Math 5707 Spring 2017 (Darij Grinberg): homework set 2
Please hand in solutions to FIVE of the seven problems.

See the lecture notes for relevant material. If you reference results from the lecture notes, please **mention the date and time** of the version of the notes you are using (as the numbering changes during updates).

Exercise 1. Let G and H be two simple graphs. The *Cartesian product* of G and H is a new simple graph, denoted $G \times H$, which is defined as follows:

- The vertex set $\operatorname{V}(G \times H)$ of $G \times H$ is the Cartesian product $\operatorname{V}(G) \times \operatorname{V}(H)$.
- A vertex (g, h) of $G \times H$ is adjacent to a vertex (g', h') of $G \times H$ if and only if we have
  - **either** $g = g'$ and $hh' \in \operatorname{E}(H)$,
  - **or** $h = h'$ and $gg' \in \operatorname{E}(G)$.

  (In particular, exactly one of the two equalities $g = g'$ and $h = h'$ has to hold when (g, h) is adjacent to (g', h').)

(a) Recall the n-dimensional cube graph Q_n defined for each $n \in \mathbb{N}$. (Its vertices are n-tuples $(a_1, a_2, \ldots, a_n) \in \{0, 1\}^n$, and two such vertices are adjacent if and only if they differ in exactly one entry.) Prove that $Q_n \cong Q_{n-1} \times Q_1$ for each positive integer n. (Thus, Q_n can be obtained from Q_1 by repeatedly forming Cartesian products; i.e., it is a "Cartesian power" of Q_1.)

(b) Assume that each of the graphs G and H has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian path.

(c) Assume that both numbers $|\operatorname{V}(G)|$ and $|\operatorname{V}(H)|$ are > 1, and that at least one of them is even. Assume again that each of the graphs G and H has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian cycle.

Exercise 2. Let n be a positive integer. Recall that K_n denotes the complete graph on n vertices. This is the graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $\mathcal{P}_2(V)$ (so each two distinct vertices are connected).

Find Eulerian circuits for the graphs K_3, K_5, and K_7.

Exercise 3. Let n be a positive integer, and K be a nonempty finite set. Let $k = |K|$. A *de Bruijn sequence* of order n on K means a list $(c_0, c_1, \ldots, c_{k^n-1})$ of k^n elements of K such that

(1) for each n-tuple $(a_1, a_2, \ldots, a_n) \in K^n$ of elements of K, there exists a **unique** $r \in \{0, 1, \ldots, k^n - 1\}$ such that $(a_1, a_2, \ldots, a_n) = (c_r, c_{r+1}, \ldots, c_{r+n-1})$.

Here, the indices are understood to be cyclic modulo k^n; that is, c_q (for $q \geq k^n$) is defined to be $c_{q\%k^n}$, where $q\%k^n$ denotes the remainder of q modulo k^n.

(Note that the condition (1) can be restated as follows: If we write the elements $c_0, c_1, \ldots, c_{k^n-1}$ on a circular necklace (in this order), so that the last element c_{k^n-1} is followed by the first one, then each n-tuple of elements of K appears as a string of n consecutive elements on the necklace, and the position at which it appears on the necklace is unique.)

For example, $(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) = (1, 1, 2, 2, 3, 3, 1, 3, 2)$ is a de Bruijn sequence of order 2 on the set $\{1, 2, 3\}$, because for each 2-tuple $(a_1, a_2) \in \{1, 2, 3\}^2$, there exists a unique $r \in \{0, 1, \ldots, 8\}$ such that $(a_1, a_2) = (c_r, c_{r+1})$. Namely:

$$\begin{aligned} &(1,1) = (c_0, c_1); \qquad &(1,2) = (c_1, c_2); \qquad &(1,3) = (c_6, c_7); \\ &(2,1) = (c_8, c_9); \qquad &(2,2) = (c_2, c_3); \qquad &(2,3) = (c_3, c_4); \\ &(3,1) = (c_5, c_6); \qquad &(3,2) = (c_7, c_8); \qquad &(3,3) = (c_4, c_5). \end{aligned}$$

Prove that there exists a de Bruijn sequence of order n on K (no matter what n and K are).

Hint: Let D be the digraph with vertex set K^{n-1} and an arc from $(a_1, a_2, \ldots, a_{n-1})$ to $(a_2, a_3, \ldots, a_n)$ for each $(a_1, a_2, \ldots, a_n) \in K^n$ (and no other arcs). Prove that D has an Eulerian circuit.

Recall that the *indegree* of a vertex v of a digraph $D = (V, A)$ is defined to be the number of all arcs $a \in A$ whose target is v. This indegree is denoted by $\deg^-(v)$ or by $\deg^-_D(v)$ (whenever the graph D is not clear from the context).

Likewise, the *outdegree* of a vertex v of a digraph $D = (V, A)$ is defined to be the number of all arcs $a \in A$ whose source is v. This outdegree is denoted by $\deg^+(v)$ or by $\deg^+_D(v)$ (whenever the graph D is not clear from the context).

Exercise 4. Let D be a digraph. Show that $\sum\limits_{v \in \operatorname{V}(D)} \deg^-(v) = \sum\limits_{v \in \operatorname{V}(D)} \deg^+(v)$.

The next few exercises are about *tournaments*. A *tournament* is a loopless[1] digraph $D = (V, A)$ with the following property: For any two distinct vertices $u \in V$ and $v \in V$, **precisely** one of the two pairs (u, v) and (v, u) belongs to A. (In other words, any two distinct vertices are connected by an arc in one direction, but not in both.)

A 3-*cycle* in a tournament $D = (V, A)$ means a triple (u, v, w) of vertices in V such that all three pairs (u, v), (v, w) and (w, u) belong to A.

Exercise 5. Let $D = (V, A)$ be a tournament. Set $n = |V|$ and $m = \sum\limits_{v \in V} \dbinom{\deg^-(v)}{2}$.

(a) Show that $m = \sum\limits_{v \in V} \dbinom{\deg^+(v)}{2}$.

[1] A digraph (V, A) is said to be *loopless* if it has no loops. (A *loop* means an arc of the form (v, v) for some $v \in V$.)

(b) Show that the number of 3-cycles in D is $3\left(\dbinom{n}{3}-m\right)$.

Exercise 6. If a tournament D has a 3-cycle (u,v,w), then we can define a new tournament $D'_{u,v,w}$ as follows: The vertices of $D'_{u,v,w}$ shall be the same as those of D. The arcs of $D'_{u,v,w}$ shall be the same as those of D, except that the three arcs (u,v), (v,w) and (w,u) are replaced by the three new arcs (v,u), (w,v) and (u,w). (Visually speaking, $D'_{u,v,w}$ is obtained from D by turning the arrows on the arcs (u,v), (v,w) and (w,u) around.) We say that the new tournament $D'_{u,v,w}$ is obtained from the old tournament D by a 3-*cycle reversal operation*.

Now, let V be a finite set, and let E and F be two tournaments with vertex set V. Prove that F can be obtained from E by a sequence of 3-cycle reversal operations if and only if each $v \in V$ satisfies $\deg_E^-(v) = \deg_F^-(v)$. (Note that a sequence may be empty, which allows handling the case $E = F$ even if E has no 3-cycles to reverse.)

A tournament $D = (V, A)$ is called *transitive* if it has no 3-cycles.

Exercise 7. If a tournament $D = (V, A)$ has three distinct vertices u, v and w satisfying $(u,v) \in A$ and $(v,w) \in A$, then we can define a new tournament $D''_{u,v,w}$ as follows: The vertices of $D''_{u,v,w}$ shall be the same as those of D. The arcs of $D''_{u,v,w}$ shall be the same as those of D, except that the two arcs (u,v) and (v,w) are replaced by the two new arcs (v,u) and (w,v). We say that the new tournament $D''_{u,v,w}$ is obtained from the old tournament D by a 2-*path reversal operation*.

Let D be any tournament. Prove that there is a sequence of 2-path reversal operations that transforms D into a transitive tournament.