UMN, Spring 2017, Math 5707: Lecture 7 (Hamiltonian paths in digraphs)

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Contents

1. Hamiltonian paths in simple digraphs

1.1. Introduction

We shall now study some questions on Hamiltonian paths in digraphs, proving (in particular) Rédei's theorem on Hamiltonian paths in tournaments.

We let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$ of all nonnegative integers.

We recall some basic notions from graph theory:

Definition 1.1.1. A *simple digraph* is a pair (V, A), where V is a finite set, and where A is a subset of $V \times V$.

If $D = (V, A)$ is a simple digraph, then the elements of V are called the *vertices* of D, while the elements of A are called the *arcs* (or *directed edges*) of D.

We shall visually represent a simple digraph $D = (V, A)$ by a picture in which each vertex $v \in V$ is represented by a node (usually a circle with the name of v

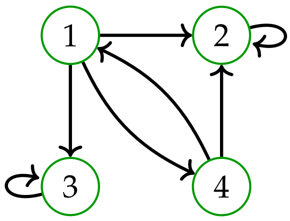
written in it), and each arc $a = (v, w)$ is represented as an arrow from the node representing the vertex v to the node representing the vertex w.

Example 1.1.2.

(a) The simple digraph

$$(\{1,2,3,4\}, \quad \{(1,2), (1,3), (1,4), (2,2), (3,3), (4,1), (4,2)\}) \tag{1}$$

has vertices $1, 2, 3, 4$ and arcs $(1,2), (1,3), (1,4), (2,2), (3,3), (4,1), (4,2)$. It can be represented by the picture



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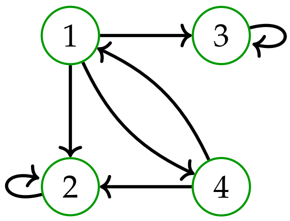
but also by the picture

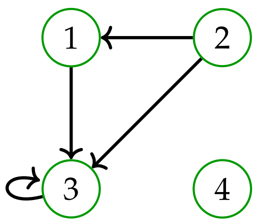


and many others.

(b) The simple digraph

$$(\{1,2,3,4\}, \quad \{\{1,3\}, (2,1), (2,3), (3,3)\}) \tag{2}$$

can be represented by the picture



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Simple digraphs are one of the most primitive notions of directed graphs (in particular, they do not allow multiple arcs); but they are exactly what we need for the following considerations. Thus, we shall simply call them "digraphs":

Convention 1.1.3. For the total of this lecture, we shall use the word "*digraph*" as a shorthand for "simple digraph".

A few more notations will be useful:

Definition 1.1.4. Let $D = (V, A)$ be a digraph.

(a) If $a = (v, w) \in A$ is an arc of D, then the vertex v is called the *source* of a, while the vertex w is called the *target* of a.

(b) We shall use the shorthand notation "vw" for any pair $(v, w) \in V \times V$ (thus, in particular, for any arc of D). Do not confuse this notation with the product of two numbers or a two-digit number.

(c) A *loop* means an arc of the form vv for some $v \in V$. In other words, a *loop* means an arc whose source is also its target.

1.2. Hamiltonian paths ("hamps")

We recall the classical concepts of walks and paths in digraphs:

Definition 1.2.1. Let $D = (V, A)$ be a digraph.

(a) A *walk* of D means a list of the form $(v_0, v_1, \ldots, v_k)$ (with $k \geq 0$), where $v_0, v_1, \ldots, v_k$ are vertices of D with the property that each $i \in \{0, 1, \ldots, k-1\}$ satisfies $v_i v_{i+1} \in A$ (that is, all of the pairs $v_0 v_1$, $v_1 v_2$, $\ldots$, $v_{k-1} v_k$ are arcs of D).

(b) A *path* of D means a walk $(v_0, v_1, \ldots, v_k)$ of D such that the vertices $v_0, v_1, \ldots, v_k$ are distinct.

(c) A walk $(v_0, v_1, \ldots, v_k)$ is said to *contain* a vertex $v \in V$ if and only if $v \in \{v_0, v_1, \ldots, v_k\}$.

(d) If $\mathbf{w} = (v_0, v_1, \ldots, v_k)$ is a walk of D, then the arcs $v_0 v_1$, $v_1 v_2$, $\ldots$, $v_{k-1} v_k$ are called the *arcs* of $\mathbf{w}$.

(e) Let u and v be two vertices of D. A *walk from u to v* means a walk $(v_0, v_1, \ldots, v_k)$ of D satisfying $v_0 = u$ and $v_k = v$.

For example, in the digraph given in (1), the 3-tuple $(4, 2, 2)$ is a walk (since 42 and 22 are arcs) but not a path (since the vertices $4, 2, 2$ are not distinct), whereas the 3-tuple $(4, 1, 2)$ is a walk and a path.

Sometimes we say "walk in D" instead of "walk of D" when D is a digraph; this language is synonymous.

We can now define a special kind of paths:

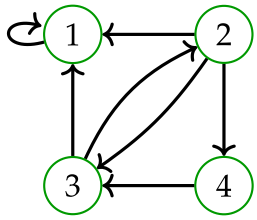
Definition 1.2.2. Let $D = (V, A)$ be a digraph.

A *Hamiltonian path* of D means a path of D that contains each vertex of D.

In other words, a *Hamiltonian path* of D means a path $(v_0, v_1, \ldots, v_k)$ of D such that $V = \{v_0, v_1, \ldots, v_k\}$.

We shall abbreviate "Hamiltonian path" as "*hamp*".

For example, the digraph



has a hamp $(4, 3, 2, 1)$, whereas the digraphs given in (2) and in (1) have no hamps.

1.3. The reverse and complement digraphs

Definition 1.3.1. Let $D = (V, A)$ be a digraph. Then:

(a) The elements of $(V \times V) \setminus A$ (that is, the pairs $(i, j) \in V \times V$ that are not arcs of D) will be called the *non-arcs* of D.

(b) The *reversal* of a pair $(i, j) \in V \times V$ is defined to be the pair (j, i).

(c) Furthermore, D^{rev} is defined as the digraph (V, A^{rev}), where

$$A^{\text{rev}} = \{(j, i) \mid (i, j) \in A\}.$$

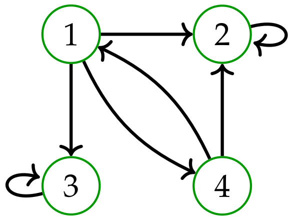
That is, D^{rev} is the digraph D with all its arcs reversed (meaning that each arc is replaced by its reversal; in other words, sources become targets, and targets become sources). We call D^{rev} the *reversal* of the digraph D.

(d) Furthermore, $\overline{D}$ is defined as the digraph $(V, \overline{A})$, where

$$\overline{A} = (V \times V) \setminus A.$$

That is, $\overline{D}$ is the digraph D with all its arcs removed and all its non-arcs added in as arcs. We call $\overline{D}$ the *complement* of the digraph D.

Example 1.3.2. Let

$D =$ 

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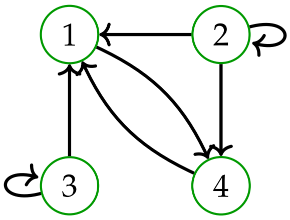
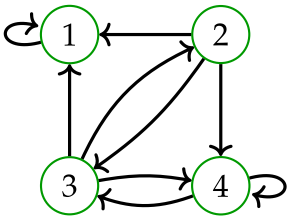
(Formally speaking, we mean "Let D be the digraph represented by this picture"; rigorously, this digraph is given by (1).)

Then,

$D^{\text{rev}} =$ 

and $\overline{D} =$ 

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Convention 1.3.3. In the following, the symbol "#" stands for the word "number" (as in "the number of"). For example,

$$(\text{\# of subsets of } \{1,2,3\}) = 2^3 = 8.$$

We will be interested in the # of hamps of a digraph. In particular, we will ask ourselves when a digraph has a hamp at all. We begin with a simple case:

Proposition 1.3.4. Let $n \in \mathbb{N}$. Let V be the set $\{1,2,\ldots,n\}$. Let A be the set

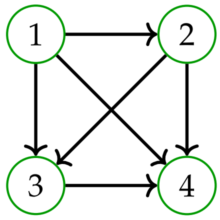
$$\{(i,j) \in V \times V \mid i < j\} = \{12,\ 13,\ 14,\ \ldots,\ 1n,\\ 23,\ 24,\ \ldots,\ 2n,\\ \ldots\\ (n-1)\,n\}$$

(where we are again using the notation ij for the pair (i,j)). Let D be the digraph (V,A). Then,

$$(\text{\# of hamps of } D) = 1.$$

Before we prove this easy fact, let us show an example: When $n = 4$, the digraph

D in Proposition 1.3.4 takes the following form:



and has the unique hamp $(1, 2, 3, 4)$.

Proof of Proposition 1.3.4. We must prove that the digraph D has a unique hamp. Clearly, $(1, 2, \ldots, n)$ is a hamp of D; thus, it remains to show that this hamp is the only hamp of D. In other words, we need to prove that any hamp of D equals $(1, 2, \ldots, n)$.

Let σ be a hamp of D. Thus, σ is a path of D that contains each vertex of D (by the definition of a "hamp"). Hence, σ contains each vertex of D exactly once (because if the list σ contained a vertex more than once, then it would not be a path). In other words, σ is a list of all vertices of D, listed without multiplicities (since σ clearly is a list of vertices of D). In other words, σ is a list of all elements of V, listed without multiplicities (since the vertices of D are the elements of V). Since $V = \{1, 2, \ldots, n\}$, this entails that σ is a list of the numbers $1, 2, \ldots, n$ in some order, without multiplicities. In particular, σ is an n-tuple. Write σ as $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Therefore, $\sigma_1\sigma_2, \sigma_2\sigma_3, \ldots, \sigma_{n-1}\sigma_n$ are arcs of D (since σ is a path of D). However, the definition of A shows that any arc ij of D satisfies $i < j$. Therefore, since we know that $\sigma_1\sigma_2, \sigma_2\sigma_3, \ldots, \sigma_{n-1}\sigma_n$ are arcs of D, we conclude that $\sigma_1 < \sigma_2 < \cdots < \sigma_n$. In other words, the list σ is strictly increasing.

But we know that σ is a list of the numbers $1, 2, \ldots, n$ in some order. Hence, σ is a strictly increasing list of the numbers $1, 2, \ldots, n$ in some order. But the only such list is $(1, 2, \ldots, n)$. Hence, we must have $\sigma = (1, 2, \ldots, n)$.

Forget that we fixed σ. We thus have shown that any hamp σ of D satisfies $\sigma = (1, 2, \ldots, n)$. In other words, any hamp of D equals $(1, 2, \ldots, n)$. This proves Proposition 1.3.4. $\square$

What happens to the # of hamps of a digraph when we reverse all arcs of the digraph? The answer is simple:

Proposition 1.3.5. Let D be a digraph. Then,

$$(\text{\# of hamps of } D^{\text{rev}}) = (\text{\# of hamps of } D).$$

Proof. The hamps of D^{rev} are just the hamps of D, walked backwards. In more detail: If $(v_0, v_1, \ldots, v_k)$ is a hamp of D, then $(v_k, v_{k-1}, \ldots, v_0)$ is a hamp of D^{rev}, and vice versa. Thus, we have a bijection

$$\begin{aligned} \{\text{hamps of } D\} &\to \{\text{hamps of } D^{\text{rev}}\}, \\ (v_0, v_1, \ldots, v_k) &\mapsto (v_k, v_{k-1}, \ldots, v_0). \end{aligned}$$

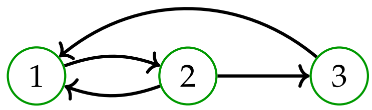
This entails that $|\{\text{hamps of } D\}| = |\{\text{hamps of } D^{\text{rev}}\}|$. In other words, we have $(\text{\# of hamps of } D) = (\text{\# of hamps of } D^{\text{rev}})$. Thus, Proposition 1.3.5 is proved. □

A more interesting question is what happens to the # of hamps of a digraph D when we pass to the complement $\overline{D}$. This # can change, but surprisingly, its change is not completely arbitrary. The following result is due to Berge ([Berge91, §10.1, Theorem 1], [Tomesc85, Problem 7.7, directed case]):

Theorem 1.3.6 (Berge)**.** Let D be a digraph. Then,

$$\left(\text{\# of hamps of } \overline{D}\right) \equiv (\text{\# of hamps of } D) \bmod 2.$$

Example 1.3.7. Let D be the following digraph:

$D =$ 

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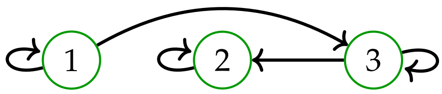
This digraph has 3 hamps: $(1,2,3)$ and $(2,3,1)$ and $(3,1,2)$.

Its complement $\overline{D}$ looks as follows:

$\overline{D} =$ 

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It has only 1 hamp: $(1,3,2)$.

Thus, in this case, Theorem 1.3.6 says that $1 \equiv 3 \bmod 2$.

Proof of Theorem 1.3.6. (We follow [Berge91, §10.1, Theorem 1].)

Write the digraph D as $D = (V, A)$ (so that V is its set of vertices, and A is its set of arcs). We WLOG assume that $V \neq \varnothing$, since otherwise the claim is obvious.

Set $n = |V|$. A *V-listing* will mean a list of elements of V that contains each element of V exactly once. (Thus, each V-listing is an n-tuple, and there are exactly $n!$ many V-listings.) Note that a V-listing is the same as a hamp of the digraph $(V, V \times V)$ (since any pair of two elements of V is an arc of this digraph). Any hamp of D or of $\overline{D}$ is a V-listing, but not every V-listing is a hamp of D or of $\overline{D}$.

If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a V-listing, then we define a set

$$P(\sigma) := \{\sigma_1\sigma_2, \ \ \sigma_2\sigma_3, \ \ \sigma_3\sigma_4, \ \ \ldots, \ \ \sigma_{n-1}\sigma_n\}.$$

(Recall that we are using Definition 1.1.4 **(b)**, so $\sigma_i\sigma_{i+1}$ means the pair (σ_i, σ_{i+1}).) Note that $P(\sigma)$ is the set of all arcs of σ (when σ is viewed as a hamp of the digraph $(V, V \times V)$). If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a V-listing, then the arcs $\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n$ are distinct (since their sources $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are distinct), and thus $P(\sigma)$ is an $(n-1)$-element set.

We make four simple observations:

> *Observation 0:* If σ is a hamp of D, then $P(\sigma)$ is a subset of A.

[*Proof of Observation 0:* Let σ be a hamp of D. Then, σ is a path of D. Hence, each of the pairs $\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n$ is an arc of D and thus belongs to A. In other words, $\{\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n\}$ is a subset of A. In other words, $P(\sigma)$ is a subset of A (since $P(\sigma) = \{\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n\}$). This proves Observation 0.]

> *Observation 1:* We can reconstruct a V-listing σ from the set $P(\sigma)$ (that is, the map $\sigma \mapsto P(\sigma)$ that sends each V-listing σ to the set $P(\sigma)$ is injective).

[*Proof of Observation 1:* Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a V-listing. Thus, $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a list of elements of V that contains each element of V exactly once. Hence, the n elements $\sigma_1, \sigma_2, \ldots, \sigma_n$ are distinct and we have $V = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Therefore, σ_1 is the unique vertex of D distinct from $\sigma_2, \sigma_3, \ldots, \sigma_n$.

The set $P(\sigma)$ consists of the pairs $\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n$ (by the definition of $P(\sigma)$). The second entries of these pairs are $\sigma_2, \sigma_3, \ldots, \sigma_n$. Hence, σ_1 is the unique vertex of D that does not appear as a second entry of any pair in $P(\sigma)$ (since σ_1 is the unique vertex of D distinct from $\sigma_2, \sigma_3, \ldots, \sigma_n$). Thus, σ_1 can be recovered from $P(\sigma)$. Furthermore, σ_2 is the unique vertex of D such that $\sigma_1\sigma_2 \in P(\sigma)$ (since $\sigma_1, \sigma_2, \ldots, \sigma_n$ are distinct); thus, σ_2 can be recovered from $P(\sigma)$ as well (once σ_1 is known). Furthermore, σ_3 is the unique vertex of D such that $\sigma_2\sigma_3 \in P(\sigma)$ (since $\sigma_1, \sigma_2, \ldots, \sigma_n$ are distinct); thus, σ_3 can be recovered from $P(\sigma)$ as well (once σ_2 is known). Proceeding likewise, we can (successively) recover $\sigma_1, \sigma_2, \ldots, \sigma_n$. Thus, we can recover the whole V-listing σ from $P(\sigma)$. This proves Observation 1.]

> *Observation 2:* Let σ be a V-listing. Then, σ is a hamp of D if and only if $P(\sigma) \subseteq A$.

[*Proof of Observation 2:* We have the following chain of logical equivalences:

$$\begin{aligned}
&(\sigma \text{ is a hamp of } D) \\
\Longleftrightarrow\ &(\sigma \text{ is a path of } D) \qquad (\text{since } \sigma \text{ contains each vertex of } D) \\
\Longleftrightarrow\ &(\sigma \text{ is a walk of } D) \qquad (\text{since the vertices in } \sigma \text{ are distinct}) \\
\Longleftrightarrow\ &(\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n \text{ are arcs of } D) \\
\Longleftrightarrow\ &(\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n \text{ belong to } A) \\
\Longleftrightarrow\ &(\{\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n\} \subseteq A) \\
\Longleftrightarrow\ &(P(\sigma) \subseteq A) \qquad (\text{since } P(\sigma) = \{\sigma_1\sigma_2, \ \sigma_2\sigma_3, \ \sigma_3\sigma_4, \ \ldots, \ \sigma_{n-1}\sigma_n\}).
\end{aligned}$$

This proves Observation 2.]

> *Observation 3:* Let σ be a V-listing. Then, σ is a hamp of $\overline{D}$ if and only if $P(\sigma) \subseteq (V \times V) \setminus A$.

[*Proof of Observation 3:* This is proved by the same argument as Observation 2, but with D and A replaced by $\overline{D}$ and $(V \times V) \setminus A$ (since $(V \times V) \setminus A$ is the set of all arcs of $\overline{D}$).]

Now, let N be the number of pairs (σ, B) where σ is a V-listing and B is a subset of A satisfying $B \subseteq P(\sigma)$. Then,

$$N = \sum_{\sigma \text{ is a } V\text{-listing}} N_\sigma, \tag{3}$$

where

$$N_\sigma := (\# \text{ of subsets } B \text{ of } A \text{ satisfying } B \subseteq P(\sigma)).$$

But we also have

$$N = \sum_{B \text{ is a subset of } A} N^B, \tag{4}$$

where

$$N^B := (\# \text{ of } V\text{-listings } \sigma \text{ such that } B \subseteq P(\sigma)).$$

(The "B" in "N^B" is not an exponent but just a superscript.)

Let us now relate these two sums to hamps. We begin with the sum in (3). We shall use the *Iverson bracket notation* – i.e., the notation $[\mathcal{A}]$ for the truth value of a statement $\mathcal{A}$. (This truth value is defined to be 1 if $\mathcal{A}$ is true, and to be 0 if $\mathcal{A}$ is false.) Clearly, if $\mathcal{A}$ and $\mathcal{B}$ are two equivalent statements, then $[\mathcal{A}] = [\mathcal{B}]$. We will use this fact without explicit mention. Also, if S is a set, and if $\mathcal{A}(s)$ is a statement for each $s \in S$, then

$$\sum_{s \in S} [\mathcal{A}(s)] = (\# \text{ of elements } s \in S \text{ satisfying } \mathcal{A}(s)). \tag{5}$$

Also, it is easy to see that

$$2^m \equiv [m = 0] \bmod 2 \tag{6}$$

for each $m \in \mathbb{N}$.

For any V-listing σ, we have

$$\begin{aligned}
N_\sigma &= (\# \text{ of subsets } B \text{ of } A \text{ satisfying } B \subseteq P(\sigma)) \\
&= (\# \text{ of subsets of } A \cap (P(\sigma))) \\
&\qquad \left(\begin{array}{c} \text{since a subset } B \text{ of } A \text{ satisfying } B \subseteq P(\sigma) \\ \text{is the same thing as a subset of } A \cap (P(\sigma)) \end{array} \right) \\
&= 2^{|A \cap (P(\sigma))|} \\
&\equiv [|A \cap (P(\sigma))| = 0] \qquad (\text{by } (6)) \\
&= [A \cap (P(\sigma)) = \varnothing] \\
&= [P(\sigma) \subseteq (V \times V) \setminus A] \qquad (\text{since } P(\sigma) \text{ is a subset of } V \times V) \\
&= \left[\sigma \text{ is a hamp of } \overline{D}\right] \bmod 2
\end{aligned} \tag{7}$$

(by Observation 3 above). Hence, (3) becomes

$$\begin{aligned} N &= \sum_{\sigma \text{ is a } V\text{-listing}} \underbrace{N_\sigma}_{\substack{\equiv [\sigma \text{ is a hamp of } \overline{D}] \mod 2 \\ \text{(by (7))}}} \\ &\equiv \sum_{\sigma \text{ is a } V\text{-listing}} [\sigma \text{ is a hamp of } \overline{D}] \\ &= \left(\# \text{ of } V\text{-listings } \sigma \text{ such that } \sigma \text{ is a hamp of } \overline{D}\right) \qquad \text{(by (5))} \\ &= \left(\# \text{ of hamps of } \overline{D}\right) \mod 2 \qquad (8) \end{aligned}$$

(because each hamp of $\overline{D}$ is a V-listing).

Now, let us study the numbers N^B more closely. Fix a subset B of A. Then,

$$N^B = (\# \text{ of } V\text{-listings } \sigma \text{ such that } B \subseteq P(\sigma)).$$

When is this # odd?

To find out, let us define another word: A *path cover* of V shall mean a set of paths in the digraph $(V, V \times V)$ (not (V, A)) such that each vertex $v \in V$ is contained in **exactly** one of these paths. (Recall that a path is allowed to consist of a single vertex, but cannot have 0 vertices.)

For example, if $V = \{1,2,3,4,5,6,7\}$, then

$$\{(1,3,5), \ (2), \ (6), \ (7,4)\}$$

is a path cover of V, and so is

$$\{(1), \ (2), \ (3,4,6,5,7)\},$$

and so is

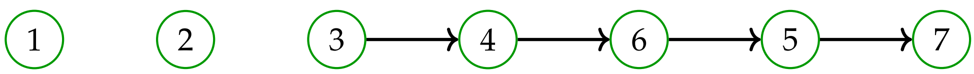
$$\{(1), \ (2), \ (3), \ (4), \ (5), \ (6), \ (7)\}$$

(in this path cover, each vertex belongs to its own path), and so is

$$\{(1,2,3,4,5,6,7)\}$$

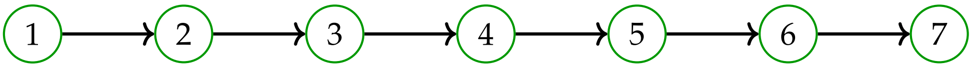
(a path cover consisting of just a single path). We can visualize these path covers by drawing the paths in the obvious manner (i.e., we represent each element of V as a node, and we draw arrows for the arcs of each path in our path cover). Thus, the four above-listed examples of path covers look as follows:

and



and

and



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Note that the notion of a path cover of V depends only on V, not on A.

If C is a path cover of V, then the paths that belong to C will be called the *C-paths*. For example, if $C = \{(1), (2), (3,4,6,5,7)\}$, then the C-paths are (1), (2) and $(3,4,6,5,7)$.

If C is a path cover of V, then we let $\operatorname{Arcs} C$ denote the set of arcs of all C-paths. In other words, we let

$$\operatorname{Arcs} C := \bigcup_{(a_1,a_2,\ldots,a_k) \text{ is a } C\text{-path}} \{a_1a_2, \ a_2a_3, \ \ldots, \ a_{k-1}a_k\}.$$

We call this set $\operatorname{Arcs} C$ the *arc set* of C. For example, the four above-listed examples of path covers have arc sets

$$\{13, \ 35, \ 74\}, \\ \{34, \ 46, \ 65, \ 57\}, \\ \varnothing, \\ \{12, \ 23, \ 34, \ 45, \ 56, \ 67\},$$

respectively.

Now, assume that N^B is odd. Thus, $N^B \neq 0$, so that there exists **some** V-listing σ such that $B \subseteq P(\sigma)$ (since N^B is the # of such V-listings). From this, we can easily see that there exists a path cover C of V such that $B = \operatorname{Arcs} C$.

[*Proof:* We have just shown that there exists **some** V-listing σ such that $B \subseteq P(\sigma)$. Consider this σ, and write it as $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Note that σ is a hamp of the digraph $(V, V \times V)$ (since σ is a V-listing). The set $P(\sigma)$ is the set of arcs of this hamp.

However, if we remove an arc from a path, then this path breaks into two smaller paths[1]. Thus, if we remove several arcs from a path, then this path breaks into several smaller paths. Hence, in particular, if we remove some arcs from a hamp of $(V, V \times V)$, then this hamp breaks into several smaller paths, and the latter paths form a path cover of V (because each $v \in V$ is contained in exactly one of them)[2].

[1] For instance, removing the arc 34 from the path $(1,2,3,4,5)$ breaks it into the two smaller paths $(1,2,3)$ and $(4,5)$.

[2] For instance, if we remove the arcs 23 and 34 from the hamp $(1,2,3,4,5,6)$ (assuming that $V = \{1,2,3,4,5,6\}$), then this hamp breaks into three paths $(1,2)$, (3) and $(4,5,6)$, which form the path cover $\{(1,2), (3), (4,5,6)\}$ of V.

In other words, if we remove some arcs from a hamp τ of $(V, V \times V)$, then the set of all remaining arcs of τ is the arc set of a path cover of V. [3]

Applying this to $\tau = \sigma$, we conclude that if we remove some arcs from σ, then the set of all remaining arcs of σ is the arc set of a path cover of V. In other words, any subset of $P(\sigma)$ is the arc set of a path cover of V (since any subset of $P(\sigma)$ can be obtained by removing some arcs from σ). Thus, B is the arc set of a path cover of V (since B is a subset of $P(\sigma)$). [4] In other words, there exists a path cover C of V such that $B = \operatorname{Arcs} C$.]

So we have shown that there exists a path cover C of V such that $B = \operatorname{Arcs} C$. Consider this C.

Let r be the number of C-paths. Thus, C consists of r paths, and each vertex $v \in V$ is contained in exactly one of these r paths. Note that there exists at least one C-path (since $V \neq \varnothing$, but each vertex $v \in V$ must be contained in a C-path). In other words, $r \geq 1$.

Now, what are the V-listings σ that satisfy $B \subseteq P(\sigma)$? These are the V-listings σ that satisfy $\operatorname{Arcs} C \subseteq P(\sigma)$ (since $B = \operatorname{Arcs} C$). In other words, these are the V-listings σ with the property that each arc of each C-path is also an arc of σ (since $\operatorname{Arcs} C$ is the set of all arcs of all C-paths, whereas $P(\sigma)$ is the set of all arcs of σ). In other words, these are the V-listings σ with the property that if a vertex a is followed by a vertex b on some C-path, then a is also followed by b in the V-listing σ. Hence, each C-path must appear as a contiguous block on such a V-listing σ, with its vertices appearing in the same order in σ as they do on the C-path. Therefore, each V-listing σ that satisfies $B \subseteq P(\sigma)$ can be constructed as follows:

1. Start with the empty list $()$.

2. Pick some C-path, and append all its vertices (in the order in which they appear on this C-path) to the end of the list.

---

[3] We can even describe the latter path cover explicitly: Let $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ be a hamp of $(V, V \times V)$, and let us remove the arcs

$$\tau_{i_1}\tau_{i_1+1}, \quad \tau_{i_2}\tau_{i_2+1}, \quad \ldots, \quad \tau_{i_p}\tau_{i_p+1}$$

from τ, where $i_1, i_2, \ldots, i_p$ are some elements of $\{1, 2, \ldots, n-1\}$ satisfying $i_1 < i_2 < \cdots < i_p$. Then, the hamp τ breaks into the $p+1$ smaller paths

$$\begin{aligned}
&(\tau_1, \tau_2, \ldots, \tau_{i_1}), \\
&(\tau_{i_1+1}, \tau_{i_1+2}, \ldots, \tau_{i_2}), \\
&(\tau_{i_2+1}, \tau_{i_2+2}, \ldots, \tau_{i_3}), \\
&\ldots, \\
&\left(\tau_{i_p+1}, \tau_{i_p+2}, \ldots, \tau_n\right),
\end{aligned}$$

and these $p+1$ smaller paths form a path cover of V. The set of all remaining arcs of τ is the arc set of this path cover.

[4] For example, if $\sigma = (1, 2, 3, 4, 5, 6)$ and $B = \{12, 45, 56\}$, then B is obtained by removing the arcs 23 and 34 from the hamp σ, and thus B is the arc set of the path cover $\{(1, 2), (3), (4, 5, 6)\}$.

3. Then, pick another C-path, and do the same for its vertices.

4. Then, pick another C-path, and do the same for its vertices.

5. And so on, until all C-paths have been listed.

In other words, each V-listing σ that satisfies $B \subseteq P(\sigma)$ can be obtained by concatenating[5] the C-paths in some order[6]. The only freedom we have is to choose this order. For this, we have $r!$ options (since the number of C-paths is r). Each of these $r!$ options yields a different V-listing (because the C-paths are nonempty and have no vertex in common).

Thus, $N^B = r!$ (since N^B is the # of V-listings σ such that $B \subseteq P(\sigma)$). Since N^B is odd, we thus see that $r!$ is odd. However, a factorial $m!$ is always even for $m > 1$; therefore, we must have $r \leq 1$ (since $r!$ is odd). Combining this with $r \geq 1$, we obtain $r = 1$. In other words, there is only one C-path. This C-path must contain each $v \in V$ (because C is a path cover of V, and thus each $v \in V$ is contained in a C-path), and thus is a hamp of the digraph $(V, V \times V)$. Let τ be this hamp. Thus, τ is the only C-path; in other words, $C = \{\tau\}$. Hence, $\operatorname{Arcs} C$ is the set of arcs of the path τ. In other words, $\operatorname{Arcs} C = P(\tau)$. Thus, $B = \operatorname{Arcs} C = P(\tau)$, so that $P(\tau) = B \subseteq A$ (since B is a subset of A); in other words, each arc of τ belongs to A. Hence, τ is a path of D. Therefore, τ is a hamp of D (since τ is a hamp of $(V, V \times V)$).

Now, forget that we assumed that N^B is odd. We thus have shown that

$$\text{if } N^B \text{ is odd, then } B = P(\tau) \text{ for some hamp } \tau \text{ of } D. \tag{9}$$

The converse of this statement holds as well:

$$\text{if } B = P(\tau) \text{ for some hamp } \tau \text{ of } D, \text{ then } N^B \text{ is odd} \tag{10}$$

(and actually N^B equals 1 in this case).

[*Proof of (10):* Assume that $B = P(\tau)$ for some hamp τ of D. Consider this τ. Thus, τ is a V-listing σ such that $B \subseteq P(\sigma)$. Furthermore, it is easy to see that τ

[5] *Concatenating* several lists $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ means combining them into a single list, which begins with the entries of $\mathbf{a}_1$ (in the order in which they appear in $\mathbf{a}_1$), continues with the entries of $\mathbf{a}_2$ (in the order in which they appear in $\mathbf{a}_2$), and so on. For instance, concatenating three lists (a_1, a_2, a_3), (b_1, b_2) and (c_1, c_2, c_3) yields the list $(a_1, a_2, a_3, b_1, b_2, c_1, c_2, c_3)$. Since the C-paths are lists (of vertices), we can concatenate them.

[6] For an example, let us assume that $V = \{1, 2, 3, 4, 5, 6\}$ and $P = \{(1, 3, 2), (4), (6, 5)\}$, so that $r = 3$. The path cover P looks as follows:

Then, the V-listings σ that satisfy $B \subseteq P(\sigma)$ are

$$\begin{array}{lll} (1,3,2,4,6,5), & (1,3,2,6,5,4), & (4,1,3,2,6,5), \\ (4,6,5,1,3,2), & (6,5,1,3,2,4), & (6,5,4,1,3,2). \end{array}$$

is the **only** such V-listing[7]. Therefore, there is exactly 1 such V-listing. In other words, the # of such V-listings is 1. In other words, N^B is 1 (since N^B is this #). Hence, N^B is odd. This proves (10).]

Combining (9) with (10), we obtain the following: The number N^B is odd if and

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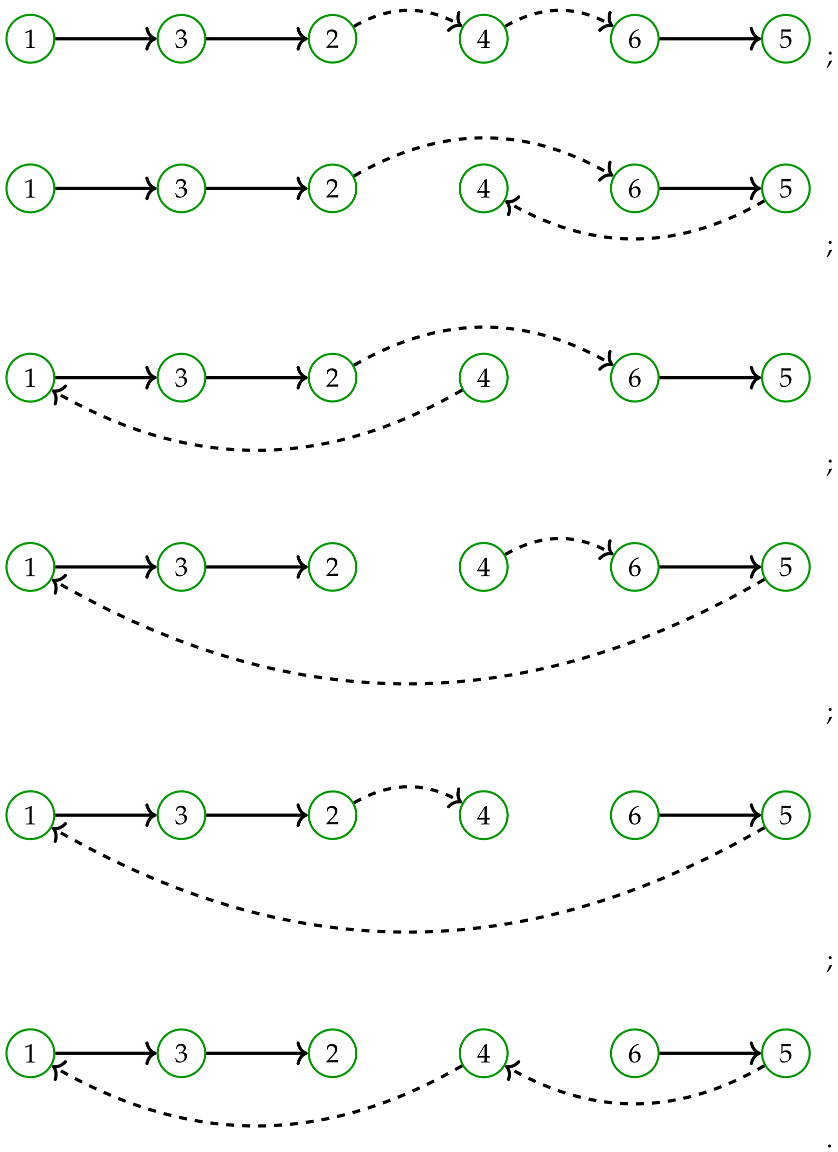
Here is how they look like (the dashed arrows connect different C-paths):



[7]*Proof.* Let σ be a V-listing such that $B \subseteq P(\sigma)$. We must show that $\sigma = \tau$.

From $B = P(\tau)$, we obtain $P(\tau) = B \subseteq P(\sigma)$. However, the sets $P(\sigma)$ and $P(\tau)$ are two finite sets of the same size (since they are both $(n-1)$-element sets). Thus, if one of them is a subset of the other, then they must be equal. Hence, from $P(\tau) \subseteq P(\sigma)$, we obtain $P(\tau) = P(\sigma)$. Using Observation 1, we thus conclude that $\tau = \sigma$. Hence, $\sigma = \tau$, qed.

only if $B = P(\tau)$ for some hamp τ of D. Therefore,

$$\left[N^B \text{ is odd}\right] = [B = P(\tau) \text{ for some hamp } \tau \text{ of } D].$$

However, it is easy to see that $m \equiv [m \text{ is odd}] \mod 2$ for each integer m. Thus,

$$N^B \equiv \left[N^B \text{ is odd}\right] = [B = P(\tau) \text{ for some hamp } \tau \text{ of } D] \mod 2. \qquad (11)$$

Forget that we fixed B. We thus have proved the congruence (11) for each subset B of A. Hence, (4) becomes

$$\begin{aligned} N &= \sum_{B \text{ is a subset of } A} \underbrace{N^B}_{\substack{\equiv [B = P(\tau) \text{ for some hamp } \tau \text{ of } D] \mod 2 \\ \text{(by (11))}}} \\ &\equiv \sum_{B \text{ is a subset of } A} [B = P(\tau) \text{ for some hamp } \tau \text{ of } D] \\ &= (\text{\# of subsets } B \text{ of } A \text{ such that } B = P(\tau) \text{ for some hamp } \tau \text{ of } D) \qquad \text{(by (5))} \\ &= (\text{\# of sets of the form } P(\tau) \text{ for some hamp } \tau \text{ of } D) \\ &\qquad \left(\begin{array}{c} \text{since each set of the form } P(\tau) \text{ for some hamp } \tau \text{ of } D \\ \text{is a subset of } A \text{ (by Observation 0, applied to } \sigma = \tau) \end{array}\right) \\ &= (\text{\# of hamps } \tau \text{ of } D) \mod 2 \end{aligned}$$

(because Observation 1 shows that different hamps τ yield different sets $P(\tau)$). Therefore,

$$(\text{\# of hamps } \tau \text{ of } D) \equiv N \equiv \left(\text{\# of hamps of } \overline{D}\right) \mod 2$$

(by (8)). Thus,

$$\left(\text{\# of hamps of } \overline{D}\right) \equiv (\text{\# of hamps } \tau \text{ of } D) = (\text{\# of hamps of } D) \mod 2.$$

This proves Theorem 1.3.6. □

1.4. Tournaments

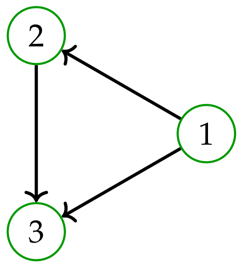
We now define two more restrictive classes of digraphs:

Definition 1.4.1. A digraph D is said to be *loopless* if it has no loops (i.e., it has no arcs of the form (v, v)).

Definition 1.4.2. A *tournament* is defined to be a loopless digraph D that satisfies the following axiom:

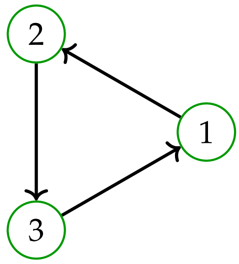
Tournament axiom: For any two distinct vertices u and v of D, **exactly** one of the two pairs (u, v) and (v, u) is an arc of D.

Example 1.4.3. The following digraph is a tournament:



.

The following digraph is a tournament as well:



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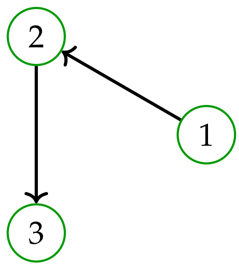
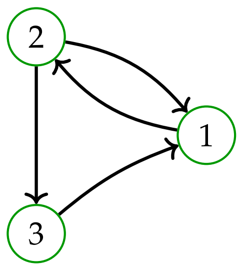
However, the following digraph is not a tournament:



,

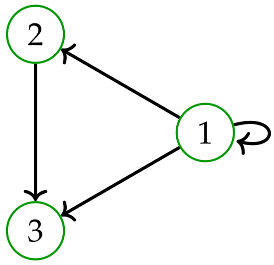
because the tournament axiom is not satisfied for $u = 1$ and $v = 3$ (since neither $(1,3)$ nor $(3,1)$ is an arc of the digraph). Nor is the following digraph a tournament:



,

because the tournament axiom is not satisfied for $u = 1$ and $v = 2$ (since both

$(1,2)$ and $(2,1)$ are arcs of the digraph). Finally, the digraph



is not a tournament either, since it is not loopless (having the loop $(1,1)$).

The digraph D in Proposition 1.3.4 always is a tournament.

A tournament can be viewed as a model for the outcome of a round-robin tournament between a number of contestants (assuming that each contest ends in a victory by one of the contestants). The vertices are the contestants, and the arcs encode the winner of each contest (namely, if contestant u wins against contestant v, then we encode it as an arc (u,v)). This is the reason for the name "tournament".

Here is a quick consequence of the definition of a tournament:

Proposition 1.4.4. Let D be a tournament. Then, the arcs of $\overline{D}$ that are not loops are precisely the arcs of D^{rev}.

Proof. The definition of D^{rev} shows that the arcs of D^{rev} are precisely the reversals of the arcs of D.

The definition of $\overline{D}$ shows that the arcs of $\overline{D}$ are precisely the non-arcs of D.

The digraph D is a tournament. Thus, D is loopless (by the definition of a tournament), i.e., has no loops. In other words, none of the arcs of D is a loop. Hence, none of the reversals of the arcs of D is a loop either (since the reversal of an arc a is a loop only when a itself is a loop). In other words, none of the arcs of D^{rev} is a loop (since the arcs of D^{rev} are precisely the reversals of the arcs of D).

For any two distinct vertices u and v of D, we have the following chain of logical equivalences:

$$\begin{aligned}
&\left((u,v) \text{ is an arc of } \overline{D}\right) \\
\iff &\left((u,v) \text{ is a non-arc of } D\right) && \left(\begin{array}{c}\text{since the arcs of } \overline{D} \text{ are precisely} \\ \text{the non-arcs of } D\end{array}\right) \\
\iff &\left((u,v) \text{ is not an arc of } D\right) && \left(\text{by the definition of a ``non-arc''}\right) \\
\iff &\left((v,u) \text{ is an arc of } D\right) && \left(\begin{array}{c}\text{since the tournament axiom tells us that} \\ \textbf{exactly} \text{ one of the two} \\ \text{pairs } (u,v) \text{ and } (v,u) \text{ is an arc of } D\end{array}\right) \\
\iff &\left((u,v) \text{ is an arc of } D^{\text{rev}}\right) && \left(\begin{array}{c}\text{since the arcs of } D^{\text{rev}} \text{ are precisely} \\ \text{the reversals of the arcs of } D\end{array}\right).
\end{aligned}$$

Thus, the arcs of $\overline{D}$ that are not loops are precisely the arcs of D^{rev} that are not loops. Since none of the arcs of D^{rev} is a loop, we can simplify this as follows: The arcs of $\overline{D}$ that are not loops are precisely the arcs of D^{rev}. This proves the Proposition 1.4.4. $\square$

We note that Proposition 1.4.4 also has a converse (which we shall not use and thus won't prove either):

Proposition 1.4.5. Let D be a loopless digraph. Then, D is a tournament if and only if the arcs of $\overline{D}$ that are not loops are precisely the arcs of D^{rev}.

Here are three other obvious properties of tournaments:

Proposition 1.4.6. Let D be a tournament. Then, D^{rev} is a tournament as well.

Proposition 1.4.7. Let $D = (V, A)$ be a tournament, and let $vw \in A$ be an arc of D. Let D' be the digraph obtained from D by reversing the arc vw (that is, replacing it by wv). (In other words, let $D' = (V,\ \ (A \setminus \{vw\}) \cup \{wv\})$.) Then, D' is again a tournament.

Proposition 1.4.8. Let D be a tournament with n vertices. Then, D has exactly $\dbinom{n}{2}$ many arcs.

With so many arcs, one might hope that a tournament has better chances than a random digraph to have a hamp (Hamiltonian path). And indeed:

Theorem 1.4.9 (Rédei's Little Theorem)**.** Any tournament has a hamp. Here, we agree to consider the empty list $()$ as a hamp of the empty tournament with 0 vertices.

We will now briefly outline a quick proof of this theorem, but it is not strictly needed since we will later prove a much stronger result (Theorem 1.6.1) from which Theorem 1.4.9 will also follow.

Proof of Theorem 1.4.9 (sketched). We shall prove this by strong induction on the number of vertices of the tournament. Thus, we fix a tournament $D = (V, A)$, and assume that all tournaments with fewer vertices than D have hamps. Now we want to find a hamp of D.

If V is empty, then $()$ is a hamp (according to our agreement). Hence, we WLOG assume that V is not empty. Choose any $v \in V$. Let

$$X := \{u \in V \mid uv \in A\} \qquad \text{and} \qquad Y := \{u \in V \mid vu \in A\}.$$

By the definition of a tournament, the sets X, Y and $\{v\}$ are disjoint, and their union is V. Hence, the two sets X and Y have smaller size than V.

Now, consider the two tournaments $(X, A \cap (X \times X))$ and $(Y, A \cap (Y \times Y))$. These two tournaments have fewer vertices than D (since the two sets X and Y have smaller size than V), and thus have hamps (by the induction hypothesis). Let $(x_1, x_2, \ldots, x_a)$ and $(y_1, y_2, \ldots, y_b)$ be these hamps (these can be empty lists if X or Y is empty). Then, it is easy to see that $(x_1, x_2, \ldots, x_a, u, y_1, y_2, \ldots, y_b)$ is a hamp of D. Thus, D has a hamp. This completes the induction step, and thus Theorem 1.4.9 is proved. □

1.5. Hamiltonian cycles in tournaments

Encouraged by Theorem 1.4.9, we can ask a stronger question: Is it true that any tournament has a Hamiltonian **cycle**? Let us first define this concept:

Definition 1.5.1. Let $D = (V, A)$ be a digraph.

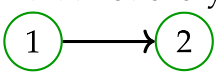
(a) A *closed walk* of D means a walk $(v_0, v_1, \ldots, v_k)$ of D satisfying $v_k = v_0$.

(b) A *cycle* of D means a closed walk $(v_0, v_1, \ldots, v_k)$ of D such that $k \geq 1$ and such that the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct.

(c) A *Hamiltonian cycle* of D means a cycle of D that contains each vertex of D.

In other words, a *Hamiltonian cycle* of D means a cycle $(v_0, v_1, \ldots, v_k)$ of D such that $V = \{v_0, v_1, \ldots, v_k\}$.

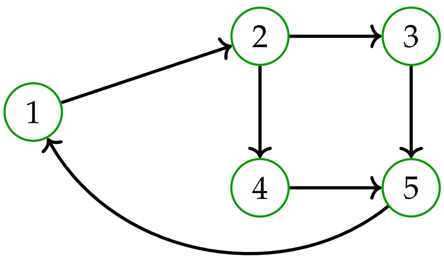
For example, in the digraph D constructed in Example 1.3.7, the 3-tuple $(1, 2, 1)$ is a cycle (but not a Hamiltonian one, since it fails to contain the vertex 3), and the 4-tuple $(2, 3, 1, 2)$ is a Hamiltonian cycle.

Now, it is clear that not every tournament has a Hamiltonian cycle; for example, the tournament 

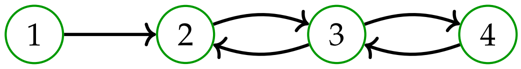
has none. One reason for this is obvious:

Definition 1.5.2. Let $D = (V, A)$ be a digraph with at least one vertex. We say that the digraph D is *strongly connected* if for every two vertices u and v in V, there exists a walk from u to v in D.

Example 1.5.3. The digraph



is strongly connected, whereas the digraph



is not (for example, it has no walk from 2 to 1).

Proposition 1.5.4. Let D be a digraph. If D has a Hamiltonian cycle, then D is strongly connected.

Proof (sketched). Assume that D has a Hamiltonian cycle. Then, for any two vertices u and v of D, we can obtain a walk from u to v by walking along this cycle. Thus, D is strongly connected. □

Proposition 1.5.4 gives only a necessary, not a sufficient condition for the existence of a Hamiltonian cycle. However, it turns out that it is also sufficient when the digraph is a tournament with at least two vertices:

Theorem 1.5.5 (Camion's theorem)**.** Let D be a strongly connected tournament with at least two vertices. Then, D has a Hamiltonian cycle.

Before we prove this, we show a simple proposition about strongly connected digraphs:

Proposition 1.5.6. Let $D = (V, A)$ be a strongly connected digraph. Then:

(a) If V has at least two vertices, then D has a cycle.

(b) Each arc $a \in A$ is contained in at least one cycle of D.

Proof of Proposition 1.5.6 (sketched). **(b)** Let $a = uv \in A$ be an arc. Then, there is a walk from v to u in D (since D is strongly connected). Hence, there is a path from v to u in D as well (by the "if there is a walk, then there is a path" theorem[8]). Pick such a path and combine it with the arc a to get a cycle that contains the arc a. Thus, Proposition 1.5.6 **(b)** is proved.

(a) Assume that V has at least two vertices. Thus, V has two distinct vertices u and v. Consider these u and v. Then, there is a walk from v to u in D (since D is strongly connected). This walk must have at least one arc (since u and v are distinct). Hence, D has an arc. From Proposition 1.5.6 **(b)**, we conclude that this arc is contained in at least one cycle of D. Hence, D has a cycle. This proves Proposition 1.5.6 **(a)**. □

We are now ready to prove Theorem 1.5.5:

[8]We have previously stated this theorem for undirected graphs, but it exists in the same form (and with the same proof) for digraphs.

Proof of Theorem 1.5.5. (We are following [Berge91, §10.2, Theorem 4].)

Write D as (V, A). Proposition 1.5.6 **(a)** shows that D has a cycle. Thus, D has a cycle of maximum length[9] (since the total set of cycles of D is finite[10]). Let

$$\mathbf{c} = (v_0, v_1, \ldots, v_k) \qquad (\text{with } v_k = v_0)$$

be a cycle (of D) having maximum length. We claim that $\mathbf{c}$ is a Hamiltonian cycle.

To prove this, we assume the contrary. Thus, $\mathbf{c}$ does not contain some vertex of D. Our goal is to obtain a contradiction by finding a cycle that is longer than $\mathbf{c}$.

Let $C = \{v_0, v_1, \ldots, v_k\}$ be the set of all vertices of the cycle $\mathbf{c}$. Thus, C is a proper subset of V (since $\mathbf{c}$ does not contain some vertex of D). Hence, $V \setminus C \neq \varnothing$.

The vertices $w \in V \setminus C$ are precisely the vertices not contained in the cycle $\mathbf{c}$. Thus, they are distinct from each of $v_0, v_1, \ldots, v_k$. Hence, for each vertex $w \in V \setminus C$, exactly one of the pairs wv_0 and v_0w must belong to A (by the tournament axiom). In other words, the set $V \setminus C$ is the union of its two subsets

$$X := \{w \in V \setminus C \mid wv_0 \in A\} \qquad \text{and}$$
$$Y := \{w \in V \setminus C \mid v_0w \in A\},$$

and furthermore these two subsets X and Y are disjoint. Thus, $X \cup Y = V \setminus C$ and $X \cap Y = \varnothing$.

We shall now prove the following:

Observation 1: Let $w \in X$. Then, $wv_i \in A$ for each $i \in \{0, 1, \ldots, k\}$.

[*Proof of Observation 1:* The claim we must prove is "$wv_i \in A$ for each $i \in \{0, 1, \ldots, k\}$". Substituting $k - i$ for i in this claim, we can restate it as "$wv_{k-i} \in A$ for each $i \in \{0, 1, \ldots, k\}$".

We shall prove this restated claim by induction on i:

Induction base: From $w \in X$, we immediately obtain $wv_0 \in A$. In view of $v_{k-0} = v_k = v_0$, we can rewrite this as $wv_{k-0} \in A$. Hence, the claim "$wv_{k-i} \in A$ for each $i \in \{0, 1, \ldots, k\}$" holds for $i = 0$.

Induction step: Let $j \in \{1, 2, \ldots, k\}$. Assume that $wv_{k-(j-1)} \in A$. We must show that $wv_{k-j} \in A$.

Assume the contrary. Thus, $wv_{k-j} \notin A$.

Let $r = k - j$; thus, $r \in \{0, 1, \ldots, k-1\}$. Also, $wv_r = wv_{k-j} \notin A$. We have $w \neq v_r$ (since $w \in X \subseteq X \cup Y = V \setminus C$). Thus, by the tournament axiom, we see that exactly one of the pairs wv_r and v_rw must belong to A. Hence, $v_rw \in A$ (since $wv_r \notin A$).

Also, from $r = k - j$, we obtain $r + 1 = k - j + 1 = k - (j-1)$, so that $wv_{r+1} = wv_{k-(j-1)} \in A$. Hence, we can "detour" our cycle $\mathbf{c}$ to pass through w, obtaining a longer cycle $(v_0, v_1, \ldots, v_r, w, v_{r+1}, v_{r+2}, \ldots, v_k)$ (because $v_rw \in A$ and $wv_{r+1} \in A$). However, this contradicts the fact that $\mathbf{c}$ is a cycle having **maximum length**. This contradiction shows that our assumption was wrong; hence, we have shown that $wv_{k-j} \in A$. This completes the induction step; thus, we have proved Observation 1.]

[9] The *length* of a walk $(v_0, v_1, \ldots, v_k)$ is defined to be the number k.

[10] This is because a cycle cannot have length $> |V|$.

Observation 2: Let $w \in Y$. Then, $v_i w \in A$ for each $i \in \{0, 1, \ldots, k\}$.

[*Proof of Observation 2:* We shall prove this by induction on i:

Induction base: From $w \in Y$, we immediately obtain $v_0 w \in A$. Hence, Observation 2 is proved for $i = 0$.

Induction step: Let $j \in \{1, 2, \ldots, k\}$. Assume that $v_{j-1} w \in A$. We must show that $v_j w \in A$.

Assume the contrary. Thus, $v_j w \notin A$. However, $w \neq v_j$ (since $w \in Y \subseteq X \cup Y = V \setminus C$). Thus, by the tournament axiom, we see that exactly one of the pairs $w v_j$ and $v_j w$ must belong to A. Hence, $w v_j \in A$ (since $v_j w \notin A$). Hence, we can "detour" our cycle $\mathbf{c}$ to pass through w, obtaining a longer cycle $(v_0, v_1, \ldots, v_{j-1}, w, v_j, v_{j+1}, \ldots, v_k)$ (because $v_{j-1} w \in A$ and $w v_j \in A$). However, this contradicts the fact that $\mathbf{c}$ is a cycle having **maximum length**. This contradiction shows that our assumption was wrong; hence, we have shown that $v_j w \in A$. This completes the induction step; thus, we have proved Observation 2.]

Observation 3: Let $w \in X$ and $z \in C$. Then, $zw \notin A$.

[*Proof of Observation 3:* We have $z \in C = \{v_0, v_1, \ldots, v_k\}$, so that $z = v_i$ for some $i \in \{0, 1, \ldots, k\}$. Consider this i. Observation 1 yields $w v_i \in A$ (since $w \in X$). In other words, $wz \in A$ (since $z = v_i$). However, $w \neq z$ (since $w \in X \subseteq X \cup Y = V \setminus C$ and $z \in C$). Thus, by the tournament axiom, exactly one of the two pairs wz and zw must belong to A. Hence, from $wz \in A$, we obtain $zw \notin A$. This proves Observation 3.]

Observation 4: Let $w \in Y$ and $z \in C$. Then, $wz \notin A$.

[*Proof of Observation 4:* We have $z \in C = \{v_0, v_1, \ldots, v_k\}$, so that $z = v_i$ for some $i \in \{0, 1, \ldots, k\}$. Consider this i. Observation 2 yields $v_i w \in A$ (since $w \in Y$). In other words, $zw \in A$ (since $z = v_i$). However, $w \neq z$ (since $w \in Y \subseteq X \cup Y = V \setminus C$ and $z \in C$). Thus, by the tournament axiom, exactly one of the two pairs wz and zw must belong to A. Hence, from $zw \in A$, we obtain $wz \notin A$. This proves Observation 4.]

Observation 5: We have $X \neq \varnothing$.

[*Proof of Observation 5:* There exists some vertex $q \in V \setminus C$ (since $V \setminus C \neq \varnothing$). Consider this q. Since D is strongly connected, there exists a walk from q to v_0. This walk must cross from the set $V \setminus C$ into the set C at some point[11] (since $q \in V \setminus C$ whereas $v_0 \in C$). Hence, there exists an arc wz whose source w belongs to $V \setminus C$ and whose target z belongs to C. Consider this arc. Thus, $w \in V \setminus C$ and $z \in C$ and $wz \in A$. If we had $w \in Y$, then Observation 4 would yield $wz \notin A$, which would contradict $wz \in A$. Hence, we cannot have $w \in Y$. Thus, $w \notin Y$. However, $w \in V \setminus C = X \cup Y$. Combining this with $w \notin Y$, we obtain $w \in (X \cup Y) \setminus Y \subseteq X$. Hence, $X \neq \varnothing$. Thus, Observation 5 is proved.]

[11] By this we mean the following: One of the arcs of this walk must have its source in $V \setminus C$ and its target in C.

Observation 6: We have $Y \neq \varnothing$.

[*Proof of Observation 6:* This is very similar to the proof of Observation 5 above: There exists some vertex $q \in V \setminus C$ (since $V \setminus C \neq \varnothing$). Consider this q. Since D is strongly connected, there exists a walk from v_0 to q. This walk must cross from the set C into the set $V \setminus C$ at some point (since $v_0 \in C$ whereas $q \in V \setminus C$). Hence, there exists an arc zw whose source z belongs to C and whose target w belongs to $V \setminus C$. Consider this arc. Thus, $z \in C$ and $w \in V \setminus C$ and $zw \in A$. If we had $w \in X$, then Observation 3 would yield $zw \notin A$, which would contradict $zw \in A$. Hence, we cannot have $w \in X$. Thus, $w \notin X$. However, $w \in V \setminus C = X \cup Y$. Combining this with $w \notin X$, we obtain $w \in (X \cup Y) \setminus X \subseteq Y$. Hence, $Y \neq \varnothing$. Thus, Observation 6 is proved.]

Observation 7: There exists an arc $yx \in A$ with $y \in Y$ and $x \in X$.

[*Proof of Observation 7:* There exists at least one vertex $u \in X$ (by Observation 5) and at least one vertex $v \in Y$ (by Observation 6). Consider these u and v. From $u \in X$, we obtain $u \notin Y$ (since $X \cap Y = \varnothing$) and thus $u \in V \setminus Y$. Since D is strongly connected, there exists a walk from v to u. This walk must cross from the set Y into the set $V \setminus Y$ at some point (since $v \in Y$ and $u \in V \setminus Y$). Hence, there exists an arc $yx \in A$ whose source y belongs to Y and whose target x belongs to $V \setminus Y$. Consider this arc yx. Thus, $y \in Y$ and $x \in V \setminus Y$ and $yx \in A$. If we had $x \in C$, then Observation 4 (applied to $w = y$ and $z = x$) would yield $yx \notin A$, which would contradict $yx \in A$. Hence, we cannot have $x \in C$. Thus, $x \notin C$, so that $x \in V \setminus C = X \cup Y$. Since we also have $x \notin Y$ (because $x \in V \setminus Y$), we thus conclude that $x \in (X \cup Y) \setminus Y \subseteq X$. Hence, we have found an arc $yx \in A$ with $y \in Y$ and $x \in X$. This proves Observation 7.]

We are almost done now. Observation 7 shows that there exists an arc $yx \in A$ with $y \in Y$ and $x \in X$. Consider this arc. Observation 1 (applied to $w = x$ and $i = 1$) yields $xv_1 \in A$. Observation 2 (applied to $w = y$ and $i = 0$) yields $v_0 y \in A$. Thus, we can "detour" our cycle $\mathbf{c}$ to pass through y and x, obtaining a longer cycle $(v_0, y, x, v_1, v_2, v_3, \ldots, v_k)$ (because $v_0 y \in A$ and $yx \in A$ and $xv_1 \in A$). However, this contradicts the fact that $\mathbf{c}$ is a cycle having **maximum length**. This contradiction finishes our proof that $\mathbf{c}$ is a Hamiltonian cycle. Thus, Theorem 1.5.5 follows. □

1.6. Rédei's theorem

We now come to the highlight of this lecture, a result of L. Rédei from 1933 ([Redei33, §I]):

Theorem 1.6.1 (Rédei's Strong Theorem)**.** Let D be a tournament. Then, (# of hamps of D) is odd. Here, we agree to consider the empty list $()$ as a hamp of the empty tournament with 0 vertices.

Theorem 1.6.1 clearly implies Theorem 1.4.9 (because if the # of hamps of D is odd, then this # is clearly nonzero, and therefore D has a hamp). However, Theorem 1.6.1 is much harder to prove than Theorem 1.4.9. The proof we shall give below is not Rédei's original proof (which relied on subtle manipulation of determinants[12]), but rather Berge's proof from [Berge91, §10.2, Theorem 6] (which also appears in [Tomesc85, solution to problem 7.8]).[13] We have already done most of the hard work when we proved Theorem 1.3.6, which will come useful in the proof; but we will need one more lemma:

Lemma 1.6.2. Let $D = (V, A)$ be a tournament, and let $vw \in A$ be an arc of D. Let D' be the digraph obtained from D by reversing the arc vw (that is, replacing it by wv). (In other words, let $D' = (V, \ (A \setminus \{vw\}) \cup \{wv\})$.) Then,

$$(\text{\# of hamps of } D) \equiv (\text{\# of hamps of } D') \bmod 2.$$

Proof of Lemma 1.6.2. (We follow [Berge91, §10.2, Theorem 6].)

The digraph D is a tournament, thus loopless. Hence, the arc vw is not a loop. In other words, $v \neq w$. Hence, the tournament axiom entails that exactly one of the two pairs vw and wv is an arc of D. Hence, wv is not an arc of D (since vw is an arc of D).

Define two further digraphs D_0 and D_2 by

$$D_0 := (\text{the digraph } D \text{ with the arc } vw \text{ removed}) = (V, \ A \setminus \{vw\})$$

and

$$D_2 := (\text{the digraph } D \text{ with the arc } wv \text{ added}) = (V, \ A \cup \{wv\}).$$

Note that neither D_0 nor D_2 is a tournament.

The digraph D_0 is the digraph D with the arc vw removed. Hence, the digraph $\overline{D_0}$ is the digraph $\overline{D}$ with the arc vw added.

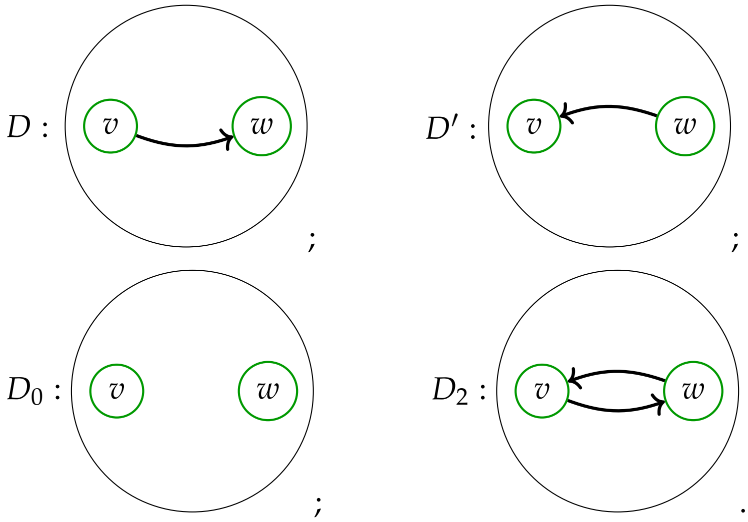
The digraph D_2 is the digraph D with the arc wv added. Hence, the digraph $(D_2)^{\text{rev}}$ is the digraph D^{rev} with the arc vw added.

Here are visualizations of the four digraphs D, D', D_0 and D_2 (we are only showing the arcs between the vertices v and w, since all other arcs are exactly the

[12] An English translation of this proof can be found in Moon's booklet [Moon13, proof of Theorem 14].

[13] Another proof appears in [Lass02, Corollaire 5.1].

same in all four digraphs):



We shall use the following notation: If two digraphs E_1 and E_2 with the same set of vertices have the same arcs except possibly the loops (i.e., if the arcs of E_1 that are not loops are precisely the arcs of E_2 that are not loops), then we shall write $E_1 \overset{\circ}{=} E_2$. In other words, two digraphs E_1 and E_2 satisfy $E_1 \overset{\circ}{=} E_2$ if and only if they are "equal up to loops" (i.e., they have the same vertices and the same arcs except possibly for the loops). In other words, two digraphs E_1 and E_2 satisfy $E_1 \overset{\circ}{=} E_2$ if and only if one can be obtained from the other by adding and removing loops.

It is clear that if $\mathbf{p}$ is a path of a digraph, then none of the arcs of $\mathbf{p}$ is a loop (because the vertices of a path have to be distinct, but a loop would contribute two equal vertices to $\mathbf{p}$). In other words, a loop cannot be an arc of any path. Thus, if we add or remove a loop to a digraph, then the paths of the digraph do not change; in particular, the hamps of the digraph do not change. Hence, if E_1 and E_2 are two digraphs satisfying $E_1 \overset{\circ}{=} E_2$, then

$$(\text{\# of hamps of } E_1) = (\text{\# of hamps of } E_2) . \qquad (12)$$

However, D is a tournament; thus, Proposition 1.4.4 yields that the arcs of $\overline{D}$ that are not loops are precisely the arcs of D^{rev}. Hence, $\overline{D} \overset{\circ}{=} D^{\text{rev}}$ (but we don't generally have $\overline{D} = D^{\text{rev}}$, since the digraph $\overline{D}$ has loops whereas the digraph D^{rev} does not). This entails $\overline{D_0} \overset{\circ}{=} (D_2)^{\text{rev}}$ (because the digraph $\overline{D_0}$ is the digraph $\overline{D}$ with the arc vw added, whereas the digraph $(D_2)^{\text{rev}}$ is the digraph D^{rev} with the arc vw added). Therefore, (12) (applied to $E_1 = \overline{D_0}$ and $E_2 = (D_2)^{\text{rev}}$) yields

$$\left(\text{\# of hamps of } \overline{D_0}\right) = \left(\text{\# of hamps of } (D_2)^{\text{rev}}\right) = (\text{\# of hamps of } D_2)$$

(by Proposition 1.3.5, applied to D_2 instead of D). Hence,

$$\begin{aligned}(\text{\# of hamps of } D_2) &= \left(\text{\# of hamps of } \overline{D_0}\right) \\ &\equiv (\text{\# of hamps of } D_0) \bmod 2 \qquad (13)\end{aligned}$$

(by Theorem 1.3.6, applied to D_0 instead of D).

However, recall that D_2 is the digraph D with the arc wv added (and this arc wv is not an arc of D). Hence, the hamps of D are exactly the hamps of D_2 that do not use[14] the arc wv. Therefore,

$$\begin{aligned}
&(\text{\# of hamps of } D) \\
&= (\text{\# of hamps of } D_2 \text{ that do not use the arc } wv) \\
&= (\text{\# of hamps of } D_2) \\
&\qquad - (\text{\# of hamps of } D_2 \text{ that use the arc } wv) . \qquad (14)
\end{aligned}$$

However, the digraph D_2 is the digraph D' with the arc vw added (this follows by comparing the definitions of D_2 and D'). Thus, the digraph D' is the digraph D_2 with the arc vw removed (since vw is not an arc of D'). Thus, the hamps of D' are exactly the hamps of D_2 that do not use the arc vw. In particular, any hamp of D' is a hamp of D_2. Therefore, any hamp of D' that uses the arc wv is a hamp of D_2 that uses the arc wv.

On the other hand, a path of D_2 cannot use both arcs vw and wv simultaneously[15]. Thus, any path of D_2 that uses the arc wv cannot use the arc vw. Hence, in particular, any hamp of D_2 that uses the arc wv cannot use the arc vw, and thus must be a hamp of D' (since the hamps of D' are exactly the hamps of D_2 that do not use the arc vw). Thus, any hamp of D_2 that uses the arc of wv is a hamp of D' that uses the arc wv. Conversely, as we have already shown, any hamp of D' that uses the arc wv is a hamp of D_2 that uses the arc wv. Combining the results of the previous two sentences, we see that the hamps of D_2 that use the arc wv are precisely the hamps of D' that use the arc wv. Hence,

$$\begin{aligned}
&(\text{\# of hamps of } D_2 \text{ that use the arc } wv) \\
&= \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) . \qquad (15)
\end{aligned}$$

The digraph D_0 is the digraph D' with the arc wv removed (this follows by comparing the definitions of D_0 and D'). Hence, the hamps of D_0 are precisely the hamps of D' that do not use the arc wv. Therefore,

$$\begin{aligned}
&(\text{\# of hamps of } D_0) \\
&= \left(\text{\# of hamps of } D' \text{ that do not use the arc } wv\right) \\
&= \left(\text{\# of hamps of } D'\right) \\
&\qquad - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) . \qquad (16)
\end{aligned}$$

[14] A walk $\mathbf{w}$ is said to *use* an arc a if a is an arc of $\mathbf{w}$.

[15] since the vertices of a path must be distinct, but having both vw and wv as arcs would cause at least one of the vertices v and w to appear twice

Now, (14) becomes

$$\begin{aligned}
&(\text{\# of hamps of } D) \\
&= \underbrace{(\text{\# of hamps of } D_2)}_{\substack{\equiv(\text{\# of hamps of } D_0) \bmod 2 \\ \text{(by (13))}}} - \underbrace{(\text{\# of hamps of } D_2 \text{ that use the arc } wv)}_{\substack{=(\text{\# of hamps of } D' \text{ that use the arc } wv) \\ \text{(by (15))}}} \\
&\equiv (\text{\# of hamps of } D_0) - \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) \\
&\equiv (\text{\# of hamps of } D_0) + \left(\text{\# of hamps of } D' \text{ that use the arc } wv\right) \\
&\qquad\qquad (\text{since } x - y \equiv x + y \bmod 2 \text{ for any two integers } x \text{ and } y) \\
&= \left(\text{\# of hamps of } D'\right) \bmod 2 \qquad (\text{by (16)}).
\end{aligned}$$

This proves Lemma 1.6.2. □

At last, we are now ready to prove Rédei's Strong Theorem:

Proof of Theorem 1.6.1. Write the digraph D as $D = (V, A)$. We WLOG assume that $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ (indeed, we can always achieve this by renaming the vertices of D).

We want to show that $(\text{\# of hamps of } D)$ is odd. Lemma 1.6.2 shows that if we reverse any arc of D (that is, if we pick some arc vw of D and replace it by the arc wv), then the number $(\text{\# of hamps of } D)$ remains unchanged modulo 2 (that is, it stays even if it was even, and stays odd if it was odd). Thus, of course, the same holds if we reverse **several** arcs of D (because we can perform these reversals one by one, and our digraph remains a tournament throughout the process[16]). Since we are only interested in this number modulo 2 (after all, we are trying to show that it is odd), we can therefore WLOG assume that

$$\begin{aligned}
A = \{(i,j) \in V \times V \mid i < j\} = \{&12,\ 13,\ 14,\ \ldots,\ 1n, \\
&23,\ 24,\ \ldots,\ 2n, \\
&\ldots \\
&(n-1)\,n\}
\end{aligned}$$

(because we can always achieve this situation by reversing each arc ij of D that satisfies $i > j$). Assume this. Then, Proposition 1.3.4 yields

$$(\text{\# of hamps of } D) = 1.$$

Thus, $(\text{\# of hamps of } D)$ is odd. This proves Theorem 1.6.1. □

One might wonder whether Theorem 1.6.1 has a converse: Does every odd positive integer equal the # of hamps of some tournament? Surprisingly, the answer is "no": By a mix of theoretical reasoning and computer-assisted brute force, it has been proved that a tournament cannot have exactly 7 hamps, nor can it have exactly 21 hamps. Each other odd number between 1 and 80555 has been verified to appear as # of hamps of some tournament, but the question for higher numebrs is still open. See [MO232751] for more about this peculiar question.

[16]Here we are using Proposition 1.4.7.

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