5707 Spring 2017 Lecture 5

Hamiltonian paths & cycles

Def. Let $G=(V,E)$ be a simple graph.

A Hamiltonian path in G means a walk in G which contains each vertex of G exactly once. It is obviously a path.

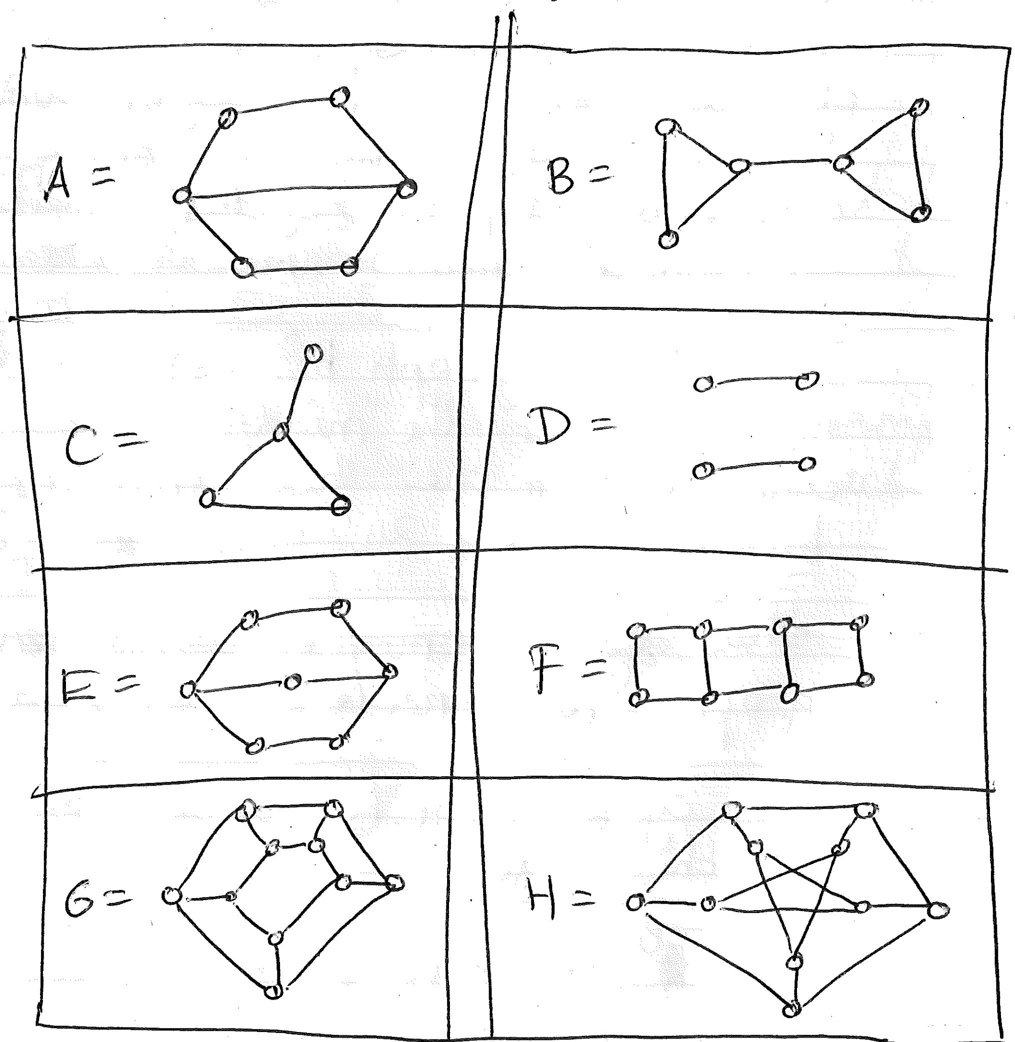
A Hamiltonian cycle in G means a cycle $(v_0, v_1, \ldots, v_k)$ in G such that each vertex of G appears exactly once among $v_0, v_1, \ldots, v_{k-1}$.

Some graphs have Hamiltonian paths, others don't. Of course, having a Hamiltonian cycle is even stronger than having a Hamiltonian path: If $(v_0, v_1, \ldots, v_k)$ is a Hamiltonian cycle of G, then $(v_0, v_1, \ldots, v_{k-1})$ is a Hamiltonian path of G.

Convention: In the following, we abbreviate

- "Hamiltonian path" as "hamp";
- "Hamiltonian cycle" as "hamc".

Examples: Which of the following graphs have hamps? Which have hamcs?



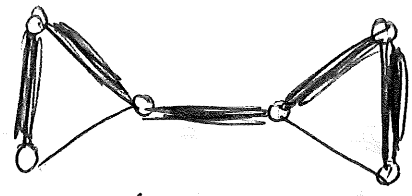
Answers: First of all, if $\exists$ hamc, then $\exists$ hamp (since we can remove the last vertex ~~& edge~~ from a hamc to get a hamp).

Second, a graph that is not connected has neither a hamp nor a hamc (clearly).

These simplify our job. Nevertheless, finding hamc and hamps or proving their non-existence is a hard task; there are no fast algorithms. You can try a mix of brute force (checking all paths / cycles; there are only finitely many) and wit.

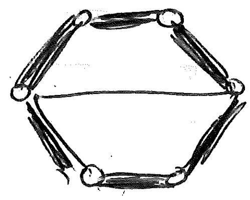
The simple graph B has no hamcs, since the "middle edge" (the one connecting the two triangles) is a bridge (i.e., an edge which, if removed, would render the graph disconnected), and therefore any cycle must be wholly "on one side" of this edge.

But ~~A~~ B does have a hamp:



(shown here by drawing the edges of the hamp thickly).

The graph A has a hamc:

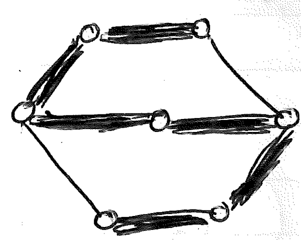


and thus also a hamp.

The graph C has a hamp, but no hamc. The reasons are similar to those for B.

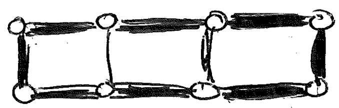
The graph D has neither hamp nor hamc, since it is disconnected.

The graph E has a hamp:



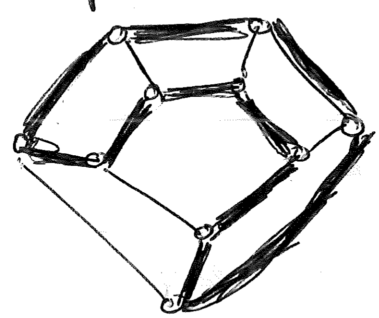
but no hamc (checking this does require some work, though).

The graph F has a hamc:



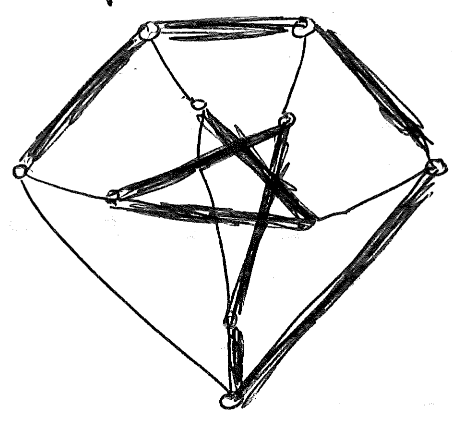
and hence a hamp as well.

The graph G has a hamc:



and hence a hamp as well.

The graph H has a hamp:



but no hamc (this is not obvious — see the Wikipedia page on "Petersen graph").

As already mentioned, finding hamcs or proving that there are none is hard. This is related to the "traveling salesman problem" (TSP), which asks for a hamc with "minimum weight" (each edge has a number assigned to it, called its weight, and you want to find a hamc minimizing the total weight of the edges it uses, provided that a hamc exists).
There is a lot of literature on this (mostly in computer science).

We shall show some necessary criteria and some sufficient criteria (but no necessary-and-sufficient criteria) for the existence of hamps and hamcs. Here is the most famous one:

Thm. 1 (Ore). Let $G=(V,E)$ be a simple graph. Set $n=|V|$.
Assume that $n \geq 3$.

Assume that $\deg x + \deg y \geq n$ for any two non-adjacent vertices x and y of G.

Then, G has a hamc.

There are various proofs of Thm. 1 around; see, e.g., [Harju 14, Theorem 3.6] or

[Guicha16, Theorem 5.3.2].
We shall give a somewhat unconventional proof (following the "Algorithm" section on the Wikipedia page on "Ore's theorem"):

<u>Proof of Thm. 1.</u> A <u>listing</u> of V shall mean a list of elements of V, containing each element exactly once. It must clearly be an n-tuple.

The <u>hamness</u> of a listing $(v_1, v_2, \ldots, v_n)$ of V will mean the number of all $i \in \{1, 2, \ldots, n\}$ such that $v_i v_{i+1} \in E$. Here, we set $v_{n+1} = v_1$.

Now, I claim that if you have a listing of hamness $< n$, then you can slightly modify it to get a listing of larger hamness.

[<u>Proof:</u> Let $(v_1, v_2, \ldots, v_n)$ be a listing of hamness $< n$.
We want to modify it to get a listing of larger hamness.

$\exists\, i \in \{1, 2, \ldots, n\}$ such that $v_i v_{i+1} \notin E$ (since the listing $(v_1, v_2, \ldots, v_n)$ has

hamness $< n$). Pick such an i.
Thus, v_i and v_{i+1} are non-adjacent.
Hence, $\deg(v_i) + \deg(v_{i+1}) \geq n$
(by the $\deg x + \deg y \geq n$ condition of Thm. 1).

Since $\deg(v_i) = \left|\{w \in V \mid v_i w \in E\}\right|$

$$= \left|\{j \in \{1, 2, \ldots, n\} \mid v_i v_j \in E\}\right|$$

$$= \left|\{j \in \{1, 2, \ldots, n\} \setminus \{i\} \mid v_i v_j \in E\}\right|$$

(since $j = i$ could not satisfy $v_i v_j \in E$ anyway)

and $\deg(v_{i+1}) = \left|\{w \in V \mid v_{i+1} w \in E\}\right|$

$$= \left|\{j \in \{1, 2, \ldots, n\} \mid v_{i+1} v_{j+1} \in E\}\right|$$

(since $(v_2, v_3, \ldots, v_n, v_{n+1})$ is a listing of V (since $v_{n+1} = v_1$))

$$= \left|\{j \in \{1, 2, \ldots, n\} \setminus \{i\} \mid v_{i+1} v_{j+1} \in E\}\right|$$

(since $j = i$ could not satisfy $v_{i+1} v_{j+1} \in E$ anyway),

this rewrites as

$|\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_i v_j \in E\}|$

$+ |\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_{i+1} v_{j+1} \in E\}|$

$\geq n$.

Thus, the two subsets
$\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_i v_j \in E\}$ and
$\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_{i+1} v_{j+1} \in E\}$ of
the $(n-1)$-element set $\{1,2,\ldots,n\} \setminus \{i\}$
~~must overlap~~ have their sizes add up to something $\geq n$. Hence, these two subsets must overlap.

In other words, $\exists\, j \in \{1,2,\ldots,n\} \setminus \{i\}$ satisfying both $v_i v_j \in E$ and $v_{i+1} v_{j+1} \in E$. Consider such a j.

Now, consider a new listing ~~$(w_1, w_2, \ldots, w_n)$~~ obtained from the old listing $(v_1, v_2, \ldots, v_n)$ as follows:

- First, rotate the old listing so that it starts with v_{i+1}. Thus, you get the listing $(v_{i+1}, v_{i+2}, \ldots, v_n, v_1, v_2, \ldots, v_i)$.
- Then, reverse the part of the listing starting at v_{i+1} and ending at v_j. Thus, you get the new listing

$$(\underbrace{v_j, v_{j-1}, \ldots\ldots, v_{i+1}}_{\substack{\text{This is the reversed part,} \\ \text{which may and may not} \\ \text{"wrap around" (i.e. contain} \\ \ldots, v_1, v_n \text{ somewhere).}}}, \underbrace{v_{j+1}, v_{j+2}, \ldots, v_i}_{\substack{\text{This is the part} \\ \text{that was not} \\ \text{reversed.}}}).$$

This is the new listing we want.

I claim that the new listing has larger hamness than the old one.

Indeed, rotating the listing clearly has not changed its hamness. But reversing part of it has: After the reversal, the edges $v_i v_{i+1}$ and $v_j v_{j+1}$ no longer count towards the hamness (if they ever existed to begin with), but the edges $v_i v_j$ and $v_{i+1} v_{j+1}$ started counting towards the hamness.

This is a good bargain, because it means that the hamness gained +2 (thanks to the edges $v_i v_j$ and $v_{i+1} v_{j+1}$, which both exist), while only losing 0 or -1 (since the edge $v_i v_{i+1}$ did not exist and thus was not lost, whereas the edge $v_j v_{j+1}$ might have existed and thus might have been lost). Thus, the hamness of the new listing is larger than the hamness of the old listing either by 1 or by 2.

(Visually, it is best to represent a listing $(v_1, v_2, \ldots, v_n)$ by drawing the vertices $v_1, v_2, \ldots, v_n$ on a circle in this order. The hamness then counts how often successive vertices on the circle are adjacent. We rotated all vertices and then reflected part of them to increase the hamness.)]

Thus, we can start with <u>any</u> listing of V and keep modifying it until its hamness becomes $\geq n$.

But once its hamness is $\geq n$, we

have found a hamc. So we are done. $\square$

<u>Cor. 2 (Dirac).</u> Let $G = (V, E)$ be a simple graph. Let $n = |V|$.

Assume that $n \geq 3$.
Assume that $\deg x \geq n/2$ $\forall x \in V$.
Then, G has a hamc.

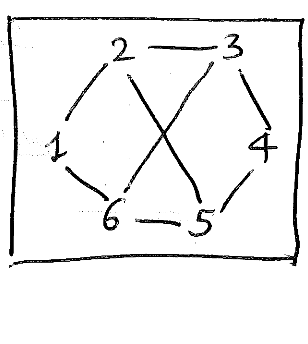
<u>Proof.</u> Apply Thm. 1. $\square$

So much for sufficient criteria.
What about necessary ones?

<u>Prop. 3.</u> Let $G = (V, E)$ be a simple graph.

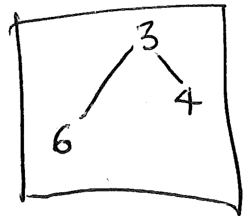
For each subset S of V, we let $G \setminus S$ be the subgraph of G obtained by removing all vertices in S (and all edges that have at least one endpoint in S).

[For example, if

$G =$ 

and $S = \{1, 2, 5\}$,

then

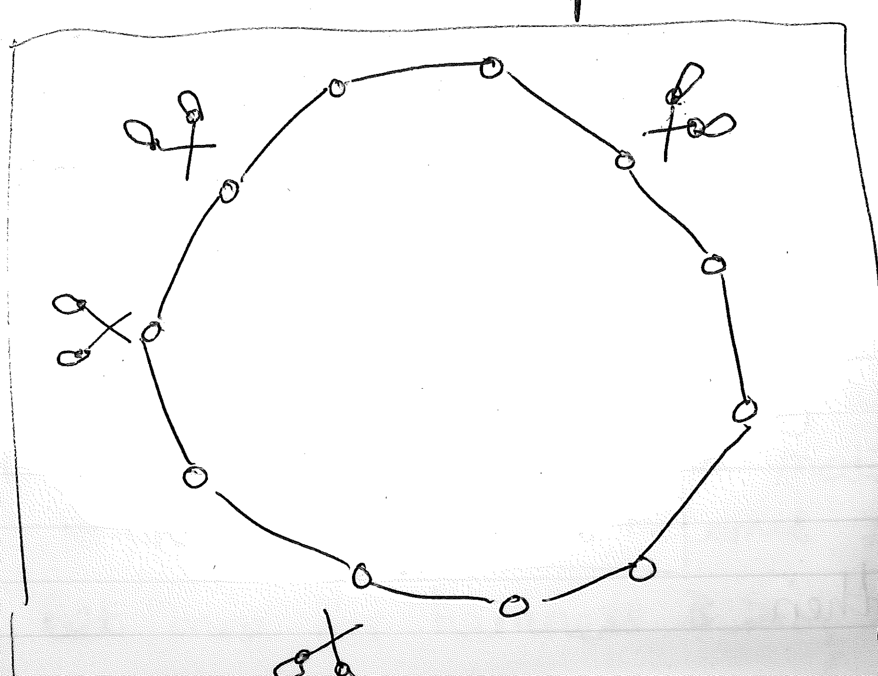
$G \setminus S =$  .]

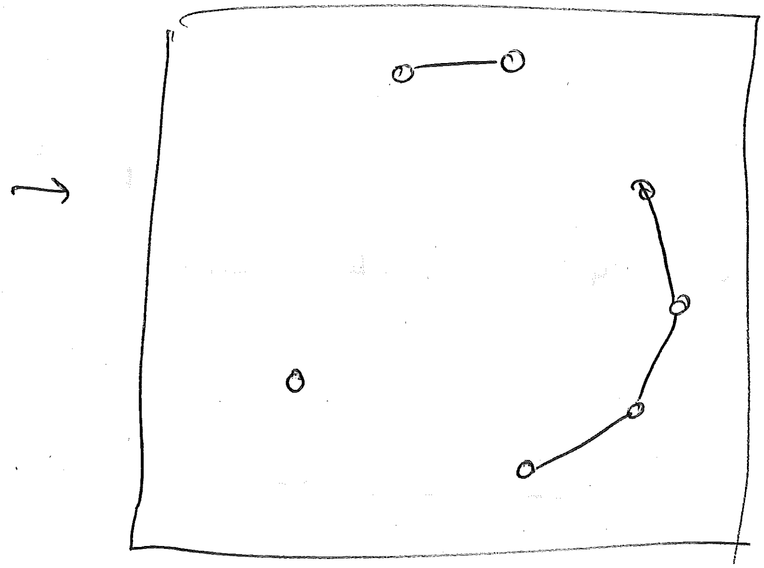
Also, we let $b_0(H)$ denote the number of connected components of a simple graph H.

(a) If G has a hamc, then every $S \subseteq V$ satisfies $b_0(G \setminus S) \leq |S|$, if S is nonempty.

(b) If G has a hamp, then every $S \subseteq V$ satisfies $b_0(G \setminus S) \leq |S| + 1$.

Proof. (a) Cutting $|S|$ vertices out of a cycle, we split the cycle into at most $|S|$ connected components:





of course, our graph G might not be a cycle, but if it has a hamc, then we can conclude that $G \setminus S$ will have $\leq |S|$ connected components just using the (surviving) edges of the hamc alone. Taking into account all the other edges can only decrease the # of components.

(b) Similar.

Now, let us move on to concrete examples of graphs having hamcs.

Def. Let $n \in \mathbb{N}$. The *n-hypercube* Q_n is the simple graph with vertex set $\{0,1\}^n = \{(a_1, a_2, \ldots, a_n) \mid \text{each } a_i \in \{0,1\}\}$ and edge set defined as follows:

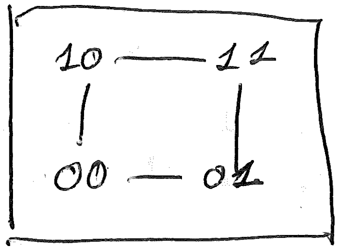
An $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ is adjacent to an $(b_1, b_2, \ldots, b_n) \in \{0,1\}^n$ if and only if there exists *exactly* one $i \in \{1, 2, \ldots, n\}$ satisfying $a_i \neq b_i$.

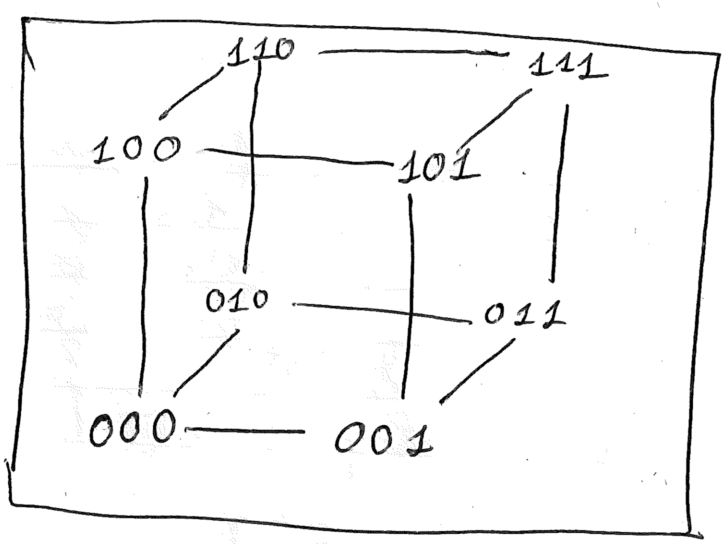
The elements of $\{0,1\}^n$ are often called *bitstrings*, and their entries are called their *bits*. So two bitstrings are adjacent in Q_n if and only if they differ in exactly one bit.

We often write a bitstring $(a_1, a_2, \ldots, a_n)$ as $a_1 a_2 \cdots a_n$.

Examples:

$$Q_1 = \boxed{0 \text{ ——— } 1}.$$

$Q_2 =$ 

$Q_3 =$ 

Q_4 looks like a tesseract, "etc".

Q_0 has only one vertex: the empty bitstring.

<u>Thm. 4 (Gray).</u> Let $n \geq 2$. Then, the graph Q_n has a hamc.

Such hamcs are called <u>Gray codes</u>.

They are circular lists of bitstrings such that two consecutive bitstrings in the list

always differ in exactly one bit.
See the Wikipedia article on "Gray code" for applications.

<u>First proof of Thm. 4.</u> Use the notations from HW2 Exercise 1.

Now, notice that $|V(Q_n)| = 2^n$ is even $\forall n \geq 1$.
Also, Q_1 has a hamp (namely, $(0, 1)$).

Hence, we can prove by induction on n that Q_n has a hamp $\forall n \geq 1$.
(In the induction step, use HW2 Exercise ~~1~~ parts (a) and (b)).

Using this, we can furthermore prove that Q_n has a hamc $\forall n \geq 2$.

(Indeed, use HW 2 Exercise 1 (a) to see that $Q_n \cong Q_{n-1} \times Q_1$; then observe that Q_{n-1} and Q_1 have hamps, that both numbers $|V(Q_{n-1})|$ and $|V(Q_1)|$ are ~~[illegible]~~ > 1, and ~~then~~ that at least one of them is even. Hence HW 2 Exercise 1 (c) shows that $Q_{n-1} \times Q_1$ has a hamc. This gives you a hamc of Q_n via the isomorphism $Q_n \cong Q_{n-1} \times Q_1$.) $\square$

<u>Second proof of Thm. 4.</u> Here is a more concrete proof:

We shall show something stronger:

<u>Claim 1:</u> For each $n \geq 1$, the n-hypercube Q_n has a hamp from $00\cdots0$ to $100\cdots0$.

(Keep in mind that $00\cdots0$ and $100\cdots0$ are bitstrings, not numbers:

$$00\cdots0 = (\underbrace{0, 0, \ldots, 0}_{n \text{ zeroes}}),$$

$$100\cdots0 = (1, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ zeroes}}).)$$

[Proof of Claim 1: Induction over n.

Induction base: Fix $N \geq 2$.

Assume that Claim 1 holds for $n = N-1$. Thus, ~~a hamp~~ Q_{N-1} has a hamp from

$$\underbrace{00\cdots0}_{N-1 \text{ zeroes}} \text{ to } 1\underbrace{00\cdots0}_{N-2 \text{ zeroes}}.$$

Let $\mathbf{p}$ be such a hamp.

By attaching a 0 to the front of each bitstring (=vertex) in $\mathbf{p}$, we obtain a path $\mathbf{q}$ from

$$\underbrace{00\cdots0}_{N \text{ zeroes}} \text{ to } 01\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$$

in Q_N.

But by attaching a 1 to the front of each bitstring (=vertex) in $\mathbf{p}$, we obtain a path $\mathbf{r}$ from

$$1\underbrace{00\cdots0}_{N-1 \text{ zeroes}} \text{ to } 11\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$$

in Q_N.

Now we can assemble a hamp from $\underbrace{00\cdots0}_{N \text{ zeroes}}$ to $1\underbrace{00\cdots0}_{N-1 \text{ zeroes}}$ in Q_N as follows:

- Start at $\underbrace{00\cdots0}_{N \text{ zeroes}}$, and follow the path q to its end (i.e., to $01\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$).
- Then, move to the adjacent vertex $11\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$.
- Then, walk the path r backwards, ending up at $1\underbrace{00\cdots0}_{N-1 \text{ zeroes}}$.

So Claim 1 holds for $n = N$, too. $\square$]

Now, Claim 1 gives us a hamp from

$00\cdots0$ to $100\cdots0$ in Q_n. Attach the edge $\{100\cdots0, 00\cdots0\}$ to it to obtain a hamc in Q_n.

(Here we use $n \geq 2$, because we want this edge to be unused by the hamp! For $n \geq 2$, it works, because the hamp cannot have used this edge (it must have taken $2^n - 1 > 1$ steps to get from $00\cdots0$ to $100\cdots0$, whereas this edge would have gotten there in 1 step). But for $n=1$, this is exactly what goes wrong.) $\square$

Note that the n-hypercube Q_n can be redefined in terms of subsets of $\{1,2,\ldots,n\}$. Namely: Let G_n be the simple graph whose vertex set is the powerset $\mathcal{P}(\{1,2,\ldots,n\})$ of $\{1,2,\ldots,n\}$, and whose vertices S and T are adjacent if and only if S and T differ in exactly one element (i.e., $|(S\setminus T) \cup (T\setminus S)| = 1$).

Then, $G_n \cong Q_n$, because the map

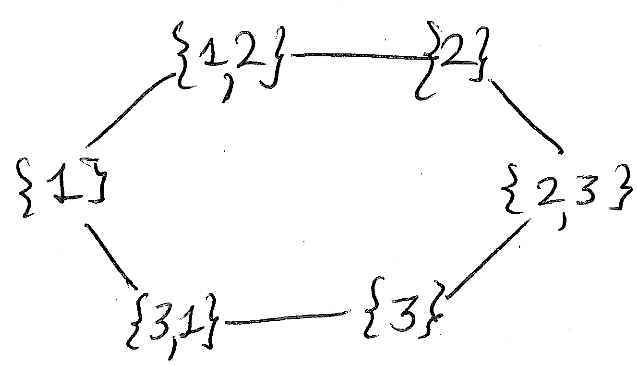
$$\{0,1\}^n \longrightarrow \mathcal{P}(\{1,2,\dots,n\}),$$

$$a_1 a_2 \cdots a_n \longmapsto \{i \in \{1,2,\dots,n\} \mid a_i = 1\}$$

is an isomorphism $Q_n \to G_n$.

So Thm. 4 shows that for $n \geq 2$, we can list all subsets of $\{1,2,\dots,n\}$ in a circular list such that two consecutively appearing subsets always ~~have just 1 element~~ differ in exactly one element.

It was a long-standing question whether the same can be done with the subsets of $\{1,2,\dots,n\}$ having size $(n \pm 1)/2$ when n is odd. For example, for $n=3$, we can do it:



In other words, if n is odd, and if Q'_n is the induced subgraph of Q_n on the subset

$$\left\{ a_1 a_2 \cdots a_n \in \{0,1\}^n \;\middle|\; \sum_{i=1}^{n} a_i \in \left\{ \frac{n+1}{2}, \frac{n-1}{2} \right\} \right\},$$

then does Q'_n have a hamc

(for $n \geqslant 2$)?
Apparently, this question has been finally answered by Mütze in 2014 (arXiv: 1404.4442).
Some variants of this question (for other sizes etc.) are still open!