University of Minnesota

Math 5707: Graph Theory, Spring 2017 Homework 2

Nicholas Rancourt (edited by DG)

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1 Exercise 1

1.1 Problem

Let G and H be two simple graphs. The *Cartesian product* of G and H is a new simple graph, denoted $G \times H$, which is defined as follows:

- The vertex set $\operatorname{V}(G \times H)$ of $G \times H$ is the Cartesian product $\operatorname{V}(G) \times \operatorname{V}(H)$.
- A vertex (g, h) of $G \times H$ is adjacent to a vertex (g', h') of $G \times H$ if and only if we have
  - **either** $g = g'$ and $hh' \in \operatorname{E}(H)$,
  - **or** $h = h'$ and $gg' \in \operatorname{E}(G)$.

  (In particular, exactly one of the two equalities $g = g'$ and $h = h'$ has to hold when (g, h) is adjacent to (g', h').)

(a) Recall the n-dimensional cube graph Q_n defined for each $n \in \mathbb{N}$. (Its vertices are n-tuples $(a_1, a_2, \ldots, a_n) \in \{0, 1\}^n$, and two such vertices are adjacent if and only if

they differ in exactly one entry.) Prove that $Q_n \cong Q_{n-1} \times Q_1$ for each positive integer n. (Thus, Q_n can be obtained from Q_1 by repeatedly forming Cartesian products; i.e., it is a "Cartesian power" of Q_1.)
(b) Assume that each of the graphs G and H has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian path.
(c) Assume that both numbers $|\mathrm{V}(G)|$ and $|\mathrm{V}(H)|$ are > 1, and that at least one of them is even. Assume again that each of the graphs G and H has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian cycle.

1.2 Solution to parts (a) and (b)

Proof of Part (a). To show that $Q_n \cong Q_{n-1} \times Q_1$, I will first show that their vertex sets are the same, provided that we identify each pair $((x_1, x_2, \ldots, x_{n-1}), x_n) \in \{0,1\}^{n-1} \times \{0,1\}$ with the n-tuple $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$. Then, we will show that their edge sets are the same.

The equality of their vertex sets follows directly from the definition of $\mathrm{V}(Q_{n-1} \times Q_1)$:

$$\begin{aligned}\mathrm{V}(Q_{n-1} \times Q_1) &= \{0,1\}^{n-1} \times \{0,1\} \\ &= \{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0,1\} \text{ for } 1 \leq i \leq n-1, \text{ and } x_n \in \{0,1\}\} \\ &= \{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0,1\} \text{ for } 1 \leq i \leq n\} \\ &= \{0,1\}^n = \mathrm{V}(Q_n).\end{aligned}$$

To show the equality of the edge sets, I will first show that $\mathrm{E}(Q_{n-1} \times Q_1) \subseteq \mathrm{E}(Q_n)$, then that $\mathrm{E}(Q_n) \subseteq \mathrm{E}(Q_{n-1} \times Q_1)$. Let $\{(x_1, y_1), (x_2, y_2)\} \in \mathrm{E}(Q_{n-1} \times Q_1)$, where $x_1, x_2 \in \mathrm{V}(Q_{n-1})$ and $y_1, y_2 \in \mathrm{V}(Q_1)$. Note that since $\mathrm{V}(Q_{n-1} \times Q_1) = \mathrm{V}(Q_n)$, $(x_1, y_1) \in \mathrm{V}(Q_n)$ and $(x_2, y_2) \in \mathrm{V}(Q_n)$. By the definition of $\mathrm{E}(Q_{n-1} \times Q_1)$, there are two cases to consider:

- **Case 1:** $x_1 = x_2$ and $y_1 y_2 \in \mathrm{E}(Q_1)$. (This is the "either" case from the definition.) In this case, $x_1 = x_2 = (x_{1,1}, x_{1,2}, \ldots, x_{1,n-1})$. The graph Q_1 has only one edge: $\{0,1\}$, so $\{y_1, y_2\} = \{0,1\}$ and $y_1 \neq y_2$. Then $(x_1, y_1) = (x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, y_1)$ and $(x_2, y_2) = (x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, y_2)$ differ in only one entry (y_1 and y_2). Therefore, $\{(x_1, y_1), (x_2, y_2)\} \in \mathrm{E}(Q_n)$.

- **Case 2:** $y_1 = y_2$ and $x_1 x_2 \in \mathrm{E}(Q_{n-1})$. (This is the "or" case from the definition.) Since $x_1 x_2 \in \mathrm{E}(Q_{n-1})$, the first $n-1$ entries in (x_1, y_1) and (x_2, y_2) differ in exactly one entry. But $y_1 = y_2$, so (x_1, y_1) and (x_2, y_2) differ in exactly one entry as well. Therefore, $\{(x_1, y_1), (x_2, y_2)\} \in \mathrm{E}(Q_n)$.

Case 1 and Case 2 together show that $\mathrm{E}(Q_{n-1} \times Q_1) \subseteq \mathrm{E}(Q_n)$. It must now be shown that $\mathrm{E}(Q_n) \subseteq \mathrm{E}(Q_{n-1} \times Q_1)$. Let $\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n})\} \in \mathrm{E}(Q_n)$. Since $(x_{1,1}, x_{1,2}, \ldots, x_{1,n})$ and $(x_{2,1}, x_{2,2}, \ldots, x_{2,n})$ must differ in exactly one entry, there are two cases to consider.

- **Case 1:** $(x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}) = (x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})$ and $x_{1,n} \neq x_{2,n}$. This implies that $\{x_{1,n}, x_{2,n}\} = \{0, 1\} \in \mathrm{E}(Q_1)$. $(x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}) = (x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})$ and $\{x_{1,n}, x_{2,n}\} \in \mathrm{E}(Q_1)$ satisfy the "either" condition for
$$\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n})\} \in \mathrm{E}(Q_{n-1} \times Q_1).$$

- **Case 2:** $x_{1,n} = x_{2,n}$ and $(x_{1,1}, x_{1,2}, \ldots, x_{1,n-1})$ differs from $(x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})$ in exactly one entry. This implies that $\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})\} \in \mathrm{E}(Q_{n-1})$. Together with $x_{1,n} = x_{2,n}$, this satisfies the "or" condition for
$$\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n})\} \in \mathrm{E}(Q_{n-1} \times Q_1).$$

Case 1 and Case 2 together show that $\mathrm{E}(Q_n) \subseteq \mathrm{E}(Q_{n-1} \times Q_1)$. Therefore, $\mathrm{E}(Q_n) = \mathrm{E}(Q_{n-1} \times Q_1)$. Since $\mathrm{V}(Q_{n-1} \times Q_1) = \mathrm{V}(Q_n)$ and $\mathrm{E}(Q_n) = \mathrm{E}(Q_{n-1} \times Q_1)$, it follows that $Q_n \cong Q_{n-1} \times Q_1$. □

Proof of Part (b). Let $n = |\mathrm{V}(G)|$ and $m = |\mathrm{V}(H)|$. Since both G and H have a Hamiltonian path, there is a listing $(v_1, v_2, \ldots, v_n)$ of the vertices of G such that $v_i v_{i+1} \in \mathrm{E}(G)$ for all $1 \leq i \leq n-1$, and a listing $(w_1, w_2, \ldots, w_m)$ of the vertices of H such that $w_i w_{i+1} \in \mathrm{E}(H)$ for all $1 \leq i \leq m-1$. It follows from the definition of $\mathrm{E}(G \times H)$ that the following holds:

- $\{(v_i, w_j), (v_i, w_{j+1})\} \in \mathrm{E}(G \times H)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$, and
- $\{(v_i, w_j), (v_{i+1}, w_j)\} \in \mathrm{E}(G \times H)$ for all $1 \leq i \leq n-1$ and $1 \leq j \leq m$.

Thus we may construct a Hamiltonian path as below:

$$((v_1, w_1), (v_1, w_2), \ldots, (v_1, w_m), (v_2, w_m), (v_2, w_{m-1}), \ldots, (v_2, w_1), (v_3, w_1), (v_3, w_2), \ldots, (v_{n-1}, w_a), (v_n, w_a), \ldots, (v_n, w_b)),$$

where $a = 1$ if $n \equiv 1 \mod 2$ and $a = m$ if $n \equiv 0 \mod 2$; and $b = m$ if $n \equiv 1 \mod 2$ and $b = 1$ if $n \equiv 0 \mod 2$. It is easily verified that each consecutive pair of vertices is adjacent in $G \times H$, and that each of the $n \cdot m$ vertices of $G \times H$ appears exactly once. Indeed, this path fully traverses the $n \times m$ matrix M where the entry $m_{i,j} = (v_i, w_j)$, descending row by row in alternating (left/right) directions. □

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2 Exercise 2

2.1 Problem

Let n be a positive integer. Recall that K_n denotes the complete graph on n vertices. This is the graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $\mathcal{P}_2(V)$ (so that each two distinct vertices are connected). Find Eulerian circuits for the graphs K_3, K_5, and K_7.

2.2 Solution

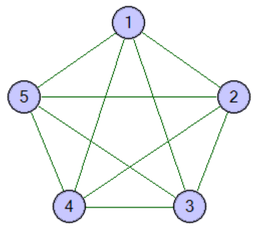
We shall represent walks as lists of edges, omitting the vertices.

- For K_3, the edge set contains only the three edges $\{1,2\}$, $\{2,3\}$, and $\{3,1\}$. The edges in this order are already an Eulerian circuit: $(\{1,2\},\{2,3\},\{3,1\})$.

- In the representation of the graph K_5 below, an Eulerian circuit can be created by starting at 1 and first following the edges clockwise around the outer pentagon back to 1, then following the edges of the inner pentagram back to 1:

$$(\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\},\\ \{1,3\},\{3,5\},\{5,2\},\{2,4\},\{4,1\}).$$

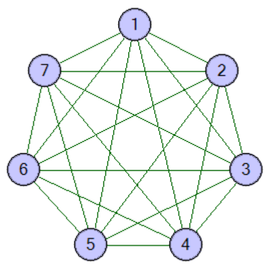
  Note that all $\binom{5}{2} = 10$ edges are included, with the first 5 edges $\{i,j\}$ having $(j-i) \equiv 1 \mod 5$ and the next 5 edges having $(j-i) \equiv 2 \mod 5$.



- In the representation of the graph K_7 below, an Eulerian circuit can be created by starting at 1 and first following the edges clockwise around the outer heptagon back to 1, then following the edges of the first inner heptagram clockwise by skipping 1 vertex each time back to 1, and finally following the other inner heptagram clockwise by skipping two vertices each time back to 1:

$$(\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,1\},\\ \{1,3\},\{3,5\},\{5,7\},\{7,2\},\{2,4\},\{4,6\},\{6,1\},\\ \{1,4\},\{4,7\},\{7,3\},\{3,6\},\{6,2\},\{2,5\},\{5,1\}).$$

  Note that all $\binom{7}{2} = 21$ edges are included, with the first 7 edges $\{i,j\}$ having $(j-i) \equiv 1 \mod 7$, the next 7 edges having $(j-i) \equiv 2 \mod 7$, and the final 7 edges having $(j-i) \equiv 3 \mod 7$.



4 Exercise 4

4.1 Problem

Let D be a digraph. Show that $\sum\limits_{v \in \operatorname{V}(D)} \deg^-(v) = \sum\limits_{v \in \operatorname{V}(D)} \deg^+(v)$.

4.2 Solution

Proof. Let $V = \operatorname{V}(D)$ and $A = \operatorname{A}(D)$. (We use the notation $\operatorname{A}(D)$ for the set of all arcs of D.)

Using the definition of $\deg^-(v)$, the sum on the left hand side can be written as

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} |\{a \in A \mid v \text{ is the target of } a\}|.$$

Next, using Proposition 0.3.a from homework set 1, the sum can be rewritten again as

$$\sum_{v \in V} |\{a \in A \mid v \text{ is the target of } a\}| = \sum_{v \in V} \sum_{u \in V} [(u, v) \in A].$$

Finally, the order of the summation can be flipped, and we can reverse the above process to arrive at the conclusion:

$$\begin{aligned}
\sum_{v \in V} \sum_{u \in V} [(u, v) \in A] &= \sum_{u \in V} \sum_{v \in V} [(u, v) \in A] \\
&= \sum_{u \in V} |\{a \in A \mid u \text{ is the source of } a\}| \\
&= \sum_{u \in V} \deg^+(u) = \sum_{v \in V} \deg^+(v).
\end{aligned}$$

□

5 Exercise 5

5.1 Problem

Let $D = (V, A)$ be a tournament. Set $n = |V|$ and $m = \sum\limits_{v \in V} \binom{\deg^+(v)}{2}$.

(a) Show that $m = \sum\limits_{v \in V} \binom{\deg^-(v)}{2}$.

(b) Show that the number of 3-cycles in D is $3 \cdot \left(\binom{n}{3} - m\right)$.

5.2 Solution

Proof of Part (a). To begin, note that for every $v \in V$, by the definition of a tournament we have that for each of the $n-1$ vertices $u \in V$ such that $u \neq v$, either $(u, v) \in A$ or $(v, u) \in A$ but not both. Thus we have for each $v \in V$ the equality $\deg^+(v) + \deg^-(v) = n - 1$, and therefore

$$\deg^+(v) = n - 1 - \deg^-(v). \tag{1}$$

Also, since there is exactly one arc between each pair of vertices, we have $|A| = \binom{n}{2}$. But $|A| = \sum\limits_{u \in V} \sum\limits_{v \in V} [(u, v) \in A] = \sum\limits_{v \in V} \deg^+(v)$, so we have

$$\sum_{v \in V} \deg^+(v) = \binom{n}{2} = \frac{n \cdot (n-1)}{2}. \tag{2}$$

Using (1), we can rewrite the definition of m as

$$m = \sum_{v \in V} \binom{n - 1 - \deg^-(v)}{2}.$$

Expanding the binomial coefficient and performing algebra:

$$\begin{aligned} m =& \sum_{v \in V} \frac{(n - 1 - \deg^-(v))(n - 2 - \deg^-(v))}{2} \\ =& \frac{1}{2} \sum_{v \in V} \left(\deg^-(v)^2 - \deg^-(v)\right) + \frac{1}{2} \sum_{v \in V} \left(4 \cdot \deg^-(v) - 2n \cdot \deg^-(v) + n^2 - 3n + 2\right) \\ =& \sum_{v \in V} \frac{\deg^-(v)(\deg^-(v) - 1)}{2} + (2 - n) \sum_{v \in V} \deg^-(v) + \frac{1}{2} \sum_{v \in V} \left(n^2 - 3n + 2\right). \end{aligned}$$

We can now make use of (2) and the fact that $|V| = n$ to evaluate the second and third sums:

$$\begin{aligned} =& \sum_{v \in V} \binom{\deg^-(v)}{2} + \frac{(2 - n)n(n-1)}{2} + \frac{n(n-1)(n-2)}{2} \\ =& \sum_{v \in V} \binom{\deg^-(v)}{2}. \end{aligned}$$

□

Proof of Part (b). Let

$$S = \{s \in \mathcal{P}_3(V) \mid \text{the sub-digraph } D\mid_s \text{ contains at least one 3-cycle}\} \qquad \text{and}$$
$$T = \{t \in \mathcal{P}_3(V) \mid \text{the sub-digraph } D\mid_t \text{ contains no 3-cycles}\}.$$

(Here, $D\mid_s$ denotes the sub-digraph of D obtained by removing all vertices that don't lie in s, and removing all arcs that don't connect two vertices in s.)

Consider an arbitrary $s \in S$. There exists a 3-cycle (s_1, s_2, s_3) in the sub-digraph $D\mid_s$, so (s_1, s_2), (s_2, s_3), and (s_3, s_1) all belong to A. But then (s_2, s_3, s_1) and (s_3, s_1, s_2) are also 3-cycles in D. On the other hand, since D is a tournament, none of (s_2, s_1), (s_3, s_2), or (s_1, s_3) belong to A. Hence none of (s_3, s_2, s_1), (s_2, s_1, s_3), or (s_1, s_3, s_2) are 3-cycles in D. Therefore, for each $s \in S$, the sub-digraph $D\mid_s$ contains exactly three 3-cycles. Hence, the total number of 3-cycles in D is $3|S|$ (because clearly, each 3-cycle in D must be contained in $D\mid_s$ for a unique $s \in S$). It thus remains to be shown that $|S| = \binom{n}{3} - m$.

The problem can be simplified further by noting that $|\mathcal{P}_3(V)| = \binom{n}{3}$. Since $|S| = |\mathcal{P}_3(V)| - |T|$, the goal is reduced to showing that $|T| = m$.

Consider an arbitrary $t = \{t_1, t_2, t_3\} \in T$. Since the sub-digraph $D\mid_t$ must contain exactly one arc between each pair of vertices, we have $\sum_{i=1}^{3} \deg^{-}_{D|_t}(t_i) = 3$.

Now, we can't have $\deg^{-}_{D|_t}(t_i) = 1$ for all $1 \leq i \leq 3$, or we would have a 3-cycle. Nor can we have $\deg^{-}_{D|_t}(t_i) = 3$ for any $1 \leq i \leq 3$, since this would contradict the definition of a tournament. Therefore we must have $\deg^{-}_{D|_t}(t_i) = 2$ for exactly one $1 \leq i \leq 3$. There are arcs from the other two elements of t having t_i as their target.

Hence, each subset $t \in T$ has exactly one element t_i satisfying $\deg^{-}_{D|_t}(t_i) = 2$, and there are arcs from the other two elements of t having t_i as their target. Consequently, each $t \in T$ gives rise to two arcs of D having a common target. Conversely, any two arcs of D having a common target are obtained from exactly one $t \in T$. To see this, pick any two arcs $(u, v) \in A$ and $(w, v) \in A$ with a common target v. The vertices of these arcs define a unique element of $\mathcal{P}_3(V)$, namely $\{u, v, w\}$. Clearly the sub-digraph $D\mid_{\{u,v,w\}}$ does not contain a 3-cycle, so $\{u, v, w\} \in T$.

Each choice of two arcs having a common target defines exactly one element of T, and each element of T contains exactly one such vertex. Hence, the elements in T can be counted by totaling the distinct (unordered) pairs of arcs in A with a common vertex as their target. But this is precisely what $m = \sum_{v \in V} \binom{\deg^{-}(v)}{2}$ counts (because we can count them by first choosing their common target v, and then we have $\binom{\deg^{-}(v)}{2}$ options for choosing the two arcs). So we have $|T| = m$, and therefore

$$\#\text{ 3-cycles in } D = 3(|\mathcal{P}_3(V)| - |T|) = 3\left(\binom{n}{3} - m\right).$$

□

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7 Exercise 7

7.1 Problem

Let D be any tournament. Prove that there is a sequence of 2-path reversal operations that transforms D into a transitive tournament.

7.2 Solution

Proof. Let $V = \operatorname{V}(D)$, $A = \operatorname{A}(D)$, and $n = |V|$. If D contains no 3-cycles, it is already transitive.

It suffices to show that as long as a 3-cycle exists, a 2-path reversal operation can be chosen that will decrease the number of 3-cycles. Assume that D contains at least one 3-cycle. By the results of Exercise 5 (b), the number of 3-cycles is given by

$$3\left(\binom{n}{3} - \sum_{v \in V} \binom{\deg^+(v)}{2}\right) = 3\left(\binom{n}{3} - \sum_{v \in V} \binom{\deg^-(v)}{2}\right)$$

(since Exercise 5 (a) yields $\sum\limits_{v\in V} \binom{\deg^+(v)}{2} = \sum\limits_{v\in V} \binom{\deg^-(v)}{2}$). Hence, in order to decrease the number of 3-cycles, we need to increase the sum $\sum\limits_{v\in V} \binom{\deg^-(v)}{2}$.

Pick any 3-cycle (u, v, w) in D. Consider two cases: either the indegrees of u, v, and w are all equal, or they are not all equal.

- **Case 1:** $\deg^-(u) = \deg^-(v) = \deg^-(w)$. Set $d = \deg^-(u)$. In this case, performing the 2-path reversal that replaces the arcs (u, v) and (v, w) with (v, u) and (w, v) will increase $\deg^-(u)$ by 1, decrease $\deg^-(w)$ by 1, and leave the indegrees of all other vertices unchanged.

  Let m be the value of the sum $\sum\limits_{v\in V} \binom{\deg^-(v)}{2}$ before the 2-path reversal, and m'' be the value after. Then we have:

  $$m'' = m + \left[\binom{d+1}{2} - \binom{d}{2}\right] + \left[\binom{d-1}{2} - \binom{d}{2}\right] = m + (d) + (1-d) = m + 1.$$

  Thus this 2-path reversal operation reduces the number of 3-cycles by 3 (since the number of 3-cycles is $3\left(\binom{n}{3} - \sum\limits_{v\in V} \binom{\deg^-(v)}{2}\right)$).

- **Case 2:** $\deg^-(u)$, $\deg^-(v)$, and $\deg^-(w)$ are not all equal. There are three 3-cycles containing the vertices u, v, and w. Since the indegrees are not all equal, at least one 3-cycle (x, y, z) among the three has $\deg^-(x) > \deg^-(z)$. Fix such a 3-cycle, and set $d = \deg^-(x)$ and $c = \deg^-(z)$. Performing the 2-path reversal operation that replaces the arcs (x, y) and (y, z) with (y, x) and (z, y) will increase $\deg^-(x)$ by 1, decrease $\deg^-(z)$ by 1, and leave the indegrees of all other vertices unchanged. As in Case 1, let m be the value of the sum $\sum_{v \in V} \binom{\deg^-(v)}{2}$ before the 2-path reversal, and m'' be the value after. We now have

$$m'' = m + \left[\binom{d+1}{2} - \binom{d}{2} \right] + \left[\binom{c-1}{2} - \binom{c}{2} \right] = m + (d) + (1 - c).$$

  Since $d > c$, this yields $m'' - m \geq 2$. Hence this 2-path reversal reduces the number of 3-cycles by at least 6 (since the number of 3-cycles is $3 \left(\binom{n}{3} - \sum_{v \in V} \binom{\deg^-(v)}{2} \right)$).

In either case, a 2-path reversal operation on the vertices of the 3-cycle can be chosen to reduce the total number of 3-cycles. Since there are a finite number of 3-cycles, a finite number of 2-path reversal operations will reduce the number of 3-cycles to 0. $\square$