Mathematics 5707 Homework 1

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Exercise 1. Let G be a simple graph. A triangle in G means a set $\{a, b, c\}$ of three distinct vertices a, b, and c of G such that ab, bc, and ca are edges of G. An antitriangle in G means a set $\{a, b, c\}$ of three distinct vertices a, b, and c of G such that none of ab, bc, and ca is an edge of G. A triangle-or-antitriangle in G is a set that is either a triangle or an antitriangle.

(a) Assume that $|\mathbf{V}(G)| \geq 6$. Prove that G has at least two triangle-or-anti-triangles.

Proof: To begin, assume, without loss of generality, that $|\mathbf{V}(G)| = 6$, and note that there are $\binom{6}{3} = 20$ unordered triples of vertices in G. We wish to count the number of ordered triples (a, b, c) such that $ab \in \mathbf{E}(G)$ and $bc \notin \mathbf{E}(G)$. Observe that each vertex v has $\deg v = 0, 1, 2, 3, 4$, or 5. If $\deg v$ is 0 or 5, then v is the second vertex in 0 such triples. If $\deg v$ is 1 or 4, then v is the second vertex in 4 such triples. If $\deg v$ is 2 or 3, then v is the second vertex in 6 such triples. Thus, there are at most 36 such ordered triples. Note that each unordered triple of vertices that does not form a triangle-or-antitriangle corresponds to two such ordered triples, so at most 18 unordered triples of vertices do not form triangle-or-antitriangles, so there are at least two triangle-or-antitriangles in G. ■

(b) Assume that $|\mathbf{V}(G)| = m + 6$ for some $m \in \mathbb{N}$. Prove that G has at least $m + 1$ triangle-or-anti-triangles.

Proof: Let $\mathbf{V}(G) = \{v_1, v_2, ..., v_6, u_1, ..., u_m\}$. Now, take $\{v_1, ..., v_6\}$. By the above, we can find at least one triangle-or-antitriangle among these six vertices. Suppose, without loss of generality, that this triangle-or-antitriangle includes v_1. Then, we take the vertices $\{v_2, ..., v_6, u_1\}$. Again, there is at least one triangle-or-antitriangle among these six vertices, and since v_1 is not included, this triangle-or-antitriangle must be distinct from the one found previously. By repeating this process of finding a triangle-or-antitriangle, removing one of its vertices, and replacing it by one of the u_k, we find at least one new triangle-or-antitriangle for each of $\{u_1, ..., u_m\}$, plus one in which none of the u_k were used.

Thus, we find $m+1$ triangle-or-antitriangles. ■

Exercise 2. Let G be a simple graph. Let $n = |\mathbf{V}(G)|$ be the number of vertices of G. Assume that $|\mathbf{E}(G)| < \frac{n(n-2)}{4}$. Prove that there exist three distinct vertices a, b, and c of G such that none of ab, bc, and ca are edges of G.

Proof: Suppose that for all distinct $a, b, c \in \mathbf{V}(G)$, at least one of $ab, bc, ca \in \mathbf{E}(G)$. Then the complement $\overline{G}$ of G is a graph with no triangles. We know $\left|\mathbf{E}(\overline{G})\right| = \frac{n(n-1)}{2} - |\mathbf{E}(G)|$, so that $|\mathbf{E}(G)| = \frac{n(n-1)}{2} - \left|\mathbf{E}(\overline{G})\right|$. But by Mantel's theorem we know $\left|\mathbf{E}(\overline{G})\right| \leq \frac{n^2}{4}$ (since $\overline{G}$ has no triangles), and thus $|\mathbf{E}(G)| = \frac{n(n-1)}{2} - \left|\mathbf{E}(\overline{G})\right| \geq \frac{n(n-1)}{2} - \frac{n^2}{4} = \frac{n(n-2)}{4}$. Thus, if $|\mathbf{E}(G)| < \frac{n(n-2)}{4}$, then there exist $a, b, c \in \mathbf{V}(G)$ such that $ab, bc, ca \notin \mathbf{E}(G)$. ■

Exercise 3. Let G be a simple graph. Let w be a path in G. Prove that the edges of w are distinct.

Proof: Let $\{v_0, v_1, ..., v_k\}$ be the vertices of w. Since w is a path, its vertices are distinct. Now, the edges of w are $\{v_i v_{i+1} \mid 0 \leq i \leq k-1\}$. Note that two edges pq and rs are the same only when $p = r$ and $q = s$ or $p = s$ and $q = r$, i.e. if they connect the same pair of vertices. Now, each vertex v_i in w is connected to at most two edges in w, but these two edges must be distinct because the vertices v_{i-1} and v_{i+1} are distinct. ■

Exercise 4. Let $n \in \mathbb{N}$. What is the smallest possible size of a dominating set of the cycle graph C_{3n}?

The smallest possible size of a dominating set of C_{3n} is n.
Proof: Let the vertices of C_{3n} be $\{v_1, v_2, ..., v_{3n}\}$. We observe that since each vertex in C_{3n} has 2 neighbors, no three consecutive vertices can be excluded from a dominating set. Then, if we pick vertices such that every third vertex is in our dominating set, we have the set $\{v_3, v_6, ..., v_{3n}\}$, which has n vertices. ■

Exercise 5. Proposition 0.2 (a) If $\mathcal{A}$ and $\mathcal{B}$ are two equivalent logical statements, then $[\mathcal{A}] = [\mathcal{B}]$.

(b) If $\mathcal{A}$ is any logical statement, then $[\text{not}\,\mathcal{A}] = 1 - [\mathcal{A}]$.

(c) If $\mathcal{A}$ and $\mathcal{B}$ are two logical statements, then $[\mathcal{A} \wedge \mathcal{B}] = [\mathcal{A}][\mathcal{B}]$.

(d) If $\mathcal{A}$ and $\mathcal{B}$ are two logical statements, then $[\mathcal{A} \vee \mathcal{B}] = [\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}]$.

Proposition 0.3 Let P be a finite set. Let Q be a subset of P.

(a) Then,

$$|Q| = \sum_{p \in P} [p \in Q].$$

(b) For each $p \in P$, let a_p be a number. Then,

$$\sum_{p \in P} [p \in Q] a_p = \sum_{p \in Q} a_p.$$

(c) For each $p \in P$, let a_p be a number. Let $q \in P$. Then,

$$\sum_{p \in P} [p = q] a_p = a_q.$$

(a) Prove Proposition 0.2.

Proof of (a): If $\mathcal{A}$ and $\mathcal{B}$ are equivalent logical statements, then $\mathcal{A}$ is true if and only if $\mathcal{B}$ is true. Thus, $[\mathcal{A}] = [\mathcal{B}]$. ■

Proof of (b): If $[\mathcal{A}] = 1$, then $[\text{not}\,\mathcal{A}] = 0 = 1 - 1$. If $[\mathcal{A}] = 0$, then $[\text{not}\,\mathcal{A}] = 1 = 1 - 0$. ■

Proof of (c): If $[\mathcal{A}] = [\mathcal{B}] = 1$, $[\mathcal{A} \wedge \mathcal{B}] = 1 = [\mathcal{A}][\mathcal{B}]$. If $[\mathcal{A}] = 0$ or $[\mathcal{B}] = 0$, then $[\mathcal{A} \wedge \mathcal{B}] = 0 {=} [\mathcal{A}][\mathcal{B}]$. ■

Proof of (d): If $[\mathcal{A}] = [\mathcal{B}] = 0$, $[\mathcal{A} \vee \mathcal{B}] = 0 = [\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}]$. If $[\mathcal{A}] = 1$ or $[\mathcal{B}] = 1$, then $[\mathcal{A} \vee \mathcal{B}] = 1 = [\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}]$. ■

(b) Prove Proposition 0.3.

Proof of (a):

$$\sum_{p \in P} [p \in Q] = \sum_{p \in Q} [p \in Q] + \sum_{p \in P \setminus Q} [p \in Q] = \sum_{p \in Q} 1 + \sum_{p \in P \setminus Q} 0 = \sum_{p \in Q} 1 = |Q|\,. \blacksquare$$

Proof of (b):

$$\sum_{p \in P} [p \in Q] a_p = \sum_{p \in Q} [p \in Q] a_p + \sum_{p \in P \setminus Q} [p \in Q] a_p = \sum_{p \in Q} a_p. \blacksquare$$

Proof of (c):

$$\sum_{p \in P} [p = q] a_p = \sum_{p \neq q} 0 + \sum_{p = q} a_p = a_q. \blacksquare$$

(c) Now, let G be a simple graph. Prove that

$$\deg v = \sum_{u \in \mathbf{V}(G)} [uv \in \mathbf{E}(G)]$$

for each vertex v of G.

Proof: Let $A \subset \mathbf{V}(G)$ be the set of neighbors of v. Then,

$$\deg v = |A| = \sum_{u \in \mathbf{V}(G)} [u \in A].$$

Now, $u \in A$ is equivalent to $uv \in \mathbf{E}(G)$, so

$$\sum_{u \in \mathbf{V}(G)} [u \in A] = \sum_{u \in \mathbf{V}(G)} [uv \in \mathbf{E}(G)]. \blacksquare$$

(d) Prove that

$$2\,|\mathbf{E}(G)| = \sum_{u \in \mathbf{V}(G)} \sum_{v \in \mathbf{V}(G)} [uv \in \mathbf{E}(G)].$$

Proof:

$$2\,|\mathbf{E}(G)| = \sum_{v \in \mathbf{V}(G)} \deg v = \sum_{v \in \mathbf{V}(G)} \sum_{u \in \mathbf{V}(G)} [uv \in \mathbf{E}(G)]. \blacksquare$$

Exercise 6. Let k be a positive integer. Let G be a graph. A subset U of $\mathbf{V}(G)$ will be called k-path-dominating if for every $v \in \mathbf{V}(G)$, there exists a path of length $\leq k$ from v to some element of U. Prove that the number of all k-path-dominating subsets of $\mathbf{V}(G)$ is odd.

Proof: Consider the case of the 1-path-dominating subsets. As was proven by Brouwer, the number of such subsets is odd in any graph. Now, construct the graph G_k by adding to G edges between any two vertices that are connected by a path of length $\leq k$ in G. Then, a dominating set of G_k is a k-path-dominating subset of G, and G_k must have an odd number of dominating sets. $\blacksquare$

Exercise 7. Let G be a simple graph with $\mathbf{V}(G) \neq \emptyset$. Show that the following two statements are equivalent:

Statement 1: The graph G is connected.

Statement 2: For every two nonempty subsets A and B of $\mathbf{V}(G)$ satisfying $A \cap B = \emptyset$ and $A \cup B = \mathbf{V}(G)$, there exist $a \in A$ and $b \in B$ such that $ab \in \mathbf{E}(G)$.

Proof: First, we prove that Statement 1 implies Statement 2. Assume G is connected. Since G is connected, there exists a path between any pair of vertices $\alpha, \beta \in \mathbf{V}(G)$. Let such a path be $(\alpha, v_1, ..., v_k, \beta)$. Without loss of generality, suppose $\mathbf{V}(G)$ is divided into A and B such that $\alpha \in A$ and $\beta \in B$. Then, since the above path begins with a vertex in A and ends with a vertex in B, it must have at least one edge connecting some $a \in A$ and $b \in B$. Now, to show Statement 2 implies statement 1, suppose G is not connected, and that for every two nonempty subsets A and B of $\mathbf{V}(G)$ satisfying $A \cap B = \emptyset$ and $A \cup B = \mathbf{V}(G)$, there exist $a \in A$ and $b \in B$ such that $ab \in E(G)$. Since G is not connected, there exist vertices α and β such that no path exists from α to β. Then, we attempt to construct the sets A and B by defining $A = \{v \in \mathbf{V}(G) \mid \text{ a path from } \alpha \text{ to } v \text{ exists}\}$ and $B = \mathbf{V}(G) \setminus A$. Now, $\beta \in B$ since no path connects α to β, but since Statement 2 was assumed, there is an edge connecting β to a vertex in A, a contradiction. Hence, the above statements are equivalent. ■

Exercise 8. Let V be a nonempty finite set. Let G and H be two simple graphs such that $\mathbf{V}(G) = \mathbf{V}(H) = V$. Assume that for each $u, v \in V$,there exists a path from u to v in G or a path from u to v in H. Prove that at least one of the graphs G and H is connected.

Proof: Without loss of generality, assume G is not connected. Then, fix a vertex $u \in V$. We can divide V into two nonempty subsets: $A = \{v \in \mathbf{V}(G) \mid \text{ a path from } u \text{ to } v \text{ exists}\}$ and $B = V \setminus A$. Now, since no paths connecting elements of A to elements of B exist in G, for all $a \in A$, $b \in B$ a path from a to b exists in H. Then, for any $a_1, a_2 \in A$ (or $b_1, b_2 \in B$), a path from a_1 to a_2 (b_1 to b_2) exists in H since such a path can be constructed from the paths connecting a_1 and a_2 to any element of B (b_1 and b_2 to any element of A). Thus, H is connected. ■

Exercise 9. Let $G = (V, E)$ be a simple graph. The complement graph $\overline{G}$ of G is defined to be the simple graph $(V, \mathcal{P}_2\ (V) \setminus E)$. (Thus, two vertices u and v are adjacent in $\overline{G}$ if and only if they are not adjacent in G.) Prove that at least one of the following statements holds:

Statement 1: For each $u \in V$ and $v \in V$, there exists a path from u to v in G of length ≤ 3.

Statement 2: For each $u \in V$ and $v \in V$, there exists a path from u to v in $\overline{G}$ of length ≤ 2.

Proof: It is sufficient to show that when Statement 1 is false, Statement 2 holds. Thus, assume there exist $u, v \in V$ such that there is no path from u to v in G of length ≤ 3. Then, u and v are adjacent in $\overline{G}$. We must show that for each pair (a, b) of vertices, there exists a path of length ≤ 2 between a and b in $\overline{G}$. If the two vertices are not adjacent in G, this is trivial (since they are then adjacent in $\overline{G}$). Now, let $a, b \in V$ be a pair of adjacent vertices in G. Then, none of the paths[1] $(u, a, b, v), (u, b, a, v), (u, a, v)$, and (u, b, v) exist in G (by our assumption on u and v). Hence, we can assume without loss of generality that $ua, ub \notin \mathbf{E}(G)$, which implies $ua, ub \in \mathbf{E}(\overline{G})$. Hence, the path (a, u, b) of length 2 exists in $\overline{G}$. Therefore, Statement 2 holds whenever Statement 1 is false, so at least one of the two statements holds. ■

[1]Some of the vertices u, a, b, v might coincide. In this case, you should ignore them. For instance, you should read the path (u, a, b, v) as (u, a, v) in the case when $b = v$.