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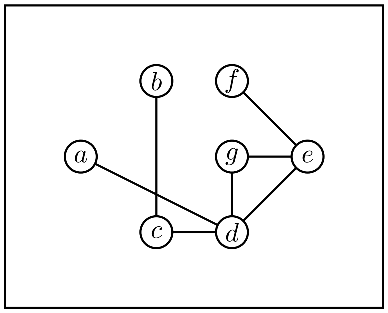
MATH 5707
Problem Set 1
2017 February 01 [1]

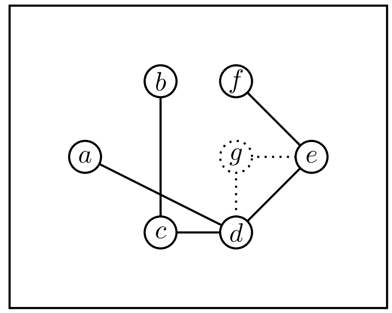
All mentioned theorems/propositions from the lecture notes are taken from the 2017-05-27 version.

Exercise 1

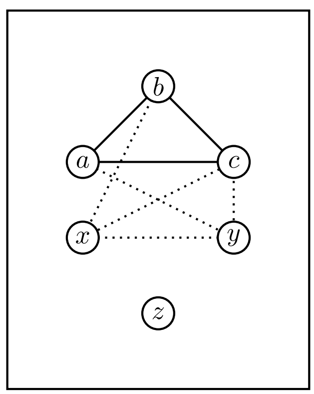
(a) It is sufficient to prove that there are at least two triangle-or-anti-triangles for $|V(G)| = 6$, because any graph with more than 6 vertices can be viewed as a graph with 6 vertices by removing the extra vertices and their edges. For example,



becomes



(the dotted vertex and edges in the second picture are understood to be absent) when we remove the vertex g. According to Proposition 2.4.1 from the notes, G already has at least one triangle-or-anti-triangle. Let this triangle-or-anti-triangle be a triangle with vertices a, b, and c. Consider the other three vertices x, y, and z. If xy, yz, and zx are all edges, then xyz is a triangle and we're done. So, assume that at least one edge, xy is a non-edge. Either two out of the three xa, xb, xc are edges, or two out of the three are non-edges. If two out of the three are edges, then we're done. So assume that two of the three edges are non-edges. Similarly, assume that two out of the three edges ya, yb, and yc are non-edges. By the pigeonhole principle, either xa and ya, xb and yb, or xc and yc are both non-edges, which forms an anti-triangle with xy.



An analogous argument for when abc is an anti-triangle comes to a similar conclusion, replacing edge with non-edge and etc.

(b) When $m = 0$, $|(V(G))| = 6$, so by Proposition 2.4.1 from the notes, G has one triangle-or-anti-triangle.

Assume there are $k+1$ triangle-or-anti-triangles when $m = k$.

Suppose there is a graph H where $|V(H)| = (k+1)+6$. According to Proposition 2.4.1 from the notes, this graph must have at least one triangle-or-anti-triangle with vertices a, b, and c. Ignore vertex a, so that you're viewing a graph with k vertices. This graph must have $k+1$ triangle-or-anti-triangles, none of which are abc because we have ignored a. Thus, abc plus these $k+1$ triangle-or-anti-triangles means that H has $(k+1)+1$ triangle-or-anti-triangles total.

Thus, by induction, G has at least $m+1$ triangle-or-anti-triangles.

Exercise 2.

A graph $G = (V, E)$ has $|V(G)| = n$. The most number of edges G could have is $\binom{n}{2}$, or $\frac{n(n-1)}{2}$. If $|E(G)| < \frac{n(n-2)}{4}$, the graph G has over $\frac{n(n-1)}{2} - \frac{n(n-2)}{4} = \frac{2n^2-2n}{4} - \frac{n^2-2n}{4} = \frac{n^2}{4}$ non-edges. Mantel's theorem (Theorem 2.5.10 in

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the notes) states that a simple graph with more than $\frac{n^2}{4}$ edges must have $a, b, c \in G$ such that ab, bc, and ca are edges. As such, it should still be true if we consider those $\frac{n^2}{4}$ edges as non-edges, and ab, bc, and ca as non-edges. Since we have more than $\frac{n^2}{4}$ non-edges, there must be $a, b, c \in G$ such that ab, bc, cb are not edges.

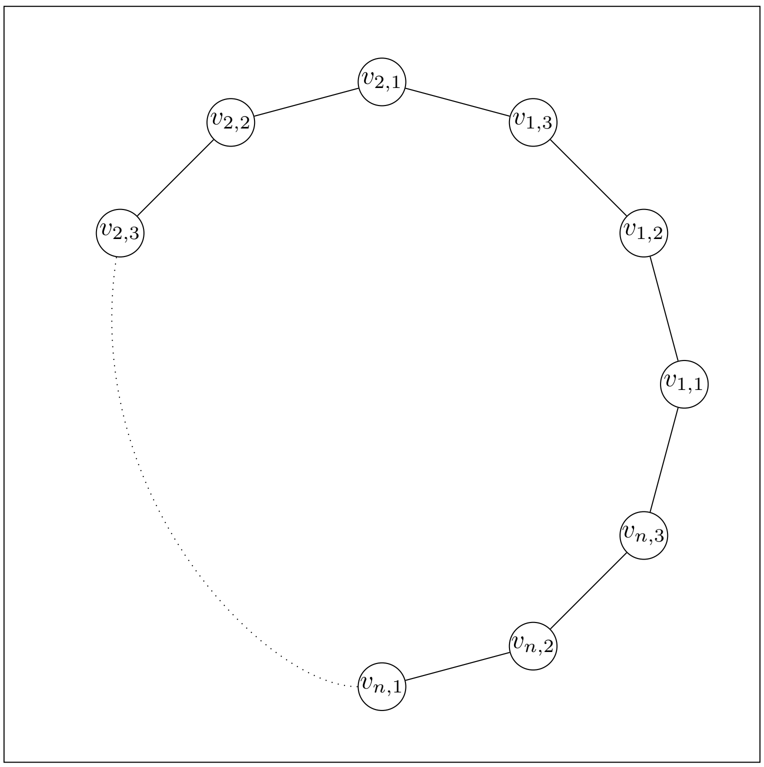
Exercise 3.

By definition, $\mathbf{w}$ in G has vertices $v_0, v_1, v_2....v_k$ such that for $0 \leq i, j \leq k$, $v_i \neq v_j$. Let $0 \leq n, m \leq k$, and $\mathbf{w}$ be $(v_0, v_1, ...v_n, v_{n+1}, ...v_m, v_{m+1}...v_k)$. If $\mathbf{w}$ has a repeated ege, then $\{v_n, v_{n+1}\} = \{v_m, v_{m+1}\}$ for some n, m. However, $v_i \neq v_j$, so $v_n \neq v_m \neq v_{m+1}$. Thus, each edge must be distinct.

Exercise 4.

A cycle is a walk $\mathbf{w}$ with vertices $(v_0, v_1, ..., v_k)$ and $v_k = v_0$ but for $0 \leq i, j < k, v_i \neq v_j$.

Label C_{3n} as follows:



Thus, C_{3n} has edges $v_{1,1}v_{1,2}, v_{1,2}v_{1,3}, \ldots, v_{n,2}v_{n,3}, v_{n,3}v_{1,1}$.

Say we're constructing a dominating set U. Each vertex v of C_{3n} has degree exactly 2. So if we choose a vertex v to be part of U, its each of its two neighbors satisfies the conditions that it has a neighbor in U.

So at most, choosing one vertex v accounts for 3 vertices: v itself, which is in U, and its 2 neighbors.

So the minimum number of vertices needed to dominate C_{3n} is $|V(C_{3n})|/3 = 3n/3 = n$.

For example, choose all the $v_{k,2}$ vertices to form U.

Exercise 5.

Proposition 0.2

(i) If $\mathcal{A}$ and $\mathcal{B}$ are true, $[\mathcal{A}] = 1, [\mathcal{B}] = 1$.
If $\mathcal{A}$ and $\mathcal{B}$ are false, $[\mathcal{A}] = 0, [\mathcal{B}] = 0$.

In either case, we obtain $[\mathcal{A}] = [\mathcal{B}]$.

(ii) If $\mathcal{A}$ is true, then $[\mathcal{A}] = 1$, $[\textit{not } \mathcal{A}] = 0$, and $1 - [\mathcal{A}] = 0$.
If $\mathcal{A}$ is false, then $[\mathcal{A}] = 0$, $[\textit{not } \mathcal{A}] = 1$, and $1 - [\mathcal{A}] = 1$.

In either case, we see that $[\textit{not } \mathcal{A}] = 1 - [\mathcal{A}]$.

(iii) If both $\mathcal{A}$ and $\mathcal{B}$ are true, then $[\mathcal{A}] = 1, [\mathcal{B}] = 1, [\mathcal{A} \wedge \mathcal{B}] = 1$, and $[\mathcal{A}][\mathcal{B}] = 1$.
If both $\mathcal{A}$ and $\mathcal{B}$ are false, then $[\mathcal{A}] = 0, [\mathcal{B}] = 0, [\mathcal{A} \wedge \mathcal{B}] = 0$, and $[\mathcal{A}][\mathcal{B}] = 0$.
If $\mathcal{A}$ is true and $\mathcal{B}$ is false, then $[\mathcal{A}] = 1, [\mathcal{B}] = 0, [\mathcal{A} \wedge \mathcal{B}] = 0$, and $[\mathcal{A}][\mathcal{B}] = 0$.
If $\mathcal{A}$ is false and $\mathcal{B}$ is true, then $[\mathcal{A}] = 0, [\mathcal{B}] = 1, [\mathcal{A} \wedge \mathcal{B}] = 0$, and $[\mathcal{A}][\mathcal{B}] = 0$.
Thus, $[\mathcal{A} \wedge \mathcal{B}] = [\mathcal{A}][\mathcal{B}]$.

(iv) If both $\mathcal{A}$ and $\mathcal{B}$ are true, then $[\mathcal{A}] = 1, [\mathcal{B}] = 1, [\mathcal{A} \vee \mathcal{B}] = 1$, and $[\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}] = 1$.
If both $\mathcal{A}$ and $\mathcal{B}$ are false, then $[\mathcal{A}] = 0, [\mathcal{B}] = 0, [\mathcal{A} \vee \mathcal{B}] = 0$, and $[\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}] = 0$.
If $\mathcal{A}$ is true and $\mathcal{B}$ is false, then $[\mathcal{A}] = 1, [\mathcal{B}] = 0, [\mathcal{A} \vee \mathcal{B}] = 1$, and $[\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}] = 1$.
If $\mathcal{A}$ is false and $\mathcal{B}$ is true, then $[\mathcal{A}] = 0, [\mathcal{B}] = 1, [\mathcal{A} \vee \mathcal{B}] = 1$, and $[\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}] = 1$.
Thus, $[\mathcal{A} \vee \mathcal{B}] = [\mathcal{A}][\mathcal{B}][\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}]$.

Proposition 0.3

(i) $|Q|$ is equivalent to the number of elements of P are in Q. Thus, each p is in Q gives the statement $p \in Q$ is true, with a truth value of 1. if p is not in Q, then $p \in Q$ is 0. So the sum $\sum_{p \in P} [p \in Q]$ accurately gives $|Q|$.

(ii) $\sum_{p \in P} [p \in Q] a_p = (0) a_p = 0$ if $p \notin Q$. Thus, if we rewrite the sum by skipping over all $p \notin Q$, we get $\sum_{p \in Q} [p \in Q] a_p$. $[p \in Q] = 1$ for all $p \in Q$. Thus, $\sum_{p \in Q} [p \in Q] a_p = \sum_{p \in Q} a_p$.

(iii) $[p = q] = 0$ when $p \neq q$. Thus, $[p = q] a_p = 0$ for all $p \in P \neq q$. When $p = q, [p = q] = 1$, so $[p = q] a_q = a_q$.

Then, $\sum_{p \in P} [p = q] a_p = 0 \cdot a_{p_1} + 0 \cdot a_{p_2} + ... + 1 \cdot a_q + ... + 0 \cdot a_p = a_q$.

(c) By Definition 2.5.1, we have $\deg v = |\{u \in V \mid uv \in E\}|$.

$[uv \in E] = 0$ if $uv \notin E$, and 1 if $uv \in G$. Thus, for every $uv \notin E$, the sum increases by nothing. For every $uv \in E$, the sum increases by 1. Thus, by definition, the $\deg v \ = \sum_{u \in V} [uv \in E]$.

(d) According to result (c),

$$\sum_{u \in V} \sum_{v \in V} [uv \in E] = \sum_{u \in V} \deg u.$$

Think of $\deg u$ as the number of edges that have u as an endpoint. That is, if $\deg u \ = 5$, then u is the endpoint of 5 edges. Then the sum of all the degrees of u are equal to twice the number of edges because each edge has two endpoints, implying that the sum of all the degrees of u double counts the edges. So,

$$2\,|E| = \sum_{u \in V} \deg u = \sum_{u \in V} \sum_{v \in V} [uv \in E].$$

Exercise 6.

A k-path-dominating subset U of $V(G)$ can be thought of as a 1-path-dominating subset of $V(G)$ by drawing an edge between all $u, v \in V$ such that the smallest path $\mathbf{w}$ from $u \to v$ has length k. Since 1-path-dominating subsets have an odd number of subsets according to Brouwer's theorem, so do k-path-dominating subsets which can be thought of as 1-path-dominating subsets!

Exercise 7.

Write our graph G as (V, E).

First let's show that Statement 1 $\Rightarrow$ Statement 2.

Assume that Statement 1 holds.

The definition of a graph G being connected is that for each $u, v \in V$, there exists a path from u to v. Suppose V was divided into nonempty subsets A, B such that for all $a \in A$ and $b \in B$, we have $ab \notin E$. Then, pick any $u \in A$ and $v \in B$. Since G is connected, there must exist a path from u to v. This path must at some point cross over from A into B (since it starts in A and ends in B). This means that there is an edge between a vertex in A and a vertex in B. This contradicts the fact that for all $a \in A$ and $b \in B$, we have $ab \notin E$.

Next let's show that Statement 2 $\Rightarrow$ Statement 1.

Assume that Statement 2 holds. Let $n = |V|$. A subset S of V shall be called *connected* if for any two vertices $u, v \in S$, there exists a path from u to v that uses only vertices in S.

We claim that for each $k \in \{1, 2, \ldots, n\}$, there exists a connected k-element subset S of V.

Indeed, we shall prove this claim by induction on k. The base case $k = 1$ is obvious (just pick any 1-element subset). In the induction step, we fix some $k \in \{1, 2, \ldots, n-1\}$ and assume that there exists a connected k-element subset A of V. We must then show that there exists a connected $(k+1)$-element subset A' of V.

Indeed, set $B = V \setminus A$. Then, A and B are subsets of V satisfying $A \cap B = \varnothing$ and $A \cup B = V$, and furthermore are nonempty (since $|A| = k \in \{1, 2, \ldots, n-1\}$). Hence, according to Statement 2, there exist $a \in A$ and $b \in B$ such that $ab \in E$. Consider these a and b. Then, it is easy to see that the $(k+1)$-element subset $A \cup \{b\}$ of V is also connected (indeed, the new vertex b is connected by an edge to $a \in A$, and thus also connected by paths to all other elements of A, since A is a connected subset). Hence, there exists a connected $(k+1)$-element subset A' of V (namely, $A \cup \{b\}$). This completes the induction step.

Thus, our claim is proven. Applying it to $k = n$, we conclude that there exists a connected n-element subset S of V. This subset must be the whole V. Thus, V is connected. In other words, the graph G is connected.

Exercise 8.

Suppose that G is not connected, that is, there is some $u_0, v_0 \in V$ such that there is no path from u_0 to v_0 in G. Thus, a path from u_0 to v_0 exists in H (by assumption).

Let a be any vertex. We claim that a path $u_0 \to a$ exists in H.

Indeed, two cases are possible:

- *Case 1:* There exists a path $a \to v_0$ in G.
- *Case 2:* There is no path $a \to v_0$ in G.

Consider Case 1 first. Here, a path $a \to v_0$ exists in G. Thus, a path $u_0 \to a$ does not. (Otherwise, a walk $(u_0, ..., a, ..., v_0)$ could be constructed from the paths $u_0 \to a$ and $a \to v_0$, which would imply that there is a path from u_0 to v_0, which we have said cannot exist.) Therefore, a path $u_0 \to a$ exists in H.

Now, consider Case 2. In this case, a path $a \to v_0$ does not exist in G. Hence, a path $a \to v_0$ must exist in H. Thus, there is a walk $(u_0, ..., v_0, ..., a)$ in H (constructed from the paths $u_0 \to v_0$ and $a \to v_0$), which means that there is a path $u_0 \to a$ in H.

Thus, in either case, a path $u_0 \to a$ exists in H.

We have proven this for each vertex a. Hence, for any two vertices a_1 and a_2, there exist paths $u_0 \to a_1$ and $u_0 \to a_2$ in H. Therefore, for any two vertices a_1 and a_2, there exist a walk $a_1 \to a_2$ in H (indeed, such a walk can be constructed by joining a path $u_0 \to a_1$ with a path $u_0 \to a_2$), and hence also a path $a_1 \to a_2$ in H. Thus, H is connected.

Exercise 9.

Let Statement 1 not hold for G, i.e. not all vertices $u, v \in G$ have a path $\mathbf{w}$ from $u \to v$ such that $|\mathbf{w}| \leq 3$.

What could the graph $\overline{G}$ look like?

For every $u, v \in G$ there are several cases.

One, u and v are not adjacent in G. Thus, $uv \notin G$ and $uv \in \overline{G}$. So path $\mathbf{w}'(u, v) \in \overline{G}$ exists, and $|\mathbf{w}'| = 1 \leq 2$.

Two, u and v are adjacent in G.

Let $A = \{a \mid au \in G \wedge a \neq v\}$ and $B = \{b \mid bv \in G \wedge b \notin A \wedge b \neq u\}$.

All $a \in A$ are connected by a a path (a, u, v, b) of length 3 to all $b \in B$, by a path (a, u, v) of length 2 to v, and by a path (a, u) of length 1 to u.

Similarly, all $b \in B$ are connected by a a path (b, v, u, a) of length 3 to all $a \in A$, by a path (b, v, u) of length 2 to u, and by a path (b, v) of length 1 to v.

Thus, there must be a vertex w s.t. $wu, wv \notin G$, otherwise Statement 1 would hold, which imples $wu, wv \in \overline{G}$, which creates a path $\mathbf{w}'$ (u, w, v) where $|\mathbf{w}'| \leq 2$.

Thus, $\forall\ u, v \in V, \exists$ a path $\mathbf{w}$ in $\overline{G}$ s.t. $|\mathbf{w}| \leq 2$.

Let Statement 2 not hold for $\overline{G}$.

For every $u, v \in \overline{G}$ there are several cases.

One, u and v are not adjacent in $\overline{G}$. Thus, $uv \notin \overline{G}$ and $uv \in G$. So path $\mathbf{w}'(u, v) \in G$ exists, and $|\mathbf{w}'| = 1 \leq 3$.

Two, u and v are adjacent in G.

Let $C = \{c \mid cv, cw \in \overline{G}\}$ (which could be empty) and $A = \{a \mid au \in \overline{G} \wedge a \neq v \wedge a \notin C\}$ and $B = \{b \mid bv \in \overline{G} \wedge b \notin A \wedge b \neq u \wedge b \notin C\}$.

All $a \in A$ are connected by a a path (a, u, v, b) of length 3 to all $b \in B$, by a path (a, u, v) of length 2 to v, by a path (a, u, c) of length 2 to c (if $C \neq \emptyset$) and by a path (a, u) of length 1 to u.

Similarly, all $b \in B$ are connected by a a path (b, v, u, a) of length 3 to all $a \in A$, by a path (b, v, u) of length 2 to u, by a path (a, v, c) of length 2 to c (if $C \neq \emptyset$) and by a path (b, v) of length 1 to v.

Thus, there must be vertices $a_1 \in A$ and $b_1 \in B$ such that $ab \notin \overline{G}$. If not, then there would be a path (a, b) of length 1 for all a and b.

As such, ab, va, ub are not in $\overline{G} \Rightarrow ab, va, ub \in G$. So there is a path $\mathbf{w}'$ (u, a, b, v) where $|\mathbf{w}'| \leq 3$.

Thus, $\forall\ u, v \in V, \exists$ a path $\mathbf{w}$ in G s.t. $|\mathbf{w}| \leq 3$.