-neighbors of u. Let G be the set of all acyclic 3-sets sourced at u. Then, the maps

$$G \to \mathcal{P}_2(U), \qquad S \mapsto S \setminus \{u\}$$

and

$$\mathcal{P}_2(U) \to G, \qquad T \mapsto T \cup \{u\}$$

are well-defined (this is easy to check: e.g., you have to apply Observation 1, and you have to argue that if S is a 3-set sourced at u, then the two elements of $S \setminus \{u\}$ are two distinct out-neighbors of u) and mutually inverse (this is essentially obvious). Hence, they provide bijections between G and $\mathcal{P}_2(U)$. Thus, $|G| = |\mathcal{P}_2(U)| = \dbinom{|U|}{2}$ (since every finite set Q and each $k \in \mathbb{N}$ satisfy $|\mathcal{P}_k(Q)| = \dbinom{|Q|}{k}$). But U is the set of all out-neighbors of u, and thus has

> *Observation 3:* Let S be an acyclic 3-set. Then, there is a **unique** vertex $u \in V$ such that S is sourced at u.

Proof of Observation 3. This can be straightforwardly verified: Write S in the form $\{a, b, c\}$. We want to know which of the six pairs ab, ba, bc, cb, ca and ac belong to A (i.e., are arcs of D). We know that exactly one of the two arcs ab and ba belongs to A (since D is a tournament); exactly one of the two arcs bc and cb belongs to A (since D is a tournament); exactly one of the two arcs ca and ac belongs to A (since D is a tournament). Hence, a total of $2 \cdot 2 \cdot 2 = 8$ cases are possible regarding the question which of the six pairs ab, ba, bc, cb, ca and ac belong to A (for example, one case is that ab, cb and ca belong to A, but ba, bc and ac do not). Two of these cases are impossible due to the requirement that S be acyclic[14]. In the remaining six cases, it is easy to check that Observation 3 holds. (For example, if ab, cb and ca belong to A, then there is a **unique** vertex $u \in V$ such that S is sourced at u; namely, this u is c. It is unique because clearly, if S is sourced at u, then u has to be an element of S, and the only element of S that works is c.) Thus, Observation 3 is proven. $\square$

Now, Observation 3 yields

$$\begin{aligned}
&(\text{the number of all acyclic 3-sets}) \\
&= \sum_{u \in V} \underbrace{(\text{the number of all acyclic 3-sets sourced at } u)}_{\substack{=\dbinom{\deg^+ u}{2} \\ \text{(by Observation 2)}}} = \sum_{u \in V} \dbinom{\deg^+ u}{2} \\
&= \sum_{v \in V} \dbinom{\deg^+ v}{2} \qquad (7)
\end{aligned}$$

(here, we renamed the summation index u as v).

On the other hand, we have the following observations, which mimic the Observations 1, 2 and 3 above (but with sources replaced by targets, and arcs changing directions), and whose proofs are analogous to those of the latter:

> *Observation 4:* Let u, v and w be three distinct vertices in V such that the arcs vu and wu are in A. Then, $\{u, v, w\}$ is an acyclic 3-set targeted at u.

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size $|U| = \deg^+ u$ (since the number of all out-neighbors of u is $\deg^+ u$). Hence, $|G| = \dbinom{|U|}{2}$ rewrites as $|G| = \dbinom{\deg^+ u}{2}$. Since G is the set of all acyclic 3-sets sourced at u, this means precisely that the number of all acyclic 3-sets sourced at u is $\dbinom{\deg^+ u}{2}$.

[14] Indeed, the two impossible cases are "ab, bc and ca belong to A, but ba, cb and ac do not" (because (a, b, c) would be a 3-cycle in this case) and "ba, cb and ac belong to A, but ab, bc and ca do not" (because (a, c, b) would be a 3-cycle in this case).

> *Observation 5:* Let $u \in V$ be a vertex. Then, the number of all acyclic 3-sets targeted at u is $\dbinom{\deg^{-} u}{2}$.

> *Observation 6:* Let S be an acyclic 3-set. Then, there is a **unique** vertex $u \in V$ such that S is targeted at u.

As I said, the proofs of Observations 4, 5 and 6 are analogous to the proofs of Observations 1, 2 and 3, and so are omitted. Now, similarly to how we proved (7) using Observations 1, 2 and 3, we can now prove the equality

$$(\text{the number of all acyclic 3-sets}) = \sum_{v \in V} \dbinom{\deg^{-} v}{2} \tag{8}$$

using Observations 4, 5 and 6. Comparing this equality with (7), we find $\sum\limits_{v \in V} \dbinom{\deg^{+} v}{2} = \sum\limits_{v \in V} \dbinom{\deg^{-} v}{2} = m$. This solves part **(a)** of the exercise.

(b) Let us make one more observation:

> *Observation 7:* Let S be a cyclic 3-set. Then, there exist exactly three 3-cycles (u, v, w) satisfying $\{u, v, w\} = S$.

Proof of Observation 7. We know that S is a cyclic 3-set. In other words, S is a 3-element subset of V whose elements in **some** order form a 3-cycle (because this is how a "cyclic 3-set" was defined). In other words, $S = \{a, b, c\}$ for some 3-cycle (a, b, c). Consider this (a, b, c). Hence, ab, bc and ca are arcs of D; therefore, ba, cb and ac are not arcs of D (since D is a tournament).

There are exactly six triples $(u, v, w) \in V^3$ satisfying $\{u, v, w\} = S$ (namely, the triples (a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b) and (c, b, a)). Among these six triples, exactly three are 3-cycles (in fact, all of the three triples (a, b, c), (b, c, a) and (c, a, b) are 3-cycles, whereas none of the three triples (a, c, b), (b, a, c) and (c, b, a) is a 3-cycle). Hence, there exist exactly three 3-cycles (u, v, w) satisfying $\{u, v, w\} = S$. This proves Observation 7. $\square$

Now, it is obvious that for each 3-cycle (u, v, w), the set $\{u, v, w\}$ is a cyclic 3-set. Hence, we can count the number of all 3-cycles as follows:

$$\begin{aligned}
&(\text{the number of all 3-cycles}) \\
&= \sum_{S \text{ is a cyclic 3-set}} \underbrace{(\text{the number of all 3-cycles } (u, v, w) \text{ satisfying } \{u, v, w\} = S)}_{\substack{=3 \\ (\text{by Observation 7})}} \\
&= \sum_{S \text{ is a cyclic 3-set}} 3 \\
&= 3 \, (\text{the number of all cyclic 3-sets}) . \qquad (9)
\end{aligned}$$

But recall that the number of all 3-sets is $\dbinom{n}{3}$. Each of these 3-sets is either cyclic or acyclic (but not both). Hence,

$$\begin{aligned}(\text{the number of all cyclic 3-sets}) &= \dbinom{n}{3} - \underbrace{(\text{the number of all acyclic 3-sets})}_{\substack{=\sum\limits_{v\in V}\dbinom{\deg^- v}{2} \\ \text{(by (8))}}} \\ &= \dbinom{n}{3} - \underbrace{\sum_{v\in V}\dbinom{\deg^- v}{2}}_{=m} = \dbinom{n}{3} - m.\end{aligned}$$

Hence, (9) rewrites as follows:

$$(\text{the number of all 3-cycles}) = 3\left(\dbinom{n}{3} - m\right).$$

This solves part **(b)** of the exercise.

[*Remark:* There is a simpler argument for **(a)**; let me briefly outline it:

$$\begin{aligned}\sum_{v\in V}\dbinom{\deg^- v}{2} - \sum_{v\in V}\dbinom{\deg^+ v}{2} &= \sum_{v\in V} \underbrace{\left(\dbinom{\deg^- v}{2} - \dbinom{\deg^+ v}{2}\right)}_{=\frac{1}{2}\left(\deg^- v - \deg^+ v\right)\left(\deg^- v + \deg^+ v + 1\right)} \\ &= \sum_{v\in V}\frac{1}{2}\left(\deg^- v - \deg^+ v\right)\underbrace{\left(\deg^- v + \deg^+ v + 1\right)}_{\substack{=n \\ \text{(why?)}}} = \frac{n}{2}\underbrace{\sum_{v\in V}\left(\deg^- v - \deg^+ v\right)}_{\substack{=\sum\limits_{v\in V}\deg^- v - \sum\limits_{v\in V}\deg^+ v = 0 \\ \text{(by Exercise 4)}}} = 0.\end{aligned}$$

However, this is of little help in proving part **(b)**.] □

0.7. Some lemmas

Before the next exercise, we prove a few simple facts that will eventually prove useful:

Lemma 0.2. Let $D = (V, A, \phi)$ be a multidigraph such that each vertex $v \in V$ satisfies $\deg^- v = \deg^+ v$. Assume that the set A is nonempty (i.e., the multidigraph D has at least one arc). Then, D has at least one cycle. (This cycle may be a one-vertex cycle, i.e., it may be of the form (v, v) for a vertex $v \in V$, provided that there is an arc from v to v.)

Proof of Lemma 0.2. We know that the set A is nonempty. In other words, the multidigraph D has at least one arc. Thus, there exists at least one path of length ≥ 1 in D (namely, the path consisting of this arc).

The set of paths of D is finite[15] and nonempty[16]. Hence, there exists a longest path in D (that is, a path in D having the maximum length). Fix such a path, and denote it by $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$. (As usual, this means that the vertices on this path are $v_0, v_1, \ldots, v_k$, and the arcs along this path are $a_1, a_2, \ldots, a_k$.) The vertices $v_0, v_1, \ldots, v_k$ are distinct (since $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ is a path).

We have $k \geq 1$ [17]. Thus, the arc a_k is well-defined. This arc a_k has target v_k. Thus, at least one arc has target v_k. Hence, $\deg^-(v_k) > 0$. But recall that each vertex $v \in V$ satisfies $\deg^- v = \deg^+ v$. Applying this to $v = v_k$, we obtain $\deg^-(v_k) = \deg^+(v_k)$. Hence, $\deg^+(v_k) = \deg^-(v_k) > 0$. Hence, there exists some arc with source v_k. Fix such an arc, and denote it by a_{k+1} (this is allowed, since so far we have only defined a_i for $i \in \{1, 2, \ldots, k\}$). Let v_{k+1} be the target of this arc (this is allowed, since so far we have only defined v_i for $i \in \{0, 1, \ldots, k\}$).

Now, $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k, a_{k+1}, v_{k+1})$ is clearly a walk in D. If this walk was a path, then it would be a longer path than $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$, which is absurd (since $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ was chosen to be a longest path in D). Hence, it is not a path. Therefore, the vertices $v_0, v_1, \ldots, v_k, v_{k+1}$ are not distinct (because if they were distinct, then the walk $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k, a_{k+1}, v_{k+1})$ would be a path). In other words, two of these vertices are equal. In other words, there exist two elements i and j of $\{0, 1, \ldots, k+1\}$ such that $i < j$ and $v_i = v_j$. Consider these i and j. Recall that the vertices $v_0, v_1, \ldots, v_k$ are distinct. Hence, j must be $k+1$ (since otherwise, $v_i = v_j$ would contradict the distinctness of $v_0, v_1, \ldots, v_k$). Thus, $v_j = v_{k+1}$. Therefore, $v_i = v_j = v_{k+1}$. Hence, $(v_i, a_{i+1}, v_{i+1}, a_{i+2}, v_{i+2}, \ldots, a_{k+1}, v_{k+1})$ is a circuit in D. This circuit is furthermore a cycle (since the vertices $v_i, v_{i+1}, \ldots, v_k$ are distinct (because $v_0, v_1, \ldots, v_k$ are distinct), and since $i < j = k+1$). Hence, there exists a cycle in D. This proves Lemma 0.2. $\square$

Corollary 0.3. Let $D = (V, A)$ be a digraph such that each vertex $v \in V$ satisfies $\deg^- v = \deg^+ v$. Assume that the set A is nonempty (i.e., the digraph D has at least one arc). Then, D has at least one cycle.

Proof of Corollary 0.3. The digraph $D = (V, A)$ gives rise to a multidigraph $D' = (V, A, \mathrm{id})$. The vertices in V have the same indegrees with respect to the latter multidigraph D' as they have with respect to the former digraph D; in other words, each $v \in V$ satisfies $\deg^-_{D'} v = \deg^-_D v$. Thus, we do not need to distinguish between $\deg^-_D v$ and $\deg^-_{D'} v$; we can use the notation $\deg^- v$ for both of these numbers.

[15]*Proof.* Recall that the vertices of a path in D must be distinct. Hence, a path in D cannot have length larger than $|V|$ (since D has only $|V|$ many vertices). Therefore, there are only finitely many paths in D (since there are only finitely many paths in D of any given length).

[16]since we just have shown that there exists at least one path of length ≥ 1 in D

[17]*Proof.* We know that there exists at least one path of length ≥ 1 in D. Hence, any longest path in D has length ≥ 1. In particular, this shows that the path $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$ has length ≥ 1 (since this path is a longest path in D). In other words, $k \geq 1$.

Applying Lemma 0.2 to D' and id instead of D and ϕ, we therefore conclude that D' has at least one cycle. If we denote this cycle by $(v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k)$, then $(v_0, v_1, \ldots, v_k)$ is a cycle of the digraph D. Thus, the digraph D has at least one cycle. This proves Corollary 0.3. $\square$

Lemma 0.4. Let V be a finite set. Let $E = (V, A)$ and $F = (V, B)$ be two tournaments with vertex set V. Let $u \in V$ and $v \in V$. Then, we have the following logical equivalence:

$$((u,v) \in B \setminus A) \iff ((v,u) \in A \setminus B). \tag{10}$$

Proof of Lemma 0.4. Let us first prove the implication

$$((u,v) \in B \setminus A) \Longrightarrow ((v,u) \in A \setminus B). \tag{11}$$

Indeed, assume that $(u,v) \in B \setminus A$ holds. Thus, $(u,v) \in B$ and $(u,v) \notin A$. The pair (u,v) belongs to B, thus is an arc of the tournament F. Therefore, (u,v) is not a loop (since tournaments have no loops). In other words, $u \neq v$. Hence, exactly one of the pairs (u,v) and (v,u) is an arc of E (since E is a tournament). In other words, exactly one of the pairs (u,v) and (v,u) belongs to A (since A is the set of the arcs of E). Since $(u,v) \notin A$, we thus have $(v,u) \in A$. On the other hand, recall again that $u \neq v$. Thus, exactly one of the pairs (u,v) and (v,u) is an arc of F (since F is a tournament). In other words, exactly one of the pairs (u,v) and (v,u) belongs to B (since B is the set of the arcs of F). Since $(u,v) \in B$, we thus have $(v,u) \notin B$. Combining $(v,u) \in A$ with $(v,u) \notin B$, we obtain $(v,u) \in A \setminus B$.

Now, forget that we assumed that $(u,v) \in B \setminus A$ holds. We thus have proven that $(v,u) \in A \setminus B$ under the assumption that $(u,v) \in B \setminus A$. In other words, we have proven the implication (11).

But we can also apply the implication (11) to v, u, B, A, F and E instead of u, v, A, B, E and F. Thus, we obtain the implication

$$((v,u) \in A \setminus B) \Longrightarrow ((u,v) \in B \setminus A). \tag{12}$$

Combining this implication with (11), we obtain the equivalence (10). Thus, Lemma 0.4 is proven. $\square$

0.8. Exercise 6: Transforming tournaments by reversing 3-cycles

Exercise 6. If a tournament D has a 3-cycle (u, v, w), then we can define a new tournament $D'_{u,v,w}$ as follows: The vertices of $D'_{u,v,w}$ shall be the same as those of D. The arcs of $D'_{u,v,w}$ shall be the same as those of D, except that the three arcs (u,v), (v,w) and (w,u) are replaced by the three new arcs (v,u), (w,v) and (u,w). (Visually speaking, $D'_{u,v,w}$ is obtained from D by turning the arrows on the arcs (u,v), (v,w) and (w,u) around.) We say that the new tournament $D'_{u,v,w}$ is obtained from the old tournament D by a 3-*cycle reversal operation*.

> Now, let V be a finite set, and let E and F be two tournaments with vertex set V. Prove that F can be obtained from E by a sequence of 3-cycle reversal operations if and only if each $v \in V$ satisfies $\deg_E^- (v) = \deg_F^- (v)$. (Note that a sequence may be empty, which allows handling the case $E = F$ even if E has no 3-cycles to reverse.)

Solution sketch to Exercise 6. Exercise 6 is [Moon13, Theorem 35].

Here is another solution:

Let us forget about V, E and F. Instead, we first introduce some more terminology:

- If (u, v) is an arc of a tournament D, then *reversing* this arc (u, v) means replacing it by the arc (v, u) (in other words, removing the arc (u, v), and adding a new arc (v, u) instead). The digraph that results from this operation is again a tournament. (Visually speaking, reversing an arc in a tournament means turning the arrow on this arc around.)

Using this terminology, our concept of "3-cycle reversal operation" can be reformulated as follows: A tournament D' is obtained from D by a 3-*cycle reversal operation* if and only if there exists a 3-cycle (u, v, w) such that D' is obtained from D by reversing the arcs (u, v), (v, w) and (w, u). If this is the case, we shall also say (more concretely) that D' is obtained from D by *reversing the* 3-*cycle* (u, v, w).

Let us introduce a new operation:

- If a tournament D has a cycle $\mathbf{c} = (v_0, v_1, \ldots, v_k)$, then we let $D''_{\mathbf{c}}$ be the tournament obtained from D by reversing the arcs $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$. In other words, we let $D''_{\mathbf{c}}$ be the tournament obtained from D by reversing all arcs of the cycle $\mathbf{c}$. We say that the new tournament $D''_{\mathbf{c}}$ is obtained from the old tournament D by *reversing the cycle* $\mathbf{c}$.

We now claim the following facts:

Observation 1: Let D be a tournament. Let $\mathbf{c}$ be a cycle of D. Then, $\mathbf{c}$ has length ≥ 3.

Proof of Observation 1. Assume the contrary. Then, $\mathbf{c}$ has length < 3. In other words, $\mathbf{c}$ has length 1 or 2.

But D is a tournament, and thus has no loops.

If the cycle $\mathbf{c}$ had length 1, then it would have the form (v, v) for some vertex v of D. Therefore, (v, v) would be an arc of D; this would imply that D has a loop; but this contradicts the fact that D has no loops. Hence, the cycle $\mathbf{c}$ cannot have length 1. Therefore, this cycle must have length 2 (since we know that $\mathbf{c}$ has length 1 or 2). Hence, this cycle $\mathbf{c}$ has the form (u, v, u) for some vertices u and v of D. Consider these u and v. Thus, both (u, v) and (v, u) are arcs of D. But this contradicts the fact that **exactly** one of the pairs (u, v) and (v, u) is an arc of D (since D is a tournament). This contradiction proves that our assumption was wrong. Hence, Observation 1 is proven. $\square$

> *Observation 2:* Let D be a tournament. Let $\mathbf{c}$ be a cycle of D. Let D'' be the tournament obtained from D by reversing the cycle $\mathbf{c}$. Then, D'' can also be obtained from D by a sequence of 3-cycle reversal operations.

Proof of Observation 2. We shall prove Observation 2 by strong induction over the length of $\mathbf{c}$. Thus, we fix an integer k, and we assume (as the induction hypothesis) that Observation 2 is proven in the case when the cycle $\mathbf{c}$ has length $< k$. We must now prove Observation 2 in the case when the cycle $\mathbf{c}$ has length k.

So let us consider the situation of Observation 2, and assume that the cycle $\mathbf{c}$ has length k. Write this cycle $\mathbf{c}$ in the form $(v_0, v_1, \ldots, v_k)$; thus, the vertices $v_0, v_1, \ldots, v_{k-1}$ of D are distinct, but $v_0 = v_k$. Observation 1 shows that $\mathbf{c}$ has length ≥ 3; in other words, we have $k \geq 3$ (since k is the length of $\mathbf{c}$).

We want to prove that D'' can be obtained from D by a sequence of 3-cycle reversal operations.

Recall that $v_0, v_1, \ldots, v_{k-1}$ are distinct. Since $k \geq 3$, this shows that v_0 and v_2 are distinct. Hence, exactly one of (v_0, v_2) and (v_2, v_0) is an arc of D (since D is a tournament). We thus are in one of the following two cases:

- *Case 1:* The pair (v_0, v_2) is an arc of D.
- *Case 2:* The pair (v_2, v_0) is an arc of D.

We consider each of these two cases separately:

- Let us consider Case 1 first. In this case, the pair (v_0, v_2) is an arc of D. Hence, $(v_0, v_2, v_3, \ldots, v_k)$ (this is just the list $(v_0, v_1, v_2, \ldots, v_k)$ with the vertex v_1 removed) is a circuit of D (since $(v_0, v_1, v_2, \ldots, v_k) = \mathbf{c}$ is a cycle of D), and furthermore is a cycle (since the vertices $v_0, v_2, v_3, \ldots, v_{k-1}$ are pairwise distinct, and since $k \geq 3$) and has length $k - 1 < k$. Denote this cycle by $\mathbf{c}'$. Let D_1 be the tournament obtained from D by reversing the cycle $\mathbf{c}'$.
  Recall that (by the induction hypothesis) Observation 2 is proven in the case when the cycle $\mathbf{c}$ has length $< k$. Hence, we can apply Observation 2 to $\mathbf{c}'$ and D_1 instead of $\mathbf{c}$ and D'' (since the cycle $\mathbf{c}'$ has length $< k$). Thus, we conclude that D_1 can also be obtained from D by a sequence of 3-cycle reversal operations.
  Next, we observe that the arc (v_0, v_2) of D has been reversed when we reversed the cycle $\mathbf{c}'$. Therefore, the tournament D_1 (unlike D) has no arc (v_0, v_2), but instead has the arc (v_2, v_0). On the other hand, the two arcs (v_0, v_1) and (v_1, v_2) have not been modified when we reversed the cycle $\mathbf{c}'$ (since these arcs are not part of the cycle $\mathbf{c}'$). Hence, these two arcs are arcs of D_1 as well. Thus, we know that (v_0, v_1), (v_1, v_2) and (v_2, v_0) are arcs of D_1. Therefore, (v_0, v_1, v_2) is a 3-cycle of D_1. Let D_2 be the tournament obtained from D_1 by reversing this 3-cycle (v_0, v_1, v_2). Thus, D_2 is obtained from D_1 by a 3-cycle reversal operation. Since D_1 (in turn) is obtained from D by a sequence of 3-cycle reversal operations, we thus conclude that D_2 is obtained

from D by a sequence of 3-cycle reversal operations.

But $D_2 = D''$ [18]. But we know that D_2 is obtained from D by a sequence of 3-cycle reversal operations. In other words, D'' is obtained from D by a sequence of 3-cycle reversal operations. This proves what we wanted to prove in Case 1.

- The argument in Case 2 is closely similar to the one we gave for Case 1. The only difference is the following: In Case 1, we have **first** reversed the cycle $\mathbf{c}' = (v_0, v_2, v_3, \ldots, v_k)$, thus obtaining a tournament D_1, and **then** reversed the 3-cycle (v_0, v_1, v_2) in D_1, thus obtaining a new tournament D_2 which was equal to D''. In contrast, this time, we have to proceed the other way round: We have to **first** reverse the 3-cycle (v_0, v_1, v_2), thus obtaining a tournament D_1, and **then** reverse the cycle $\mathbf{c}' = (v_0, v_2, v_3, \ldots, v_k)$ in D_1, thus obtaining a new tournament D_2 which again is equal to D''. Apart from this, nothing changes.

Hence, in either case, we have shown that D'' can be obtained from D by a sequence of 3-cycle reversal operations. In other words, Observation 2 holds for our D and $\mathbf{c}$. This completes the induction step. Hence, Observation 2 is proven by strong induction. $\square$

> *Observation 3:* Let V be a finite set. Let E and F be two tournaments with vertex set V. Assume that F can be obtained from E by a sequence of 3-cycle reversal operations. Then, each $v \in V$ satisfies $\deg_E^- (v) = \deg_F^- (v)$.

Proof of Observation 3. Fix $x \in V$. We shall prove that

$$\deg_E^- (x) = \deg_F^- (x) . \tag{13}$$

It is clearly enough to prove (13) in the case when F can be obtained from E by **one** 3-cycle reversal operation (because then, the validity of (13) in the general case would follow by induction). So we WLOG assume that F can be obtained from E by **one** 3-cycle reversal operation. In other words, there exists a 3-cycle (u, v, w) of

[18] *Proof.* Recall how D_2 was obtained from D:

- First, we obtained D_1 from D by reversing the cycle $\mathbf{c}' = (v_0, v_2, v_3, \ldots, v_k)$ in D. This amounts to reversing the arcs $(v_0, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k)$.
- Then, we obtained D_2 from D_1 by reversing the 3-cycle (v_0, v_1, v_2). This amounts to reversing the arcs (v_0, v_1), (v_1, v_2) and (v_2, v_0).

Thus, in total, we have reversed the arcs $(v_0, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k)$ and then the three arcs $(v_0, v_1), (v_1, v_2), (v_2, v_0)$ to obtain D_2 from D. Clearly, the reversal of the arc (v_0, v_2) was undone by the (later) reversal of the arc (v_2, v_0); therefore, we can forget about these two reversals. Hence, D_2 is obtained from D by reversing the arcs $(v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k), (v_0, v_1), (v_1, v_2)$. But this is tantamount to reversing the cycle $\mathbf{c}$ (since $\mathbf{c} = (v_0, v_1, v_2, \ldots, v_k)$). Hence, D_2 is obtained from D by reversing the cycle $\mathbf{c}$. But D'' is obtained from D in exactly the same way (i.e., by reversing the cycle $\mathbf{c}$). Hence, $D_2 = D''$.

E such that F can be obtained from E by reversing the 3-cycle (u, v, w). Consider this (u, v, w).

If $x \notin \{u, v, w\}$, then the arcs of E having target x are precisely the arcs of F having target x (since F can be obtained from E by reversing the 3-cycle (u, v, w), but this reversal clearly does not affect the arcs having target x). Therefore, if $x \notin \{u, v, w\}$, then $\deg_E^- (x) = \deg_F^- (x)$. In other words, (13) is proven in the case when $x \notin \{u, v, w\}$. Hence, we WLOG assume that we don't have $x \notin \{u, v, w\}$. In other words, we have $x \in \{u, v, w\}$. In other words, either $x = u$ or $x = v$ or $x = w$. We WLOG assume that $x = u$ (since the other two cases are similar).

Reversing the 3-cycle (u, v, w) removes the arcs $(u, v), (v, w), (w, u)$ from the digraph E while adding the arcs $(v, u), (w, v), (u, w)$. Therefore, in total, one arc having target u is removed (namely, the arc (w, u)), and one arc having target u is added (namely, the arc (v, u)). As a consequence, the number of arcs of F having target u is obtained from the number of arcs of E having target u by subtracting 1 and then adding 1 back. In other words, $\deg_F^- (u) = \deg_E^- (u) - 1 + 1 = \deg_E^- (u)$. Hence, $\deg_E^- (u) = \deg_F^- (u)$. Since $x = u$, this rewrites as $\deg_E^- (x) = \deg_F^- (x)$. Thus, (13) is proven.

Now, forget that we fixed x. We thus have shown that each $x \in V$ satisfies $\deg_E^- (x) = \deg_F^- (x)$. Renaming x as v in this statement, we conclude that each $v \in V$ satisfies $\deg_E^- (v) = \deg_F^- (v)$. This proves Observation 3. $\square$

Observation 4: Let V be a finite set. Let $E = (V, A)$ and $F = (V, B)$ be two digraphs with vertex set V. Let $v \in V$.

(a) We have $\deg_{(V, A \setminus B)}^- (v) - \deg_{(V, B \setminus A)}^- (v) = \deg_E^- (v) - \deg_F^- (v)$.

(b) Assume that E and F are tournaments. Assume furthermore that $\deg_E^- (v) = \deg_F^- (v)$. Then, $\deg_{(V, A \setminus B)}^- (v) = \deg_{(V, A \setminus B)}^+ (v)$.

Proof of Observation 4. Recall first that

$$(\text{the number of all } u \in V \text{ satisfying } (u, v) \in A) = \deg_E^- (v) \tag{14}$$

[19]. Hence,

$$\sum_{\substack{u\in V;\\(u,v)\in A}} 1 = \underbrace{(\text{the number of all } u \in V \text{ satisfying } (u,v) \in A)}_{=\deg_E^-(v)} \cdot 1$$
$$= \deg_E^-(v) \cdot 1 = \deg_E^-(v). \qquad (15)$$

The same argument (applied to F and B instead of E and A) shows that

$$\sum_{\substack{u\in V;\\(u,v)\in B}} 1 = \deg_F^-(v). \qquad (16)$$

Furthermore, the same argument that we used to prove (15) can be applied to $(V, A \setminus B)$ and $A \setminus B$ instead of E and A. As a result, we find that

$$\sum_{\substack{u\in V;\\(u,v)\in A\setminus B}} 1 = \deg_{(V,A\setminus B)}^-(v). \qquad (17)$$

Finally, the same argument that we used to prove (15) can be applied to $(V, B \setminus A)$ and $B \setminus A$ instead of E and A. As a result, we find that

$$\sum_{\substack{u\in V;\\(u,v)\in B\setminus A}} 1 = \deg_{(V,B\setminus A)}^-(v). \qquad (18)$$

[19] *Proof of* (14). Recall that $E = (V, A)$ is a digraph (not a multidigraph); therefore, the arcs of E are pairs of vertices in V. Hence, it is easy to check that the two maps

$$(\text{the set of all } u \in V \text{ satisfying } (u,v) \in A) \to (\text{the set of all arcs of } E \text{ whose target is } v),$$
$$u \mapsto (u,v)$$

and

$$(\text{the set of all arcs of } E \text{ whose target is } v) \to (\text{the set of all } u \in V \text{ satisfying } (u,v) \in A),$$
$$a \mapsto (\text{the source of } a)$$

are mutually inverse. Thus, they are inverse bijections between the set of all arcs of E whose target is v and the set of all $u \in V$ satisfying $(u,v) \in A$. Consequently, the size of the latter set equals the size of the former set. In other words, the number of all $u \in V$ satisfying $(u,v) \in A$ equals the number of all arcs of E whose target is v. But since the latter number is $\deg_E^-(v)$ (in fact, this is how $\deg_E^-(v)$ is defined), this rewrites as follows: The number of all $u \in V$ satisfying $(u,v) \in A$ equals $\deg_E^-(v)$. This proves (14).

Now, (15) yields

$$\begin{aligned}
&\deg^-_E(v) \\
&= \sum_{\substack{u \in V;\\ (u,v)\in A}} 1 = \underbrace{\sum_{\substack{u \in V;\\ (u,v)\in A;\\ (u,v)\in B}} 1}_{= \sum_{\substack{u\in V;\\ (u,v)\in A\cap B}} 1} + \underbrace{\sum_{\substack{u \in V;\\ (u,v)\in A;\\ (u,v)\notin B}} 1}_{= \sum_{\substack{u\in V;\\ (u,v)\in A\setminus B}} 1} \\
&\qquad \left(\text{since each } u \in V \text{ satisfies either } (u,v) \in B \text{ or } (u,v) \notin B\text{, but not both}\right) \\
&= \sum_{\substack{u \in V;\\ (u,v)\in A\cap B}} 1 + \sum_{\substack{u \in V;\\ (u,v)\in A\setminus B}} 1. \qquad (19)
\end{aligned}$$

Meanwhile, (16) yields

$$\begin{aligned}
&\deg^-_F(v) \\
&= \sum_{\substack{u \in V;\\ (u,v)\in B}} 1 = \underbrace{\sum_{\substack{u \in V;\\ (u,v)\in B;\\ (u,v)\in A}} 1}_{= \sum_{\substack{u\in V;\\ (u,v)\in A\cap B}} 1} + \underbrace{\sum_{\substack{u \in V;\\ (u,v)\in B;\\ (u,v)\notin A}} 1}_{= \sum_{\substack{u\in V;\\ (u,v)\in B\setminus A}} 1} \\
&\qquad \left(\text{since each } u \in V \text{ satisfies either } (u,v) \in A \text{ or } (u,v) \notin A\text{, but not both}\right) \\
&= \sum_{\substack{u \in V;\\ (u,v)\in A\cap B}} 1 + \sum_{\substack{u \in V;\\ (u,v)\in B\setminus A}} 1. \qquad (20)
\end{aligned}$$

Subtracting (20) from (19), we find

$$\begin{aligned}
&\deg^-_E(v) - \deg^-_F(v) \\
&= \left(\sum_{\substack{u \in V;\\ (u,v)\in A\cap B}} 1 + \sum_{\substack{u \in V;\\ (u,v)\in A\setminus B}} 1 \right) - \left(\sum_{\substack{u \in V;\\ (u,v)\in A\cap B}} 1 + \sum_{\substack{u \in V;\\ (u,v)\in B\setminus A}} 1 \right) \\
&= \underbrace{\sum_{\substack{u \in V;\\ (u,v)\in A\setminus B}} 1}_{\substack{=\deg^-_{(V,A\setminus B)}(v)\\ \text{(by (17))}}} - \underbrace{\sum_{\substack{u \in V;\\ (u,v)\in B\setminus A}} 1}_{\substack{=\deg^-_{(V,B\setminus A)}(v)\\ \text{(by (18))}}} = \deg^-_{(V,A\setminus B)}(v) - \deg^-_{(V,B\setminus A)}(v).
\end{aligned}$$

This proves Observation 4 **(a)**.

(b) The summation sign $\sum\limits_{\substack{u\in V;\\ (u,v)\in B\setminus A}}$ in (18) can be rewritten as $\sum\limits_{\substack{u\in V;\\ (v,u)\in A\setminus B}}$ (because of the equivalence (10)). Thus, (18) rewrites as follows:

$$\sum_{\substack{u\in V;\\ (v,u)\in A\setminus B}} 1 = \deg^-_{(V,B\setminus A)}(v). \tag{21}$$

On the other hand,

$$(\text{the number of all } u \in V \text{ satisfying } (v,u) \in A) = \deg^+_E (v) \tag{22}$$

[20]. Hence,

$$\begin{aligned}\sum_{\substack{u\in V;\\ (v,u)\in A}} 1 &= \underbrace{(\text{the number of all } u \in V \text{ satisfying } (v,u) \in A)}_{=\deg^+_E(v)} \cdot 1 \\ &= \deg^+_E (v) \cdot 1 = \deg^+_E (v). \end{aligned} \tag{23}$$

The same argument (applied to $(V, A \setminus B)$ and $A \setminus B$ instead of E and A) shows that

$$\sum_{\substack{u\in V;\\ (v,u)\in A\setminus B}} 1 = \deg^+_{(V,A\setminus B)}(v).$$

Comparing this with (21), we find $\deg^-_{(V,B\setminus A)}(v) = \deg^+_{(V,A\setminus B)}(v)$. Now, Observation 4 **(a)** yields $\deg^-_{(V,A\setminus B)}(v) - \deg^-_{(V,B\setminus A)}(v) = \deg^-_E(v) - \deg^-_F(v) = 0$ (since $\deg^-_E(v) = \deg^-_F(v)$). Hence, $\deg^-_{(V,A\setminus B)}(v) = \deg^-_{(V,B\setminus A)}(v) = \deg^+_{(V,A\setminus B)}(v)$. This proves Observation 4 **(b)**. $\square$

> *Observation 5:* Let V be a finite set. Let $E = (V, A)$ and $F = (V, B)$ be two tournaments with vertex set V such that $A \neq B$. Assume that each $v \in V$ satisfies $\deg^-_E(v) = \deg^-_F(v)$. Then:
>
> **(a)** The digraph $(V, A \setminus B)$ has at least one cycle.
>
> **(b)** Each cycle of the digraph $(V, A \setminus B)$ is also a cycle of E.
>
> **(c)** Let $(v_0, v_1, \ldots, v_k)$ be a cycle of the digraph $(V, A \setminus B)$. Then,
>
> $$\{(v_0,v_1), (v_1,v_2), \ldots, (v_{k-1},v_k)\} \subseteq A \setminus B \qquad \text{and}$$
> $$\{(v_1,v_0), (v_2,v_1), \ldots, (v_k,v_{k-1})\} \subseteq B \setminus A.$$

Proof of Observation 5. **(a)** The set $A \setminus B$ is nonempty [21]. Furthermore, each vertex $v \in V$ satisfies $\deg^-_{(V,A\setminus B)} v = \deg^+_{(V,A\setminus B)} v$ [22]. Thus, Corollary 0.3 (applied to

[20]The proof of (22) is analogous to that of (14), and so is left to the reader.

[21]*Proof.* Assume the contrary. Thus, $A \setminus B = \varnothing$. Hence, $A \subseteq B$. Since $A \neq B$, we thus conclude that A is a proper subset of B. Hence, there exists some $(u,v) \in B \setminus A$. Consider this (u,v). From $(u,v) \in B \setminus A$, we obtain $(v,u) \in A \setminus B$ (by the equivalence (10)). Hence, $(v,u) \in A \setminus B = \varnothing$, which is absurd. Hence, we have obtained a contradiction. Therefore, our assumption was false, qed.

[22]*Proof.* Let $v \in V$. The hypothesis of Observation 5 yields $\deg^-_E(v) = \deg^-_F(v)$. Hence, Observation 4 **(b)** shows that $\deg^-_{(V,A\setminus B)}(v) = \deg^+_{(V,A\setminus B)}(v)$. Qed.

$(V, A \setminus B)$ and $A \setminus B$ instead of D and A) yields that the digraph $(V, A \setminus B)$ has at least one cycle. This proves Observation 5 **(a)**.

(b) Each arc of the digraph $(V, A \setminus B)$ is also an arc of (V, A) (since $A \setminus B \subseteq A$). In other words, each arc of the digraph $(V, A \setminus B)$ is also an arc of E (since $E = (V, A)$). Therefore, each cycle of the digraph $(V, A \setminus B)$ is also a cycle of E. This proves Observation 5 **(b)**.

(c) The arcs $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ are the arcs of the cycle $(v_0, v_1, \ldots, v_k)$ of the digraph $(V, A \setminus B)$, and thus are arcs of the digraph $(V, A \setminus B)$. In other words, they belong to $A \setminus B$. In other words, $\{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\} \subseteq A \setminus B$.

On the other hand, each $i \in \{0, 1, \ldots, k-1\}$ satisfies $(v_{i+1}, v_i) \in B \setminus A$ [23]. In other words, $\{(v_1, v_0), (v_2, v_1), \ldots, (v_k, v_{k-1})\} \subseteq B \setminus A$. The proof of Observation 5 **(c)** is now complete. $\square$

> *Observation 6:* Let V be a finite set. Let $E = (V, A)$ and $F = (V, B)$ be two tournaments with vertex set V such that $A \neq B$. Assume that each $v \in V$ satisfies $\deg^-_E(v) = \deg^-_F(v)$.
>
> Let $\mathbf{c}$ be a cycle of the digraph $(V, A \setminus B)$. Thus, $\mathbf{c}$ is also a cycle of E (by Observation 5 **(b)**). Let $E'' = (V, A'')$ be the tournament obtained from E by reversing the cycle $\mathbf{c}$. (This is well-defined, since $\mathbf{c}$ is a cycle of E.) Then:
>
> **(a)** We have $|A'' \setminus B| < |A \setminus B|$.
>
> **(b)** Each $v \in V$ satisfies $\deg^-_{E''}(v) = \deg^-_F(v)$.
>
> **(c)** The tournament E'' can be obtained from E by a sequence of 3-cycle reversal operations.

Proof of Observation 6. Write the cycle $\mathbf{c}$ in the form $(v_0, v_1, \ldots, v_k)$ (with $v_0, v_1, \ldots, v_k \in V$). Then, Observation 5 **(c)** yields

$$\{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\} \subseteq A \setminus B \qquad \text{and} \tag{24}$$
$$\{(v_1, v_0), (v_2, v_1), \ldots, (v_k, v_{k-1})\} \subseteq B \setminus A. \tag{25}$$

Set $X = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\}$ and $Y = \{(v_1, v_0), (v_2, v_1), \ldots, (v_k, v_{k-1})\}$. Thus, the relations (24) and (25) rewrite as $X \subseteq A \setminus B$ and $Y \subseteq B \setminus A$, respectively. In particular, no element of X belongs to B (since $X \subseteq A \setminus B$); thus, $X \setminus B = X$. Also, $Y \subseteq B \setminus A \subseteq B$, so that $Y \setminus B = \varnothing$.

[23] *Proof.* Let $i \in \{0, 1, \ldots, k-1\}$. We know that the arcs $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ belong to $A \setminus B$. In particular, (v_i, v_{i+1}) belongs to $A \setminus B$. In other words, $(v_i, v_{i+1}) \in A \setminus B$. But (10) (applied to v_{i+1} and v_i instead of u and v) yields that we have the logical equivalence

$$((v_{i+1}, v_i) \in B \setminus A) \iff ((v_i, v_{i+1}) \in A \setminus B).$$

Hence, we have $(v_{i+1}, v_i) \in B \setminus A$ (since we know that $(v_i, v_{i+1}) \in A \setminus B$). Qed.

Also, clearly, $k \geq 1$ (since $(v_0, v_1, \ldots, v_k) = \mathbf{c}$ is a cycle), and thus the set X is nonempty. Hence, $|X| > 0$.

(a) Recall that the tournament E'' is obtained from E by reversing the cycle $\mathbf{c} = (v_0, v_1, \ldots, v_k)$. In other words, the tournament E'' is obtained from E by reversing the arcs $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ (because this is how "reversing the cycle $(v_0, v_1, \ldots, v_k)$" was defined). In other words, the tournament E'' is obtained from E by removing the arcs $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ and adding the new arcs $(v_1, v_0), (v_2, v_1), \ldots, (v_k, v_{k-1})$. Since the arc set of E'' is A'', whereas the arc set of E is A, we therefore have

$$A'' = \left(A \setminus \underbrace{\{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\}}_{=X} \right) \cup \underbrace{\{(v_1, v_0), (v_2, v_1), \ldots, (v_k, v_{k-1})\}}_{=Y}$$
$$= (A \setminus X) \cup Y.$$

Hence,

$$A'' \setminus B = ((A \setminus X) \cup Y) \setminus B = \left((A \setminus B) \setminus \underbrace{(X \setminus B)}_{=X} \right) \cup \underbrace{(Y \setminus B)}_{=\varnothing} = ((A \setminus B) \setminus X) \cup \varnothing$$
$$= (A \setminus B) \setminus X,$$

and therefore

$$\left| A'' \setminus B \right| = |(A \setminus B) \setminus X| = |A \setminus B| - |X| \qquad (\text{since } X \subseteq A \setminus B)$$
$$< |A \setminus B| \qquad (\text{since } |X| > 0).$$

This proves Observation 6 **(a)**.

(c) Observation 2 (applied to E and E'' instead of D and D'') shows that E'' can also be obtained from E by a sequence of 3-cycle reversal operations. This proves Observation 6 **(c)**.

(b) Let $v \in V$. Then, $\deg^-_E(v) = \deg^-_F(v)$ (by the hypothesis of Observation 6). But we know (from Observation 6 **(c)**) that the tournament E'' can also be obtained from E by a sequence of 3-cycle reversal operations. Hence, Observation 3 (applied to E'' instead of F) shows that $\deg^-_E(v) = \deg^-_{E''}(v)$. Therefore, $\deg^-_{E''}(v) = \deg^-_E(v) = \deg^-_F(v)$. This proves Observation 6 **(b)**. $\square$

> *Observation 7:* Let V be a finite set. Let E and F be two tournaments with vertex set V. Assume that each $v \in V$ satisfies $\deg^-_E(v) = \deg^-_F(v)$. Then, F can be obtained from E by a sequence of 3-cycle reversal operations.

Proof of Observation 7. We shall prove Observation 7 by strong induction over $|\mathrm{A}(E) \setminus \mathrm{A}(F)|$. Thus, we fix some $N \in \mathbb{N}$, and we assume (as the induction hypothesis) that Observation 7 is already proven in the case when $|\mathrm{A}(E) \setminus \mathrm{A}(F)| < N$. In other words, if E and F are two tournaments with vertex set V, if each $v \in V$ satisfies $\deg^-_E(v) = \deg^-_F(v)$, and if we have $|\mathrm{A}(E) \setminus \mathrm{A}(F)| < N$, then

$$F \text{ can be obtained from } E \text{ by a sequence of 3-cycle reversal operations.} \qquad (26)$$

Now, we need to prove Observation 7 in the case when $|\mathrm{A}(E) \setminus \mathrm{A}(F)| = N$. So let E and F be two tournaments wirth vertex set V, and assume that each $v \in V$ satisfies $\deg^-_E(v) = \deg^-_F(v)$. Assume furthermore that $|\mathrm{A}(E) \setminus \mathrm{A}(F)| = N$. Our goal is to prove that F can be obtained from E by a sequence of 3-cycle reversal operations.

Write the tournaments E and F in the forms $E = (V, A)$ and $F = (V, B)$. (This is possible, since both E and F have vertex set V.) Thus, $\mathrm{A}(E) = A$ and $\mathrm{A}(F) = B$. If $A = B$, then our claim is obvious[24]. Hence, we WLOG assume that we don't have $A = B$. Thus, $A \neq B$.

At this point, all we need to do is combining observations that we already have proven. Observation 5 **(a)** shows that the digraph $(V, A \setminus B)$ has at least one cycle. Fix such a cycle, and denote it by $\mathbf{c}$. Then, $\mathbf{c}$ is also a cycle of E (by Observation 5 **(b)**). Let $E'' = (V, A'')$ be the tournament obtained from E by reversing the cycle $\mathbf{c}$. (This is well-defined, since $\mathbf{c}$ is a cycle of E.) Thus, $\mathrm{A}(E'') = A''$. Observation 6 **(b)** shows that each $v \in V$ satisfies $\deg^-_{E''}(v) = \deg^-_F(v)$. Observation 6 **(a)** shows that $|A'' \setminus B| < |A \setminus B|$. Since $A'' = \mathrm{A}(E'')$, $A = \mathrm{A}(E)$ and $B = \mathrm{A}(F)$, this rewrites as $|\mathrm{A}(E'') \setminus \mathrm{A}(F)| < |\mathrm{A}(E) \setminus \mathrm{A}(F)|$. Since $|\mathrm{A}(E) \setminus \mathrm{A}(F)| = N$, this furthermore rewrites as $|\mathrm{A}(E'') \setminus \mathrm{A}(F)| < N$. Hence, we can apply (26) to E'' instead of E (since we have also shown that each $v \in V$ satisfies $\deg^-_{E''}(v) = \deg^-_F(v)$). As a result, we conclude that F can be obtained from E'' by a sequence of 3-cycle reversal operations. But the tournament E'' can (in turn) be obtained from E by a sequence of 3-cycle reversal operations (by Observation 6 **(c)**). Combining the previous two sentences, we conclude that F can be obtained from E by a sequence of 3-cycle reversal operations (indeed, we first apply the sequence of 3-cycle reversal operations that lets us obtain E'' from E, and then apply the sequence of 3-cycle reversal operations that lets us obtain F from E''). But this is exactly the claim that we wanted to prove. Hence, we have proven Observation 7 in the case when $|\mathrm{A}(E) \setminus \mathrm{A}(F)| = N$. Thus, the proof of Observation 7 (by strong induction) is complete. □

Exercise 6 now follows from Observation 3 and Observation 7. (Indeed, the claim of Exercise 6 is an "if and only if" statement. The "if" part of this statement follows from Observation 7, whereas the "only if" part follows from Observation 3.) □

0.9. Exercise 7: Transforming tournaments by reversing 2-paths

A tournament $D = (V, A)$ is called *transitive* if it has no 3-cycles.

[24] *Proof.* Assume that $A = B$. Thus, $(V, A) = (V, B)$, so that $E = (V, A) = (V, B) = F$. Hence, F can be obtained from E by a sequence of 3-cycle reversal operations (namely, by the empty sequence). But this is exactly what we have to prove.

Exercise 7. If a tournament $D = (V, A)$ has three distinct vertices u, v and w satisfying $(u, v) \in A$ and $(v, w) \in A$, then we can define a new tournament $D''_{u,v,w}$ as follows: The vertices of $D''_{u,v,w}$ shall be the same as those of D. The arcs of $D''_{u,v,w}$ shall be the same as those of D, except that the two arcs (u, v) and (v, w) are replaced by the two new arcs (v, u) and (w, v). We say that the new tournament $D''_{u,v,w}$ is obtained from the old tournament D by a *2-path reversal operation*.

Let D be any tournament. Prove that there is a sequence of 2-path reversal operations that transforms D into a transitive tournament.

Solution sketch to Exercise 7. We shall solve Exercise 7 by induction on $|V|$, where V denotes the vertex set of D.

The *induction base* (i.e., the case $|V| = 0$) is obvious (because in this case, D is already transitive, and thus the empty sequence of 2-path reversal operations transforms D into a transitive tournament).

Now, to the *induction step*. Fix a positive integer N, and assume (as the induction hypothesis) that Exercise 7 is already solved in the case when $|V| = N - 1$. Now, we must solve Exercise 7 in the case when $|V| = N$. So let us fix a tournament D with vertex set V satisfying $|V| = N$. Write D in the form $D = (V, A)$.

We say that a tournament is *sinkless* if it has no vertex that has outdegree 0.

We shall now prove the following observation:

> *Observation 1:* Assume that the tournament D is sinkless. Let $u \in V$. Then, we can apply a 2-path reversal operation to D that decreases $\deg^+ u$ by 1.

Proof of Observation 1. The tournament D is sinkless. In other words, it has no vertex that has outdegree 0. In particular, the vertex u does not have outdegree 0. Hence, there exists at least one arc (u, v) of D having source u. Consider such an arc. Furthermore, the vertex v also does not have outdegree 0 (since D has no vertex that has outdegree 0). Thus, there exists at least one arc (v, w) of D having source v. Consider such an arc.

We have $u \neq v$ (since (u, v) is an arc of D) and $v \neq w$ (since (v, w) is an arc of D). Also, if we had $u = w$, then both $(v, u) = (v, w)$ and (u, v) would be arcs of D, and this would contradict the fact that D is a tournament (indeed, a tournament has only one arc between two distinct vertices). Hence, we cannot have $u = w$. Thus, $u \neq w$. Now, the three vertices u, v and w of D are distinct (since $u \neq v$, $v \neq w$ and $u \neq w$) and satisfy $(u, v) \in A$ and $(v, w) \in A$ (since (u, v) and (v, w) are arcs of D). Hence, a new tournament $D''_{u,v,w}$ is defined. Recall that this new tournament $D''_{u,v,w}$ differs from D in that the two arcs (u, v) and (v, w) are replaced by the two new arcs (v, u) and (w, v) (due to the definition of $D''_{u,v,w}$). In particular, $D''_{u,v,w}$ is lacking the arc (u, v) that D used to have, but does not have any new arcs (i.e., arcs that D lacked) with source u. Thus, $\deg^+_{D''_{u,v,w}} u = \deg^+_D u - 1$.

But the tournament $D''_{u,v,w}$ clearly is obtained from D by a 2-path reversal operation. This 2-path reversal operation has decreased $\deg^+ u$ by 1 (since $\deg^+_{D''_{u,v,w}} u = \deg^+_D u - 1$). Hence, Observation 1 is proven. □

> *Observation 2:* We can apply a sequence of 2-path reversal operations to D that ensures the following: The tournament obtained at the end of this sequence is not sinkless.

Proof of Observation 2. If D already is not sinkless, then Observation 2 obviously holds (just apply the empty sequence). Otherwise, fix any vertex $u \in V$. (This is possible, since $|V| = N > 0$.) Observation 1 shows that we can apply a 2-path reversal operation to D that decreases $\deg^+ u$ by 1. Apply this 2-path reversal operation, and replace D by the resulting tournament. Repeat this step as often as possible (each time applying Observation 1 anew, as long as D is sinkless). This process must eventually come to an end[25], and thus we eventually end up with a tournament that is no longer sinkless.[26] This proves Observation 2. □

Now, our goal is to show that there is a sequence of 2-path reversal operations that transforms D into a transitive tournament. We achieve this by performing the following procedure:

- **First step:** We first perform a sequence of 2-path reversal operations that transforms D into a tournament that is not sinkless. Such a sequence exists because of Observation 2. Let E be the tournament obtained at the end of this step.

- **Second step:** Now, the tournament E is not sinkless. In other words, E has a vertex that has outdegree 0. Fix such a vertex, and denote it by p. Let E_1 be the tournament obtained from E by removing the vertex p and the arcs whose target or source is p. [27] Then, the number of the vertices of E_1 is $|V| - 1 = N - 1$ (since $|V| = N$). Therefore, by the induction hypothesis, we can apply Exercise 7 to E_1 instead of D. We thus conclude that there is a sequence of 2-path reversal operations that transforms E_1 into a transitive tournament. Fix such a sequence, and apply it "inside E" (i.e., apply the same operations to E, ignoring the vertex p). Let F be the tournament obtained at the end of this step.

[25] Indeed, each time we apply the 2-path reversal operation, the outdegree $\deg^+ u$ is decreased by 1; but this outdegree cannot keep decreasing by 1 indefinitely.

[26] Note that we are not guaranteed to obtain $\deg^+ u = 0$ in the final tournament. We are only guaranteed that it will not be sinkless! There may be another vertex v satisfying $\deg^+ v = 0$ instead.

[27] Formally speaking, if we write the tournament E as (V, A'), then E_1 is the tournament (V_1, A'_1), where $V_1 = V \setminus \{p\}$, where $A'_1 = \{a \in A' \mid \text{neither the source nor the target of } a \text{ is } p\}$. Of course, due to p having outdegree 0, the tournament E has no arcs with source p, and so we only need to care about arcs with target p.

The tournament F is thus obtained from D by a sequence of 2-path reversal operations. What do we know about F ?

- First, we know that the vertex p has outdegree 0 in F (because it had outdegree 0 in E, and because no arcs with source or target p have been modified by the operations that transformed E into F). In other words, the tournament F has no arcs with source p. Therefore, the tournament F has no 3-cycles that contain the vertex p (because if a 3-cycle contains p, then there must be an arc with source p).

- We furthermore know that the tournament obtained from F by removing the vertex p is transitive (because the operations that transformed E into F were chosen in such a way as to transform E_1 into a transitive tournament). In other words, the tournament obtained from F by removing the vertex p has no 3-cycles. Equivalently, the tournament F has no 3-cycles that do not contain the vertex p.

We thus have seen that the tournament F has no 3-cycles that contain the vertex p, but also has no 3-cycles that do not contain the vertex p. Hence, the tournament F has no 3-cycles at all. In other words, the tournament F is transitive.

Hence, there is a sequence of 2-path reversal operations that transforms D into a transitive tournament (namely, the sequence of operations that transformed D into F). In other words, Exercise 7 is solved in the case when $|V| = N$. This completes the induction step, and so the solution of Exercise 7 is complete. $\square$

Remark 0.5. Exercise 7 suggests an additional question: If E and F are two tournaments with the same vertex set, then is it always possible to transform E into F by a sequence of 2-path reversal operations?

The answer to this question is "no", and there is a rather neat reason for this: WLOG assume that the common vertex set of E and F is $\{1, 2, \ldots, n\}$. If D is a tournament with vertex set $\{1, 2, \ldots, n\}$, then an *inversion* of D will mean an arc (i, j) of D satisfying $i > j$. Now, it is easy to see that if we apply a 2-path reversal operation to a tournament with vertex set $\{1, 2, \ldots, n\}$, then the number of inversions of the tournament does not change modulo 2 (i.e., if this number was even, then it remains even; and if this number was odd, then it remains odd). Hence, E cannot be transformed into F by a sequence of 2-path reversal operations unless the number of inversions of E is congruent to the number of inversions of F modulo 2.

But what if they are congruent? I don't know. Feel free to comment!

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